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THE DYNAMICS OF F-QUANTUM SPACES

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ABSTRACT. The concepts of strong isomorphisms, weak isomorphism and conjugation in the dynamics of F-quantum spaces have been introduced and studied.

1. Introduction

An *abstract dynamical system* is a quadruple $F = (X, \mathcal{B}, p, f)$, where X is a nonempty set, $\mathcal{B}$ is a σ-algebra of subsets of X, p is a normalized measure on $\mathcal{B}$, and f is a measure preserving transformation on X. The study of continuous transformations, defined on a topological space (usually compact), with particular regard to properties of interest in the qualitative theory of differential equations constitutes the subject matter of topological dynamics. Many of the properties of transformation groups may just as well be isolated and studied for a single transformation and its iterates.

A *classical dynamical system* is a pair (X, σ), where X is a nonempty compact Hausdorff space, and σ is a continuous map of X into itself. Given a classical dynamical system, there exists a normalized (total measure one), positive measure μ on the class $\mathcal{B}$ of Borel sets of X such that σ preserves the measure μ, i.e. $(X, \mathcal{B}, \mu, \sigma)$ is an abstract dynamical system (cf. [1]).

A theory of F-quantum spaces and their dynamics based on F-quantum spaces ([12], [13]) was developed and studied in [7].

An *F-quantum space* is a couple $(X, \mathcal{M})$, where X is a nonempty set and $\mathcal{M}$ is a σ-algebra of fuzzy events [11].

An *F-state* on an F-quantum space is a mapping $m\colon \mathcal{M} \to [0,1]$ satisfying the conditions:

(i) $m(f \vee f') = 1$ for every $f \in \mathcal{M}$;
(ii) if $\{f_i\}$ is a sequence of pairwise orthogonal elements from $\mathcal{M}$, then $m\left(\bigvee f_i\right) = \sum m(f_i)$;

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here $f' = 1 - f$ and $\bigvee f_i = \sup f_i$. The axioms of a σ-algebra here are different from that of Klement [4], and the conditions on m are also different. An F-quantum dynamical system is described as a quadruple $(X, \mathcal{M}, m, U)$, where $U\colon \mathcal{M} \to \mathcal{M}$ is a σ-homomorphism (i.e. $U(f') = 1 - U(f)$ and $U(\bigvee f_n) = \bigvee U(f_n)$ for every $f \in \mathcal{M}$ and any sequence $\{f_n\} \in \mathcal{M}$, satisfying $m(U(f)) = m(f)$ for every f in $\mathcal{M}$).

In a series of papers [5], [6], [7], efforts were made by Markechová to generalize to F-quantum dynamical systems the notions of isomorphism and conjugation of dynamical systems in classical probability theory. Various approaches to the problem of fuzzy generalization of Kolmogorov-Sinai entropy have been also offered by Markechová [6], [7] among others [2], [3], [11]–[13]. In a recent paper [16] we have been able to develop a more satisfactory theory of entropy of F-dynamical systems on the basis of yet another approach (see also [8]–[10], [14]–[16]).

The present paper is devoted to the study of the concepts of strong isomorphism, weak isomorphism and conjugation in the theory of F-quantum spaces. The approach is based on the theory developed in [15], [16]. We prove that

$$\text{strong isomorphism} \implies \text{weak isomorphism} \implies \text{conjugacy}$$

in the dynamics of F-quantum spaces.

2. Basic definitions and results

2.1. Let X be a nonempty set and $I = [0,1]$ be the closed unit interval of the real line.

A *fuzzy set* λ in X is an element of the family I^X of all functions from X to I. For $t \in [0,1]$, the element $\lambda \in I^X$, defined by $\lambda(x) = t$ for all x in X, is denoted by $\mathbf{t}$. If $f\colon X \to Y$ is a function and $\mu \in I^Y$, then $f^{-1}(\mu)$ is a fuzzy set in X defined by $f^{-1}(\mu) = \mu \circ f$.

We write $\lambda_i \uparrow \lambda$ and say that the sequence $\{\lambda_i\}_{i=1}^{\infty}$ of fuzzy sets in X *increases* to $\lambda \in I^X$ if $\{\lambda_i(x)\}_{i=1}^{\infty}$ is monotonic increasing and converges to $\lambda(x)$ for each x in X.

The map $'\colon I^X \to I^X$ which assigns to $\lambda \in I^X$ the fuzzy set $\mathbf{1} - \lambda \in I^X$ is called the *complementation map* and it satisfies the following:

(i) $(\lambda')' = \mathbf{1} - (\mathbf{1} - \lambda) = \lambda$ for all λ in I^X;

(ii) for any sequence $\{\lambda_i\}_{i=1}^{\infty}$ of elements in $\mathcal{M}$,

$$\left(\bigvee_{i=1}^{\infty} \lambda_i\right)' = \bigwedge_{i=1}^{\infty} \lambda_i' \qquad \text{and} \qquad \left(\bigwedge_{i=1}^{\infty} \lambda_i\right)' = \bigvee_{i=1}^{\infty} \lambda_i'.$$

2.2. ([4], cf. [16]) A *fuzzy σ-algebra* $\mathcal{M}$ on a nonempty set X is a subfamily of I^X satisfying:

A1. $\mathbf{1} \in \mathcal{M}$,
A2. $\lambda \in \mathcal{M} \implies \mathbf{1} - \lambda \in \mathcal{M}$,
A3. for any sequence $\{\lambda_i\}_{i=1}^{\infty}$ of elements in $\mathcal{M}$,

$$\bigvee_{i=1}^{\infty} \lambda_i = \sup \lambda_i \in \mathcal{M}\,.$$

Arbitrary intersection of fuzzy σ-algebras on a set X is a fuzzy σ-algebra on X.

A *fuzzy probability measure* (or *F-probability measure*) on a fuzzy σ-algebra $\mathcal{M}$ is a function $m\colon \mathcal{M} \to I$ satisfying the following conditions:

M1. $m(\mathbf{1}) = 1$,
M2. $m(\mathbf{1} - \lambda) = 1 - m(\lambda)$,
M3. for $\lambda, \mu \in \mathcal{M}$, $m(\lambda \vee \mu) + m(\lambda \wedge \mu) = m(\lambda) + m(\mu)$,
M4. for any sequence $\{\lambda_i\}_{i=1}^{\infty}$ in $\mathcal{M}$ such that $\lambda_i \uparrow \lambda$, $m(\lambda) = \sup\limits_{i \in \mathbb{N}} m(\lambda_i)$.

The triple $(X, \mathcal{M}, m)$ is called an *F-probability measure space.*

2.3. Let $(X, \mathcal{M}, m)$ and $(Y, \mathcal{N}, n)$ be F-probability measure spaces. A transformation $\phi : (X, \mathcal{M}, m) \to (Y, \mathcal{N}, n)$ is called *F-measure preserving* if $\phi^{-1}(\mathcal{N}) \subseteq \mathcal{M}$ and $m(\phi^{-1}(\mu)) = n(\mu)$ for all $\mu \in \mathcal{N}$.

2.4. An *F-quantum dynamical system* is a quadruple $(X, \mathcal{M}, m, \phi)$, where $(X, \mathcal{M}, m)$ is an F-probability measure space and ϕ is an F-measure preserving transformation from $(X, \mathcal{M}, m)$ to itself.

2.5. Let $(X, \mathcal{M}, m)$ be an F-probability measure space. Define a relation $=$ (mod m) on $\mathcal{M}$ as follows:

$$\lambda = \mu \pmod{m} \iff m(\lambda) = m(\mu) = m(\lambda \wedge \mu)\,,$$

where $\lambda, \mu \in \mathcal{M}$.

If $\lambda = \mu$ (mod m), then we say that λ and μ are *m-equivalent.*

Alternatively $\lambda = \mu$ (mod m) if and only if $m(\lambda \wedge \mu) = m(\lambda \vee \mu)$. Also, $\lambda = \mu$ (mod m) implies $\lambda = \lambda \vee \mu$ (mod m) and $\lambda = \lambda \wedge \mu$ (mod m).

The relation of m-equivalence on $\mathcal{M}$ is an equivalence relation. ([15])

2.6. Let $(X, \mathcal{M}, m)$ be an F-probability measure space. We denote by $\tilde{\mathcal{M}}$ the collection of all equivalence classes induced by the relation of m-equivalence on $\mathcal{M}$; $\tilde{\mu}$ denotes the equivalence class determined by $\mu \in \mathcal{M}$. We may define

$$\tilde{\lambda} \vee \tilde{\mu} = (\lambda \vee \mu)^{\sim} \qquad \text{and} \qquad \tilde{\lambda} \wedge \tilde{\mu} = (\lambda \wedge \mu)^{\sim}\,.$$

For any sequence $\{\mu_i\}$ in $\mathcal{M}$, we define

$$\bigvee_{i=1}^{\infty} \tilde{\mu}_i = \left(\bigvee_{i=1}^{\infty} \mu_i\right)^{\sim}.$$

Since $\lambda = \mu \pmod{m}$ implies $\mathbf{1} - \lambda = \mathbf{1} - \mu \pmod{m}$, we may define

$$(\tilde{\lambda})' = (\mathbf{1} - \lambda)^{\sim}.$$

Under these operations induced from $\mathcal{M}$, $\tilde{\mathcal{M}}$ forms a fuzzy σ-algebra. Define $\tilde{m}\colon \tilde{\mathcal{M}} \to I$ by $\tilde{m}(\tilde{\mu}) = m(\mu)$. Then $\tilde{m}$ is an F-probability measure on $\tilde{\mathcal{M}}$. The pair $(\tilde{\mathcal{M}}, \tilde{m})$ is called an *F-measure algebra* ([15]).

3. Isomorphism and conjugation

Definition 3.1. Two F-quantum dynamical systems $(X, \mathcal{M}, m, \phi)$ and $(Y, \mathcal{N}, n, \psi)$ are called *strongly isomorphic* if there exists a bijective mapping $\eta\colon X \to Y$ satisfying

(i) $\lambda \in \mathcal{M}$ if and only if $\lambda \circ \eta^{-1} \in \mathcal{N}$;
(ii) $m(\lambda) = n(\lambda \circ \eta^{-1})$ for all $\lambda \in \mathcal{M}$;
(iii) the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ Y & \xrightarrow[\psi]{} & Y \end{array}$$

commutes, i.e. $\psi \circ \eta = \eta \circ \phi$.

Definition 3.2. Two F-quantum dynamical systems $(X, \mathcal{M}, m, \phi)$ and $(Y, \mathcal{N}, n, \psi)$ are called *weakly isomorphic* if there exists a bijective map $\delta\colon \mathcal{M} \to \mathcal{N}$ satisfying

(i) δ preserves lattice operations, i.e.

$$\delta\left(\bigvee_{n=1}^{\infty} \lambda_n\right) = \bigvee_{n=1}^{\infty} \delta(\lambda_n); \qquad \delta(\mathbf{1} - \lambda) = \mathbf{1} - \delta(\lambda),$$

for all $\lambda \in \mathcal{M}$, and for any sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathcal{M}$;
(ii) $m(\delta^{-1}(\mu)) = n(\mu)$ for all $\mu \in \mathcal{N}$;

(iii) the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \phi\ } & \mathcal{M} \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\ \mathcal{N} & \xrightarrow{\ \psi\ } & \mathcal{N} \end{array}$$

commutes, i.e.

$$\delta\big(U_1(\lambda)\big) = U_2\big(\delta(\lambda)\big), \qquad \lambda \in \mathcal{M};$$

here $U_1(\lambda) = \lambda \circ \phi$ and $U_2(\mu) = \mu \circ \psi$, $\lambda \in \mathcal{M}$, $\mu \in \mathcal{N}$.

THEOREM 3.3. *If two F-quantum dynamical systems $\Phi_1 = (X, \mathcal{M}, m, \phi)$ and $\Phi_2 = (Y, \mathcal{N}, n, \psi)$ are strongly isomorphic, then they are weakly isomorphic.*

Proof. Let $\eta\colon X \to Y$ be a bijective mapping satisfying 3.1.(i)–(iii). Define $\delta\colon \mathcal{M} \to \mathcal{N}$ by

$$\delta(\lambda) = \lambda \circ \eta^{-1}, \qquad \lambda \in \mathcal{M}.$$

For any $\mu \in \mathcal{N}$, put $\lambda = \mu \circ \eta$. Then $\lambda \in \mathcal{M}$, and

$$\delta(\lambda) = \delta(\mu \circ \eta) = (\mu \circ \eta) \circ \eta^{-1} = \mu$$

show that δ is surjective.

Next, let $\lambda_1, \lambda_2 \in \mathcal{M}$ such that $\lambda_1 \neq \lambda_2$. Then there exists $x \in X$ such that $\lambda_1(x) \neq \lambda_2(x)$. Let $y = \eta(x)$. Then

$$\begin{aligned} \delta(\lambda_1)(y) &= (\lambda_1 \circ \eta^{-1})(y) = (\lambda_1 \circ \eta^{-1})\big(\eta(x)\big) \\ &= \lambda_1(x) \neq \lambda_2(x) \\ &= (\lambda_2 \circ \eta^{-1})\big(\eta(x)\big) = \delta\big(\lambda_2(y)\big), \end{aligned}$$

which yields that δ is injective. Thus δ is bijective.

(i) For any sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathcal{M}$,

$$\delta\left(\bigvee_{n=1}^{\infty} \lambda_n\right) = \left(\bigvee_{n=1}^{\infty} \lambda_n\right) \circ \eta^{-1} = \bigvee_{n=1}^{\infty} (\lambda_n \circ \eta^{-1}) = \bigvee_{n=1}^{\infty} \delta(\lambda_n);$$

and, for any $\lambda \in \mathcal{M}$,

$$\delta(\lambda') = \lambda' \circ \eta^{-1} = (\lambda \circ \eta^{-1})' = \mathbf{1} - \delta(\lambda).$$

(ii) For any $\mu \in \mathcal{N}$, using 3.1.(ii), we get

$$m\big(\delta^{-1}(\mu)\big) = m(\mu \circ \eta) = n\big((\mu \circ \eta) \circ \eta^{-1}\big) = n(\mu).$$

(iii) We first prove that

$$\eta \circ \phi = \psi \circ \eta \implies \eta^{-1} \circ \psi = \phi \circ \eta^{-1}.$$

For any $y \in Y$, there exists $x \in X$ such that $\eta(x) = y$, and therefore

$$\begin{aligned}(\eta^{-1} \circ \psi)(y) &= \eta^{-1}(\psi(y)) = \eta^{-1}(\psi(\eta(x))) \\ &= \eta^{-1}(\eta(\phi(x))) = \phi(x) = \phi(\eta^{-1}(y)) = (\phi \circ \eta^{-1})(y).\end{aligned}$$

Now, for $\lambda \in \mathcal{M}$, we have

$$\begin{aligned}U_2(\delta(\lambda)) &= U_2(\lambda \circ \eta^{-1}) \\ &= (\lambda \circ \eta^{-1}) \circ \psi = \lambda \circ (\eta^{-1} \circ \psi) = \lambda \circ (\phi \circ \eta^{-1}) \\ &= U_1(\lambda) \circ \eta^{-1} = \delta(U_1(\lambda)).\end{aligned}$$

Thus Φ_1 and Φ_2 are weakly isomorphic. □

DEFINITION 3.4. ([15]) Let $(X, \mathcal{M}, m)$ and $(Y, \mathcal{N}, n)$ be F-probability measure spaces, and let $(\tilde{\mathcal{M}}, \tilde{m})$ and $(\tilde{\mathcal{N}}, \tilde{n})$ be their corresponding F-measure algebras. Then $(\tilde{\mathcal{M}}, \tilde{m})$ and $(\tilde{\mathcal{N}}, \tilde{n})$ are called *isomorphic* if there is a bijective map $\xi \colon \tilde{\mathcal{N}} \to \tilde{\mathcal{M}}$ which preserves countable joins, complements and satisfies in addition

$$\tilde{m}(\xi(\tilde{\mu})) = \tilde{n}(\tilde{\mu}) \qquad \text{for all} \quad \tilde{\mu} \in \tilde{\mathcal{N}};$$

ξ is called *F-measure algebra isomorphism.*

DEFINITION 3.5. Let $\phi \colon (X, \mathcal{M}, m) \to (X, \mathcal{M}, m)$ and $\psi \colon (Y, \mathcal{N}, n) \to (Y, \mathcal{N}, n)$ be F-measure preserving transformations. We say that ϕ is *conjugate* to ψ if there exists an F-measure algebra isomorphism $\xi \colon (\tilde{\mathcal{N}}, \tilde{n}) \to (\tilde{\mathcal{M}}, \tilde{m})$ such that $\tilde{\phi}(\xi(\tilde{\mu})) = \xi(\tilde{\psi}(\tilde{\mu}))$, $\mu \in \mathcal{N}$; here $\tilde{\phi}(\tilde{\lambda}) = (\phi(\lambda))^{\sim}$, $\tilde{\lambda} \in \tilde{\mathcal{M}}$; and $\tilde{\psi}(\tilde{\mu}) = (\psi^{-1}(\mu))^{\sim}$, $\tilde{\mu} \in \tilde{\mathcal{N}}$.

PROPOSITION 3.6. *Let $\hat{T}$ denote the family of all F-measure preserving transformations from an F-probability measure space $(X, \mathcal{M}, m)$ to itself. Then the relation of conjugacy on $\hat{T}$ is an equivalence relation.*

THEOREM 3.7. *If two F-quantum dynamical systems $(X, \mathcal{M}, m, \phi)$ and $(Y, \mathcal{N}, n, \psi)$ are weakly isomorphic, then ϕ is conjugate to ψ.*

Proof. Let $\delta \colon \mathcal{M} \to \mathcal{N}$ be a bijective mapping satisfying 3.2.(i)–(iii). Define $\xi \colon \tilde{\mathcal{N}} \to \tilde{\mathcal{M}}$ by

$$\xi(\tilde{\mu}) = (\delta^{-1}(\mu))^{\sim}, \qquad \tilde{\mu} \in \tilde{\mathcal{N}}.$$

(i) Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \tilde{\mathcal{N}}$, and $\xi(\tilde{\mu}_1) = \xi(\tilde{\mu}_2)$. Then, using 3.2.(ii), we get

$$m(\delta^{-1}(\mu_1)) = m(\delta^{-1}(\mu_2)) = m(\delta^{-1}(\mu_1) \wedge \delta^{-1}(\mu_2)) = m(\delta^{-1}(\mu_1 \wedge \mu_2)),$$

or

$$n(\mu_1) = n(\mu_2) = n(\mu_1 \wedge \mu_2),$$

i.e. $\mu_1 \sim \mu_2$, and so $\tilde{\mu}_1 = \tilde{\mu}_2$. Thus ξ is injective.

(ii) For any $\tilde{v} \in \tilde{\mathcal{M}}$, put $\mu = \delta(v) \in \mathcal{N}$. Then $\tilde{\mu} \in \tilde{\mathcal{N}}$ and $\xi(\tilde{\mu}) = \tilde{v}$. Hence ξ is surjective.

(iii) For $\tilde{\mu} \in \tilde{\mathcal{N}}$, we have

$$\tilde{m}(\xi(\tilde{\mu})) = \tilde{m}\big((\delta^{-1}(\mu))^{\sim}\big) = m\big(\delta^{-1}(\mu)\big) = n(\mu) = \tilde{n}(\tilde{\mu}).$$

Hence ξ is F-measure preserving.

(iv) Finally, for $\tilde{\mu} \in \tilde{\mathcal{N}}$, using 3.2.(iii), we get

$$\begin{aligned}\tilde{\phi}(\xi(\tilde{\mu})) &= \tilde{\phi}\big((\delta^{-1}(\mu))^{\sim}\big) = \big(\phi^{-1}(\delta^{-1}(\mu))\big)^{\sim} \\ &= \big((\delta \circ \phi)^{-1}(\mu)\big)^{\sim} = \big((\psi \circ \delta)^{-1}(\mu)\big)^{\sim} \\ &= \big(\delta^{-1}(\psi^{-1}(\mu))\big)^{\sim} \\ &= \xi\big((\psi^{-1}(\mu))^{\sim}\big) = \xi(\tilde{\psi}(\tilde{\mu})).\end{aligned}$$

Hence ϕ is conjugate to ψ. $\square$

THEOREM 3.8. *If F-quantum dynamical systems $(X, \mathcal{M}, m, \phi)$ and $(Y, \mathcal{N}, n, \psi)$ are strongly isomorphic, then ϕ is conjugate to ψ.*

Proof. The theorem follows from Theorem 3.3 and Theorem 3.7. $\square$

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