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A NOTE ON THE RANK OF SELF-DUAL POLYHEDRA

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ABSTRACT. We examine how the symmetry of a self-dual polyhedron affects its rank, answering some questions in [Jendroľ, S.: *On the symmetry groups of self-dual convex polyhedra*, Ann. Discrete Math. **51** (1992), 129–135].

A polyhedron P is said to be *self-dual* if there is an isomorphism $\delta\colon P \to P^*$, where P^* denotes the dual of P. We may regard δ as a permutation of the elements of P which sends vertices to faces and vice versa, preserving incidence. For example, the regular tetrahedron and its dual are isomorphic, and the self-dual permutation may be taken to correspond to the antipodal map.

The character of the permutation δ has only recently been considered. In [3], an example of a self-dual polyhedron is given for which no self-dual permutation has order 2. Given a self-dual polyhedron P, the least order of any self-duality is called the *rank* of P, $r(P)$. It is easy to see that $r(P)$ must be a positive power of 2.

The possible symmetries of a self-dual polyhedron were enumerated in [4], and the following result is stated which indicates how the symmetry class can affect the rank.

THEOREM 1. *If a self-dual polyhedron P has a central symmetry, then $r(P)$ is either 2 or 4.*

The symmetry does not completely determine the rank, as the following example illustrates.

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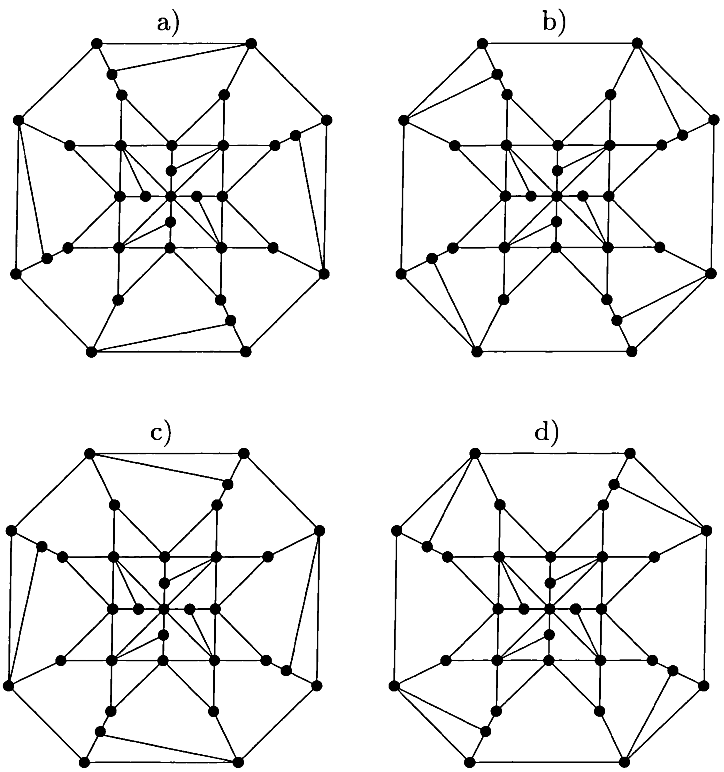

FIGURE 1. Self-dual polyhedra with fourfold rotational symmetry.

Figure 1 shows Schlegel diagrams of four self-dual polyhedra, each with symmetry group $[4]^+$. All have rank 2 except Figure 1b, which has rank 8.

In [5], it is shown that every self-dual polyhedron P corresponds to a bicolored map M on the sphere obtained by embedding the graph of P (one color) together with the graph of P^* (second color) such that the automorphism group of the map M, $\mathrm{Aut}(M)$, is one of the isometry groups of the sphere, and $[\mathrm{Aut}(M), \mathrm{Aut}(P)] = 2$. In this setting, the self-dualities correspond to the elements in $\mathrm{Aut}(M) - \mathrm{Aut}(P)$. We call $\mathrm{Aut}(M) \rhd \mathrm{Aut}(P)$ the *self-dual pairing* of P. For example, the pairing corresponding to the regular tetrahedron is $[3,4] \rhd [3.3]$, which reflects the usual embedding of the pair of dual tetrahedra in the cube (see [2] for the notation of the isometry groups of the sphere).

The self-dual pairings were catalogued in [6], and the pairing does determine the rank.

THEOREM 2.

If $\mathrm{Aut}(P) = [2]^+$ *or* $\mathrm{Aut}(P) = [2,2^+]$*, then* $r(P)$ *may be either* 2 *or* 4.

If $\mathrm{Aut}(P) = [q]^+$, $q > 2$*, then* $r(P)$ *may be either* 2 *or* q/s*, where* s *is the largest odd divisor of* q.

If $\mathrm{Aut}(P) \in \{[q]\ (q \geq 1),\ [2,2],\ [2,2]^+,\ [2^+,2^+],\ [2^+,4^+],\ [2^+,4],\ [3,3],\ [3,3]^+\}$*, then* $r(P) = 2$.

Proof. If $\mathrm{Aut}(P) = [2]^+$, then its pairing is either $[2,2]^+ \rhd [2]^+$, $[2,2^+] \rhd [2]^+$, in which case $r(P) = 2$, or $[4]^+ \rhd [2]^+$ for which $r(P) = 4$.

If $\mathrm{Aut}(P) = [2,2^+]$, then the pairing of P is either $[2,2] \rhd [2,2^+]$, so $r(P) = 2$, or $[2,4^+] \rhd [2,2^+]$, in which case $r(P) = 4$.

If $\mathrm{Aut}(P) = [q]^+$, $q > 2$, then the pairing of P is $[2,q]^+ \rhd [q]^+$ (for $q = 4$, see Figure 1a and d), $[2,q^+] \rhd [q]^+$ (see Figure 1c), in which case $r(P) = 2$, or $[2,2q^+] \rhd [q]^+$ (See Figure 1b), in which case the rank is q/s.

Because for any other pairing the rank is 2, we are done. □

In particular, if P has any symmetry excepting rotational symmetry, then $r(P)$ is 2 or 4.

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