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A COMBINATORIAL PROBLEM ARISING IN FINITE MARKOV CHAINS

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Consider a homogeneous Markov chain with the transition probability matrix P. By a constant stochastic matrix Q we mean a stochastic matrix all rows of which are identical. It is well known that $\lim_{k=\infty} P^k = Q$ for some constant matrix Q iff there is an integer k_0 such that P^{k_0} contains at least one positive column. (If P^{k_0} has a positive column, then for any integer $k > k_0$ the matrix P^k has also a positive column.)

The following pertinent question arises. Suppose that some power of a non-negative $n \times n$ matrix P has a positive column. What is the least integer k such that P^k has a positive column.

There are many known results concerning the powers of a non-negative matrix. (See, e.g., the survey paper [3], and the books [1] and [4].) As far as I can decide the question mentioned above has been explicitly treated only in the paper [8]. There is also a recent paper [5] in which a problem paralleling ours is treated (with a different motivation). Both papers contain (in essential) the result $k \leqq n^2 - 3n + 3$. Since the results of the present paper cover more than those of [5] and [8] and also the proofs are quite different it seems to be worth to publish them.

If P is a non-negative matrix, the pattern of zeros and non-zeros of P completely determines the pattern of zeros and non-zeros in every power of P. Hence the supposition that P is stochastic is irrelevant for our purposes except that P does not contain a zero row. Replacing the positive entries in P by 1 we may work with Boolean matrices, i.e. $n \times n$ matrices over the Boolean algebra $\{0, 1\}$.

Even more convenient is to work with binary relations in the following sense. (See [7]).)

Let $V = \{a_1, a_2, \dots, a_n\}$, $n \geqq 2$, be a finite set of different elements. A binary relation ϱ on V is a subset of $V \times V$. Denote by $B_n(V)$ the set of all binary relations on V.

To any $\varrho \in B_n(V)$ we asign the Boolean matrix $M_\varrho = (m_{ij})$, where $m_{ij} = 1$ iff $(a_i, a_j) \in \varrho$ and $m_{ij} = 0$ otherwise. Conversely, if M is an $n \times n$ Boolean matrix, we define ϱ_M as follows: The couple $(a_i, a_j) \in \varrho_M$ iff the element in the i-th row and j-th column in the matrix M is the element 1 (of the Boolean algebra $\{0, 1\}$).

The correspondence $\varrho \leftrightarrow M$ has the following properties. If $\varrho, \sigma \in B_n(V)$, then

$$\varrho \cup \sigma \leftrightarrow M_\varrho + M_\sigma = M_{\varrho \cup \sigma},$$
$$\varrho \cdot \sigma \leftrightarrow M_\varrho \cdot M_\sigma = M_{\varrho\sigma}.$$

If $\varrho \in B_n(V)$ and $a_i \in V$, we define

$$a_i\varrho = \{x \in V: (a_i, x) \in \varrho\},$$

$$\varrho a_j = \{y \in V: (y, a_j) \in \varrho\}.$$

Clearly

$$a_j \in a_i\varrho \Leftrightarrow a_i \in \varrho a_j \Leftrightarrow (a_i, a_j) \in \varrho.$$

If U is a non-empty subset of V, we put $U \cdot \varrho = \bigcup_{a_i \in U} a_i\varrho$ and $\varrho \cdot U$ is defined analogously.

In an intuitive manner: If A is an $n \times n$ Boolean matrix and ϱ_A the corresponding binary relation, then $a_i\varrho$ describes precisely the places of non-zeros in the i-th row of A. Analogously ϱa_j describes the places of non-zeros in the j-th column of A.

A graph-theoretical interpretation of a Boolean matrix A (and of the corresponding binary relation ϱ_A) is obvious. We may consider A as the incidence matrix of a directed graph with vertices $V = \{a_1, a_2, \ldots, a_n\}$ and $(a_i, a_j) \in \varrho$ means that there is a path of length 1 from a_i to a_j. We shall denote this graph by G_A or G_{ϱ_A}. (Note that in these directed graphs loops at the vertices are allowable.)

1. Preliminaries

We now recall some notions which are well known in the theory of non-negative matrices.

A Boolean matrix A is called reducible if there exists a permutation matrix P such that

$$P\,A\,P^{-1} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B, D are square matrices of order $\geqq 1$. Otherwise it is called irreducible. A relation $\varrho \in B_n(V)$ is called reducible iff M_ϱ is reducible. (A 1×1 matrix is irreducible.) An irreducible matrix cannot contain a zero row or a zero column.

A relation $\varrho \in B_n(V)$ is reducible iff V can be decomposed into two non-empty subsets $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that $\varrho \in (V_1 \times V_1) \cup (V_2 \times V_1) \cup (V_2 \times V_2)$.

If a column of a Boolean matrix contains no zeros, we shall say in the following that the column is positive.

Lemma 1. *If $\varrho \in B_n(V)$ is irreducible and U a non empty proper subset of V, then $U\varrho$ contains at least one element of V which is not contained in U.*

Proof. Let $U=\{a_\alpha, a_\beta, \ldots, a_\nu\}$. Suppose for an indirect proof that $\{a_\alpha, a_\beta, \ldots, a_\nu\} \cdot \varrho \subset \{a_\alpha, a_\beta, \ldots, a_\nu\}$. Let $(a_\varkappa, a_\lambda) \in \varrho$. If $a_\varkappa \in U$, we have necessarily $a_\lambda \in U$. Hence if $a_\varkappa \in U$ and $a_\lambda \in V \setminus U = \bar{U}$, then $(a_\varkappa, a_\lambda) \notin \varrho$. Therefore

$$\varrho \in (U \times U) \cup (\bar{U} \times U) \cup (\bar{U} \times \bar{U}),$$

i.e. ϱ is reducible, contrary to the assumption.

Remark. Lemma 1 also holds if $U\varrho$ is replaced by ϱU.

In particular if ϱ is irreducible, $a_i\varrho$ contains at least one element of V. Next $a_i\varrho \cup (a_i\varrho) \cdot \varrho = a_i(\varrho \cup \varrho^2)$ contains at least two different elements of V. Further $a_i(\varrho \cup \varrho^2) \cup [a_i(\varrho \cup \varrho^2)] \cdot \varrho = a_i(\varrho \cup \varrho^2 \cup \varrho^3)$ contains at least three different elements of V. Repeating this argument we immediately obtain:

Lemma 2. *If $\varrho \in B_n(V)$ is irreducible, then*

a) $a_i\varrho \cup a_i\varrho^2 \cup \ldots \cup a_i\varrho^n = V$, *for any* $a_i \in V$.

b) $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n = V \times V$.

c) *To any $a_i \in V$ there is a least integer h_i, $1 \leqq h_i \leqq n$, such that $a_i \in a_i\varrho^{h_i}$.*

Note that we also have $\varrho a_i \cup \varrho^2 a_i \cup \ldots \cup \varrho^n a_i = V$. Next by the same argument which resulted in Lemma 2a we may prove (for ϱ irreducible) that

$$a_i \cup a_i\varrho \cup \ldots \cup a_i\varrho^{n-1} = V \text{ (for any } a_i \in V).$$

This implies:

Lemma 3. *ϱ is irreducible iff G_ϱ is strongly connected.*

An irreducible Boolean matrix A is called primitive if the is an integer $t \geqq 1$ such that $A^t = I$, where I is the Boolean $n \times n$ matrix with all entries positive. Analogously a relation $\varrho \in B_n(V)$ is called primitive if there is an integer $t \geqq 1$ such that $\varrho^t = V \times V$.

Note that if ϱ is primitive, then any power of ϱ is primitive. (In contradistinction to this a power of an irreducible matrix may be reducible.)

Lemma 4. *If A is an irreducible Boolean matrix and some power of A has a positive column, then A is primitive.*

Proof. Denote $\varrho = \varrho_A$. By supposition there is an element $a^* \in V$ and an integer $s \geqq 1$ such that $\varrho^s a^* = V$. Let a_i be any element of V, $a_i \neq a^*$. Since G_ϱ is strongly connected there is a path of length s_i, $1 \leqq s_i \leqq n-1$ leading from the vertex a^* to the vertex a_i, i. e. $a^* \in \varrho^{s_i} a_i$. But then

$$\varrho^{s_i+s} a_i = \varrho^s \cdot \varrho^{s_i} a_i \supset \varrho^s a^* = V,$$

whence $\varrho^{s_i+s} a_i = V$. Putting $s_0 = \max_i s_i$, we have $\varrho^{s+s_0} a_i = V$ for any $a_i \in V$, i.e.

$\varrho^{s+s_0} = V \times V$. Hence ϱ is primitive. [We have used that $\varrho^k a_i = V$ implies $\varrho^{k+u} a_i = V$ for any integer $u \geqq 0$.]

Let now A be any Boolean square matrix. It is known and easy to see that there is a permutation matrix P such that

$$P\ A\ P^{-1} = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ A_{21} & A_2 & \dots & 0 \\ & & \dots & . \\ A_{k1} & A_{k2} & \dots & A_k \end{pmatrix},$$

where A_i $(i = 1, 2, \dots, k)$ are irreducible Boolean square matrices.

If some power of A has a positive column, the same is true for $P\ A\ P^{-1}$. By Lemma 4 in this case A_1 is necessarily primitive. Hence in the sequel it is sufficient to consider the case of an $n \times n$ matrix M of the form

$$M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

where A is primitive. We first treat the case $M = A$.

2. The case of a primitive matrix

Any $n \times n$ primitive Boolean matrix A contains at least one row and at least one column containing at least two positive elements. Hence there is an $a^* \in V$ such that $\varrho_A a^*$ contains at least two elements of V. Therefore (writing $\varrho = \varrho_A$) the equality

$$\varrho a^* \cup \varrho^2 a^* \cup \dots \cup \varrho^n a^* = V$$

can be replaced by

$$\varrho a^* \cup \varrho^2 a^* \cup \dots \cup \varrho^{n-1} a^* = V.$$

This implies that there is an integer h, $1 \leqq h \leqq n-1$, such that $a^* \in \varrho^h a^*$. Now consider the chain

$$\varrho a^* \subset \varrho^{h+1} a^* \subset \varrho^{2h+1} a^* \subset \dots \subset \varrho^{(n-2)h+1} a^*.$$

Since the first term contains at least two different elements of V we have $\varrho^{(n-2)h+1} a^* = V$. Now $(n-2)h + 1 \leqq (n-2)(n-1) + 1 = n^2 - 3n + 3$.

We have proved the first part of the following theorem.

Theorem 1. *Let P be any non-negative $n \times n$ primitive matrix. Denote $L = n^2 - 3n + 3$. Then P^L contains at least one positive column. For any $n \geqq 2$ there are matrices for which the number L cannot be replaced by a smaller one.*

To prove the second part consider the following $n \times n$ Boolean matrix W_n.

$$W_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \end{pmatrix}.$$

The corresponding graph is

$$\begin{array}{ccccc} a_1 & \longrightarrow & a_2 & \longrightarrow a_3 & \dots \\ \uparrow & \nearrow & & & \vdots \\ a_n & \longleftarrow & a_{n-1} & \longleftarrow a_{n-2} & \dots \end{array}$$

Note that the matrix W_n (Wielandt matrix) has been many times used in literature to prove various extremal properties of non-negative matrices.

The case $n=2$ (i.e. $L=1$) is trivial. So we may suppose $n \geqq 3$.

It is sufficient to prove that W_n^{L-1} does not contain a positive column. We prove more precisely (writing $\varrho = \varrho_{W_n}$) that ϱ^{L-1} does not contain the couples (a_1, a_2) $(a_2, a_3) \dots (a_{n-1}, a_n)$ (a_n, a_1). Any path leading from the vertex a_{i-1} to the vertex a_i $(i=2, 3, \dots, n)$ or from the vertex a_n to the vertex a_1 has a length of the form $l(n-1)+1+k \cdot n$, where $l \geqq 0$, $k \geqq 0$ are integers.

It is sufficient to show that an identity of the form

$$1 + kn + l(n-1) = (n-1)(n-2)$$

cannot hold. This identity can be written in the form

$$n(k+1) + l(n-1) = (n-1)^2, \tag{1}$$

which implies (for $n \geqq 3$) $(n-1) | (k+1)$, i.e. $k+1 = v(n-1)$, where $v \geqq 1$ is an integer. But then (1) implies $nv + l = n-1$, which is impossible. This proves Theorem 1.

For further purposes we prove:

Lemma 5. *The matrix* W_n^L, $L = n^2 - 3n + 3$ *contains a unique positive column (namely the second one).*

Proof. Write again $\varrho_{W_n} = \varrho$. It is sufficient to show that ϱ^L does not contain the couples $(a_1\, a_3)$ (a_2, a_4) $\dots$ (a_{n-2}, a_n) and (a_{n-1}, a_1). Any path leading from the vertex a_i to the vertex a_{i+2} $(i = 1, 2, \dots, n-2)$ or from the vertex a_{n-1} to the vertex a_1 is of the form $k(n-1) + l \cdot n + 2$. An equation

$$k(n-1) + l \cdot n + 2 = n^2 - 3n + 3$$

would imply

$$k(n-1) + n(l+1) = (n-1)^2. \tag{2}$$

Hence $(n-1)/n(l+1)$, i.e. (for $n \geqq 3$) $l+1=v(n-1)$ with an integer $v \geqq 1$. But then (2) would imply $k+vn=n-1$, which is impossible. This proves Lemma 5.

We may use Theorem 1 to prove the following well-known Corollary which will be needed in the following.

Corollary 1. *If A is an $n \times n$ primitive Boolean matrix and $S=n^2-2n+2$, then A^S has all entries positive and for any $n \geqq 2$ there are matrices for which the integer S cannot be replaced by a smaller one.*

Proof. Write again $\varrho_A=\varrho$. By Theorem 1 there is an $a^* \in V$ such that $\varrho^L a^*=V$, when $L=n^2-3n+3$. Since G_ϱ is strongly connected there is a path from a^* to a_i of length s_i, $1 \leqq s_i \leqq n-1$, i.e. $a^* \in \varrho^{s_i} a_i$. Then

$$V=\varrho^L a^* \subset \varrho^{L+s_i} a_i ,$$

whence $\varrho^{L+s_i} a_i=V$. If $s_0=\max_i s_i$, we have $\varrho^{L+s_0} a_i=V$ for any $a_i \in V$, i.e. $\varrho^{L+s_0}=V \times V$. But $L+s_0 \leqq n^2-3n+3+n-1=n^2-2n+2$. This proves the first statement.

To prove the second statement consider again the matrix $W_n (n \geqq 3)$ and denote $\varrho=\varrho_{W_n}$. It is sufficient to prove that $a_1 \notin a_1 \varrho^{S-1}$. Any path from vertex a_1 to the vertex a_1 has a length of the form either $k \cdot n (k \geqq 1)$ or $(n-1)+l(n-1)+1+k_1 n=l(n-1)+(k_1+1)n$ $(k_1 \geqq 0,\ l \geqq 0)$. Hence it is sufficient to show that the equation

$$l(n-1)+(k_1+1)n=(n-1)^2 \tag{3}$$

with $l \geqq 0$, $k_1 \geqq 0$ cannot hold. The equality (3) implies (for $n \geqq 3$) $(n-1)/(k_1+1)$, i.e. $k_1+1=v(n-1)$ with an integer $v \geqq 1$. But then (6) implies $l+vn=n-1$, which is impossible. This completes the proof of our Corollary.

Remark. It should be emphasized once more that Corollary 1 has been proved more or less independently by several authors. There are also deep considerations concerning the conditions under which S can be replaced by a smaller integer. This is done by considering the lengths of various circuits in the graph G_ϱ. (See [3].) The last method has been used in [8] to prove Theorem 1. Our method is much simpler.

A numerical example. It may be of some interest to follow on a numerical example the powers of W_n, to see how the columns are successively filled up. Take, e.g., $n=5$. Then $L=13$, $S=17$.

$$W_5=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad W_5^{12}=\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$W_5^{13}=\begin{pmatrix}1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\\0&1&1&1&1\\1&1&1&1&1\end{pmatrix},\qquad W_5^{14}=\begin{pmatrix}1&1&1&0&1\\1&1&1&1&0\\0&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix},$$

$$W_5^{15}=\begin{pmatrix}1&1&1&1&0\\0&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix},\qquad W_5^{16}=\begin{pmatrix}0&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix},$$

while W_5^{17} has all entries positive.

3. The general case

Let us consider now the matrix

$$M=\begin{pmatrix}A&0\\C&B\end{pmatrix}, \tag{4}$$

where A is an $n_1\times n_1$ primitive Boolean matrix and B an $n_2\times n_2$ Boolean matrix, $n_1+n_2=n$, $1\leqq n_1<n$. For convenience write $V=V_a\cup V_b$, where $V_a=\{a_1, a_2, \dots, a_{n_1}\}$. $V_b=\{b_1, b_2, \dots, b_{n_2}\}$.

Suppose that some power of M has a positive column. Then C cannot be a (rectangular) zero matrix. Denote $\varrho_C=\varrho_M\cap(V_b\times V_a)$, $\varrho_A=\varrho_M\cap(V_A\times V_b)$, and let there be

$$\varrho_C=\{(b'_1, a'_1), (b'_2, a'_2), \dots, (b'_u, a'_u)\},\ (b'_i\in V_b,\ a'_j\in V_a).$$

By Theorem 1 there is a vertex $a^*\in V_a$ such that $V_a=\varrho_A^L\,a^*$, where $L\leqq n_1^2-3n_1+3$.

Let $b_i\in V_b$. We first join the vertex b_i with a suitably chosen vertex b'_j by a path of length $\leqq n_2-1$. Such a path necessarily exists since $V_b\in\varrho_M^s a^{**}$ for some s and some $a^{**}\in V_a$. [If $b_i\in\{b'_1, b'_2, \dots, b'_u\}$, the path is simply of length 0.] Next we apply the path $b'_j\to a'_j$ of length 1. We have $a'_j\in b_i\varrho_M^{s_i}$, where $1\leqq s_i\leqq n_2$. Multiplying by $\varrho_M^{n_2-s_i}$ we have $a'_j\varrho_M^{n_2-s_i}\subset b_i\varrho_M^{n_2}$. Since $a'_j\varrho_M^{n_2-s_i}\neq\emptyset$, we may state: To any $b_i\in V_b$ there is at least one element $\bar{a}_i\in V_a$ such that $\bar{a}_i\in b_i\varrho_M^{n_2}$, i.e. $b_i\in\varrho_M^{n_2}\cdot\bar{a}_i$.

Now (and this is essential) since $\bar{a}_i\in V_a=\varrho_A^L a^*\subset\varrho_M^L a^*$, we have $b_i\in\varrho_M^{n_2+L}a^*$ for any $b_i\in V_b$. Hence $V_b\subset\varrho_M^{n_2+L}a^*$. Since also $V_a=\varrho_A^{n_2+L}a^*\subset\varrho_M^{n_2+L}a^*$, we have $V=V_a\cup V_b=\varrho^{n_2+L}a^*$. (This says that the column in M^{n_2+L} corresponding to a^* is positive.)

Remark. If $n_1 = 1$, we have $V_a = \{a_1\}$, $n_2 = n-1$, $b_i \in \varrho_M^{n-1} a_1$ for any $b_i \in V_b$, so that the first column in M^{n-1} is positive. (This will be used in the proof of Theorem 3.)

Now

$$n_2 + L = n - n_1 + n_1^2 - 3n_1 + 3 = n_1^2 - 4n_1 + (n+3).$$

For a fixed n the function $f(n_1) = n_1^2 - 4n_1 + (n+3)$, defined for all integers $n_1 \in \langle 1, n-1 \rangle$, achieves its minimum for $n_1 = 2$. We have $f(2) = n-1$, $f(1) = n$, $f(n-1) = n^2 - 5n + 8$. For $n \geqq 4$ we have $f(1) \leqq f(n-1)$ so that $n_2 + L \leqq n^2 - 5n + 8$. For $n = 2$ we have trivially $n_2 + L \leqq 2$. For $n = 3$ a simple consideration of all possible cases (i.e. $n_1 = 1$ and $n_1 = 2$) shows that M^2 has a positive column.

We have proved the first part of the following Theorem.

Theorem 2. *Let P be an $n \times n$ non-negative matrix having the property that some power of P has a positive column. Denote $K = n^2 - 5n + 8$. If P is not primitive, then P^K has a positive column. For any $n \geqq 3$ there are matrices for which the number K cannot be replaced by a smaller one.*

To prove the second part consider the $n \times n$ Boolean matrix

$$M = \begin{pmatrix} W_{n-1}, & 0 \\ C, & 0 \end{pmatrix},$$

where C is the $1 \times (n-1)$ matrix $(1, 0, \ldots, 0)$. Clearly M^K has a positive column. We prove that M^{K-1} does not contain a positive column. Denote $V_a = \{a_1, a_2, \ldots, a_{n-1}\}$, $V_b = \{b\}$. The corresponding graph is

$$\begin{array}{ccccc} b \longrightarrow a_1 & \longrightarrow a_2 & \longrightarrow \ldots \\ \uparrow \quad \nearrow & & \vdots \\ a_{n-1} \longleftarrow & a_{n-2} \longleftarrow & \ldots \end{array}$$

We have proved (in Lemma 5) that W_{n-1}^L with $L = (n-1)^2 - 3(n-1) + 3 = n^2 - 5n + 7 = K - 1$ contains a unique positive column, namely the second column. To prove that the bound K given in Theorem 2 is sharp it is sufficient to show that ϱ_M^{K-1} does not contain the couple (b, a_2).

Any path from the vertex b to the vertex a_2 has a length of the form $2 + k(n-1) + l(n-2)$, $k \geqq 0$, $l \geqq 0$. To show that the equation

$$2 + k(n-1) + l(n-2) = n^2 - 5n + 7 \tag{5}$$

has no solutions with non-negative integers k, l, we rewrite (5) in the form

$$(k+1)(n-1) = (n-2)(n-2-l).$$

Since (for $n \geqq 3$)$(n-1, n-2) = 1$, we have $(n-1) \mid (n-2-l)$, which is impossible since $n - 2 - l \neq 0$. This completes the proof of Theorem 2.

For $n \geqq 3$ we have $n^2 - 3n + 3 \geqq n^2 - 5n + 8$. For $n = 2$ the problem is trivial. Hence Theorem 1 and Theorem 2 imply:

Corollary 2. *Let P be any $n \times n$ non-negative matrix having the property that some power of P has a positive column. Then the least exponent k for which P^k has a positive column satisfies the inequality $k \leqq n^2 - 3n + 3$.*

4. A concluding question

Suppose that M is of the form (4) and suppose again that some power of M contains a positive column. Then there is an integer l such that M^l has all the first n_1 columns positive. We ask: What is the least such integer l.

Questions of this type have been considered under some supplementary conditions in the paper [6].

In the proof of Theorem 2 we have shown: If some power of M is positive, then to any $b_i \in V_b$ there is an $\bar{a}_i \in V_a$ such that $\bar{a}_i \in b_i \varrho_M^{n-n_1}$. Denote $S = n_1^2 - 2n_1 + 2$. By Corollary 1 we have $\bar{a}_i \varrho_A^S = V_a$ for any $\bar{a}_i \in V_a$. This implies

$$V_a = \bar{a}_i \varrho_A^S \subset \bar{a}_i \varrho_M^S \subset b_i \varrho_M^{S+n-n_1},$$

i.e. $V_b \times V_a \subset \varrho_M^{R_0}$, where $R_0 = n_1^2 - 3n_1 + (n+2)$. Since also $V_a \times V_a \subset \varrho_M^{R_0}$ we conclude that all the first n_1 columns of M^{R_0} are positive. This result holds for any $n_1 \geqq 1$. If $n_1 = 1$, we get (by the Remark in the proof of Theorem 2) a sligthly better result: $V = V_a \cup V_b = \varrho^{n-1} a_1$, e.i. M^{n-1} has the first (and unique) column positive.

We have proved the first part of the following Theorem:

Theorem 3. *Let P be a non-negative $n \times n$ matrix such that some power of P has n_1 positive column and no power of P has more than n_1 positive columns. Hereby $1 \leqq n_1 < n$. Denote*

$$R = \begin{cases} n-1 & \text{if } n_1 = 1, \\ n_1^2 - 3n_1 + (n+2) & \text{if } n_1 > 1. \end{cases} \tag{6}$$

Then P^R contains n_1 positive columns. This result is sharp in the following sense. For any couple (n_1, n), $1 \leqq n_1 < n$, there is an $n \times n$ matrix Q for which Q^{R-1} contains less than n_1 positive columns.

To prove the second part we first settle the case $n_1 = 1$. Consider the $n \times n$ Boolean matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

with the corresponding graph

$$b_{n-1} \to \dots \to b_1 \to a_1 \circlearrowleft .$$

For $i \in \{1, 2, \ldots, n-1\}$ we have $b_i \varrho_Q^{n-1} = a_1$, while $b_{n-1}\varrho_Q^{n-2}$ does not contain a_1, hence $(b_{n-1}, a_1) \notin \varrho_Q^{n-2}$.

In the following suppose $n_1 > 1$ and consider the Boolean matrix

$$Q = \begin{pmatrix} W_{n_1} & 0 \\ C & B \end{pmatrix},$$

where C is the $(n-n_1)\times n_1$ matrix

$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

and B is the $(n-n_1)\times(n-n_1)$ matrix

$$B = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

The corresponding graph

$$\begin{array}{ccccccccccc} b_{n-n_1} & \to & \ldots & \to & b_2 & \to & b_1 & \to & a_1 & \to & a_2 & \to & \ldots \\ & & & & & & & & \uparrow & \nearrow & & & \vdots \\ & & & & & & & & a_{n_1} & \leftarrow & a_{n_1-1} & \leftarrow & \ldots \end{array}$$

shows that $a_1 = b_i \varrho_Q^i$ (for $i = 1, 2, \ldots, n-n_1$), which implies $a_1 \varrho_Q^{n-n_1-i} = b_i \varrho_Q^{n-n_1}$. Hence (with $S = n_1^2 - 2n_1 + 2$)

$$V_a = a_1 \varrho_Q^S = a_1 \varrho_Q^{S+n-n_1-i} = b_i \varrho_Q^{S+n-n_1} = b_i \varrho_Q^R,$$

i.e. $V_b \times V_a \subset \varrho_Q^R$ and finally $V \times V_a \subset \varrho_Q^R$, i.e. all the first n_1 columns of Q^R are positive (and no power of Q has more then n_1 positive columns).

To prove our statement it is sufficient to show that ϱ_Q^{R-1} does not contain the couple (b_{n-n_1}, a_1).

The vertex a_1 is reached from the vertex b_{n-n_1} by paths of length either $n - n_1 + un_1$ or by paths of length $(n-n_1) + 1 + v(n_1 - 1) + (n_1 - 1)w \cdot n_1 = n + v(n_1 - 1) + w \cdot n_1$, where u, v, w are non-negative integers.

An equality of the form $n - n_1 + u \cdot n_1 = n_1^2 - 3n_1 + (n+1)$ implies $u = n_1 - 2 + \dfrac{1}{n_1}$, which is impossible for $n_1 \geqq 2$.

The equality $n+v(n_1-1)+w\cdot n_1=n_1^2-3n_1+n+1$ can be written in the form

$$v(n_1-1)+n_1(w+1)=(n_1-1)^2. \tag{7}$$

For $n_1=2$ we would have $v+2w+2=1$, which is impossible. For $n_1>2$ (7) implies $(n_1-1)\,|\,n_1(w+1)$, hence $(n_1-1)\,|\,(w+1)$, i.e. $w+1=t(n_1-1)$ with an integer $t\geqq 1$. But then $v(n_1-1)+n_1t(n_1-1)=(n_1-1)^2$ implies $v+n_1t=n_1-1$, which cannot hold. This proves Theorem 3.

We finally state a result in which n_1 does not appear explicitly. For a fixed chosen n consider the function $R=R(n_1)$ defined by (6) for all $n_1\in\{1, 2, \dots, n-1\}$. The function $R(n_1)$ is an increasing function of n_1 and we have $R(n-1)=(n-1)^2-3(n-1)+n+2=n^2-4n+6$.

This implies:

Corollary 3. *Let P be an $n\times n$ non-negative matrix such that some power of P has a positive column and P is not primitive. Denote $R_1=n^2-4n+6$. Then P^{R_1} contains the maximal possible number of positive columns. This result is sharp in the following sense. For any $n\geqq 3$ the exists a non-negative non-primitive matrix Q such that Q^{R_1-1} does not contain the maximal possible number of positive columns.*

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ОДНА КОМБИНАТОРНАЯ ЗАДАЧА, ВОЗНИКАЮЩАЯ В КОНЕЧНЫХ ЦЕПЯХ МАРКОВА

Štefan Schwarz

Резюме

Пусть P неотрицательная $n \times n$ матрица со свойством, что P^k имеет положительный столбец для некоторого натурального $k > 0$. Показывается, что наименьшее k с этим свойством удовлетворяет неравенству $k \leqq n^2 - 3n + 3$. Решаются также некоторые смежные вопросы.