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ON SOME VERSIONS OF JENSEN'S INEQUALITY ON OPERATOR ALGEBRAS

RYSZARD JAJTE

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ABSTRACT. Jensen's type inequalities are proved for convex polynomials of linear operators.

1. Introduction

The classical Jensen's inequality (for the conditional expectations) reads as follows

$$\mathbb{E}^F f(X) \geq f(\mathbb{E}^F X) \qquad \text{a.s. } \mathbb{E}|X| < \infty\,,$$

where $f\colon I \to \mathbb{R}$ is an arbitrary convex function defined on an open interval I such that $\operatorname{Prob}(X \in I) = 1$.

In the context of operator algebras, a similar result holds ([5]). Namely, if f is an operator-convex function on $(-c, c)$, and α is a normalized positive linear map on a C^*-algebra $\mathbb{A}$, then

$$\alpha f(\xi) \geq f(\alpha\xi) \tag{1}$$

for all self-adjoint operators ξ in $\mathbb{A}$ of norm less than c.

If $\mathbb{A}$ is a von Neumann algebra with a faithful normal semifinite trace τ, then, for a convex function f, the following inequality

$$\tau\big(\alpha f(\xi)\big) \geq \tau\big(f(\alpha\xi)\big) \tag{2}$$

holds ([12], [10]).

The inequality (2) is closely related to the previous results concerning the special cases (canonical trace, special positive maps, etc., see [3], [4], [11], [13]).

The main goal of this paper is to prove some results related to (1) and (2). We consider a noncommutative polynomial $W(x, x^*)$ such that $\xi \mapsto W(\xi, \xi^*)$ is a

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convex map in an algebra of operators acting in a Hilbert space. The inequalities of the form

$$\alpha f\big(W(\xi,\xi^*)\big) \geq f\big(W(\alpha\xi,\alpha\xi^*)\big)$$

and

$$\tau\big(\alpha f(W(\xi,\xi^*)\big) \geq \tau\big(f\big(W(\alpha\xi,\alpha\xi^*)\big)\big)$$

are proved, where α is a positive linear map, f is a (operator) convex and (operator) monotone function, and τ is a semifinite trace.

2. Preliminaries

We begin with some notation. Let H be a complex separable Hilbert space, and let $L(H)$ be the algebra of all bounded linear operators in H. Denote by $L^s(H)$ the self-adjoint part of $L(H)$. Let $V \subset L(H)$ be a convex set.

2.1. Definition. A (nonlinear) map $\alpha\colon V \to L^s(H)$ is said to be convex if

$$\alpha\left(\frac{\xi+\eta}{2}\right) \leq \frac{1}{2}(\alpha\xi + \alpha\eta)\,, \qquad \text{for} \quad \xi,\eta \in V\,.$$

Examples. The maps $\xi \mapsto \xi + \xi^*$, $\xi \mapsto \xi^*\xi$, $\xi \mapsto (\xi+\xi^*)^2$, $\xi \mapsto i(\xi - \xi^*)$, $\xi \mapsto -(\xi-\xi^*)^2$, $\xi \to 5\xi^*\xi + 7\xi\xi^* - \xi^2 - (\xi^*)^2$ are convex in $L(H)$. This follows from the above definition and from the operator convexity of the function $t \mapsto t^2$ (for the basic facts concerning the operator convex and operator monotone functions, we refer to [1], [6], [8]).

We shall show only the convexity of the map $\xi \mapsto \xi^*\xi$. Indeed, we have

$$\begin{aligned}\left(\frac{\xi+\eta}{2}\right)^* \frac{\xi+\eta}{2} &= \frac{1}{4}(\xi^*\xi + \eta^*\eta + \xi^*\eta + \eta^*\xi)\\ &\leq \frac{1}{4}(2\xi^*\xi + 2\eta^*\eta) = \frac{1}{2}\xi^*\xi + \frac{1}{2}\eta^*\eta\,.\end{aligned}$$

Let us notice that also the inequality

$$a^*\xi^*\xi a \geq a^*\xi^* a a^* \xi a$$

holds for $\|a\| \leq 1$ (since $a^*\xi^*(1-aa^*)\xi a \geq 0$). This is a special case of a more general inequality as we shall see in the next section.

3. Jensen's inequalities for operators

3.1. THEOREM. *Assume that $W(x,x^*)$ is a noncommutative polynomial such that the map $\xi \to W(\xi,\xi^*)$ is convex in $L(H)$. Let $W(0,0)=0$. Then, for $\|a\|\le 1$ and every $\xi\in L(H)$, the inequality*

$$a^*\big(W(\xi,\xi^*)\big)a \ge W(a^*\xi a, a^*\xi^* a) \tag{3}$$

holds.

P r o o f. We follow some general idea in [8]. Let $b\in L(H)$ such that $bb^* = 1-aa^*$. Put

$$A=\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}, \qquad B=\begin{pmatrix} a & -b\\ 0 & 0\end{pmatrix}, \qquad X=\begin{pmatrix} \xi & 0\\ 0 & 0\end{pmatrix}.$$

In the sequel, we shall briefly write $W(\xi)$ instead of $W(\xi,\xi^*)$. Before starting some calculations, let us remark that, by the condition $bb^*+aa^*=1$, the maps $Z\to A^*ZA$ and $Z\to B^*ZB$ are *-homomorphisms of matrices $\begin{pmatrix} \zeta & 0\\ 0 & 0\end{pmatrix}$ into $\begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix}$. This implies that, for the polynomial W and $X=\begin{pmatrix} \zeta & 0\\ 0 & 0\end{pmatrix}$, we have

$$A^*W(X)A = A^*\begin{pmatrix} W(\zeta) & 0\\ 0 & 0\end{pmatrix}A = W(A^*XA).$$

We have that

$$\begin{aligned}
\begin{pmatrix} W(a^*\xi a) & 0\\ 0 & W(b^*\xi b)\end{pmatrix} &= W\begin{pmatrix} a^*\xi a & 0\\ 0 & b^*\xi b\end{pmatrix}\\
&= W\Big(\frac12 A^*XA+\frac12 B^*XB\Big)\\
&\le \frac12 W(A^*XA)+\frac12 W(B^*XB)\\
&= \frac12 A^*\begin{pmatrix} W(\xi) & 0\\ 0 & 0\end{pmatrix}A+\frac12 B^*\begin{pmatrix} W(\xi) & 0\\ 0 & 0\end{pmatrix}B\\
&= \begin{pmatrix} a^*W(\xi)a & 0\\ 0 & b^*W(\xi)b\end{pmatrix}.
\end{aligned}$$

Consequently, in particular,

$$a^*W(\xi)a \ge W(a^*\xi a),$$

which ends the proof. □

3.2. THEOREM. *Let* $V = \{\xi \in L(H):\ \|\xi\| \le c\}$*, and let, as before,* $W(\xi) = W(\xi,\xi^*)$ *be a convex (noncommutative) polynomial of* ξ *and* ξ^* *such that for* $\xi \in V$*,* $\|W(\xi,\xi^*)\| < d$*. Let* f*, with* $f(0) \le 0$*, be an operator convex and operator monotone function on the interval* $(-D,D)$*, where* $D = \max(c,d)$*. Then*

$$a^* f\big(W(\xi,\xi^*)\big)a \ge f\big(W(a^*\xi a, a^*\xi^* a)\big)\,, \tag{4}$$

for $a \in L(H)$ *with* $\|a\| \le 1$*,* $\xi \in V$*.*

P r o o f. Let $bb^* = 1 - aa^*$, and let A, B and X be as in the proof of Theorem 3.1. Let us put $\varphi(\xi) = \begin{pmatrix} W(\xi) & 0 \\ 0 & 0 \end{pmatrix}$. By Theorem 3.1 and the properties of f, we have that

$$\begin{aligned}
\begin{pmatrix} f\big(W(a^*\xi a)\big) & 0 \\ 0 & f\big(W(b^*\xi b)\big) \end{pmatrix} &\le \begin{pmatrix} f\big(a^* W(\xi) a\big) & 0 \\ 0 & f\big(b^* W(\xi) b\big) \end{pmatrix} \\
&= f\begin{pmatrix} a^* W(\xi) a & 0 \\ 0 & b^* W(\xi) b \end{pmatrix} \\
&= f\Big(\tfrac{1}{2}A^*\varphi(\xi)A + \tfrac{1}{2}B^*\varphi(\xi)B\Big) \\
&\le \tfrac{1}{2} f\big(A^*\varphi(\xi)A\big) + \tfrac{1}{2} f\big(B^*\varphi(\xi)B\big) \\
&= \tfrac{1}{2}A^* f\big(\varphi(\xi)\big)A + \tfrac{1}{2}B^* f\big(\varphi(\xi)\big)B \\
&\le \begin{pmatrix} a^* f\big(W(\xi)\big)a & 0 \\ 0 & b^* f\big(W(\xi)\big)b \end{pmatrix},
\end{aligned}$$

so $f\big(W(a^*\xi a)\big) \le a^* f\big(W(\xi)\big)a$. □

As a corollary, we obtain the following:

3.3. THEOREM. *Let* $\alpha\colon L(H) \to L(H)$ *be a completely positive linear contraction, and let* f*,* W *and* ξ *be as in Theorem* 3.2*. Then*

$$\alpha f W(\xi,\xi^*) \ge f W(\alpha\xi, \alpha\xi^*)\,. \tag{5}$$

P r o o f. It is enough to apply the S t i n e s p r i n g theorem ([14]) and an obvious modification of Theorem 3.2 (comp. [7], [8]). □

The assumption that the function f is operator monotone and operator convex is rather restrictive. Putting the both sides of (5) under the sign of a semifinite trace we can obtain a version of Jensen's inequality for a function f which is only nondecreasing and convex. Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathbb{A}$. Let us recall that τ admits an extension to a linear functional on the ideal m_τ linearly spanned by the set $p_\tau = \{x \in \mathbb{A}_+ :\ \tau(x) < \infty\}$. Let $\xi = \xi_1 - \xi_2$ be the Jordan decomposition of a self-adjoint operator $\xi \in \mathbb{A}$. We say that $\tau(\xi)$ is defined if $\tau(\xi_1) < \infty$ or $\tau(\xi_2) < \infty$.

3.4. THEOREM. *Let $\mathbb{A}$ be a semifinite von Neumann algebra with a faithful normal semiinfinite trace τ, and let $\alpha\colon \mathbb{A} \to \mathbb{A}$ be a unital completely positive linear map. Let $W(x,x^*)$ be a noncommutative polynomial such that the map $\xi \to W(\xi,\xi^*)$ is convex. Assume that $\|\xi\| < a$ and $\|W(\xi,\xi^*)\| < b$. Let f be a nondecreasing and convex function on the interval $I = (-\sigma,\sigma)$, where $\sigma = \max(a,b)$. Then the inequality*

$$\tau\big(\alpha f\big(W(\xi,\xi^*)\big)\big) \geq \tau\big(f\big(W(\alpha\xi,\alpha\xi^*)\big)\big) \tag{6}$$

holds also for infinite values of τ, provided that both the sides of (6) *are defined.*

P r o o f . In the sequel, as before, we shall write $W(\xi)$ instead of $W(\xi,\xi^*)$. Let (a_n) and (b_n) be sequences of real numbers ($a_n \geq 0$) such that

$$f(u) = \sup_n (a_n u + b_n)\,, \qquad u \in I = (-\sigma,\sigma)$$

(see, e.g., [2]). Consequently,

$$f\big(W(\xi)\big) \geq a_n W(\xi) + b_n \mathbf{1}$$

and

$$\begin{aligned} \alpha f\big(W(\xi)\big) &\geq a_n \alpha W(\xi) + b_n \mathbf{1} \\ &\geq a_n W(\alpha\xi) + b_n \mathbf{1} \end{aligned}$$

(by Theorem 3.3, for $f(t) = t$).

Let

$$W(\alpha\xi) = \int_c^d \lambda\, e(\mathrm{d}\lambda)$$

be the spectral representation of $W(\alpha\xi)$.

We split the proof into two parts.

Part I. $f(s) \geq 0$, for $s \in I$.

Let $0 < \varepsilon_N \to 0$. Fix some N and take a finite partition $\big(Z_1^{(N)},\dots,Z_{m_N}^{(N)}\big)$ of the interval $[c,d]$ and real numbers $c_{n,k}^{(N)}$ ($n = 1,2,\dots,N$; $k = 1,2,\dots,m_N$) such that putting $p_j^{(N)} = e\big(Z_j^{(N)}\big)$ we have

$$\Big\| \int_c^d (a_n\lambda + b_n)\, e(\mathrm{d}\lambda) - \sum_{k=1}^{m_N} c_{n,k} p_k^{(N)} \Big\| < \varepsilon_N\,, \qquad \text{for} \quad n = 1,2,\dots,N \tag{7}$$

and

$$\Big\| \int_c^d \max_{1\leq n\leq N} (a_n\lambda + b_n)\, e(\mathrm{d}\lambda) - \sum_{k=1}^{m_N} \Big(\max_{1\leq n\leq N} c_{n,k}^{(N)} \Big) p_k^{(N)} \Big\| < \varepsilon_N\,. \tag{8}$$

For $Z_i^{(N)}$ with $\tau(p_i^{(N)}) = \infty$, we fix an increasing net $\mathbb{K}_i^{(N)}$ of projections q in $\mathbb{A}$ with $q \le p_i^{(N)}$, $\tau(q) < \infty$ and $\lim q = p_i^{(N)}$. For $Z_i^{(N)}$ with $\tau(p_i^{(N)}) < \infty$, we put $\mathbb{K}_i = \emptyset$. Let $\mathbb{B}_N$ be a von Neumann subalgebra of $\mathbb{A}$ generated by $p_i^{(N)}$ and $\mathbb{K}_i^{(N)}$, i.e., $\mathbb{B}_N = (p_i^{(N)}, \mathbb{K}_i^{(N)},\ i = 1,2,\dots,m_N)''$. Then the trace τ restricted to $\mathbb{B}_N$ is semifinite. By [15; Proposition 2.36], there is a faithful normal conditional expectation $\mathbb{E}^{\mathbb{B}_N}$ from $\mathbb{A}$ onto $\mathbb{B}_N$ such that $\tau \circ \mathbb{E}^{\mathbb{B}_N} = \tau$. Put $D = \alpha f(W(\xi))$. We have

$$\begin{aligned} &a_n W(\alpha\xi) + b_n \mathbf{1} \\ &= \Big(a_n W(\alpha\xi) + b_n \mathbf{1} - \sum_{k=1}^{m_N} c_{n,k}^{(N)} p_k^{(N)} \Big) + \sum_{k=1}^{m_N} c_{n,k}^{(N)} p_k^{(N)} \le \alpha f(W(\xi)) = D\,, \end{aligned}$$

so

$$\sum_{k=1}^{m_N} c_{n,k}^{(N)} p_k^{(N)} \le D + \varepsilon_N \mathbf{1}\,. \tag{9}$$

Consequently,

$$\sum_{k=1}^{m_N} c_{n,k}^{(N)} p_k^{(N)} \le \mathbb{E}^{\mathbb{B}_N} D + \varepsilon_N \mathbf{1}\,,$$

and finally, we get

$$\sum_{k=1}^{m_N} \Big(\max_{1\le n\le N} c_{n,k}^{(N)} \Big) p_k^{(N)} \le \mathbb{E}^{\mathbb{B}_N} D + \varepsilon_N \mathbf{1}\,. \tag{10}$$

Thus we have, for $g_N(\lambda) = \max_{1\le n\le N} (a_n\lambda + b_n)$,

$$\int_c^d g_N(\lambda)\, e(\mathrm{d}\lambda) \le D_N + 2\varepsilon_N \mathbf{1}\,, \tag{11}$$

where $D_N = \mathbb{E}^{\mathbb{B}_N} D$ $(N = 1,2,\dots)$.

The operators D_N are positive since $f \ge 0$. There is a net (N_s) such that D_{N_s} converges weakly to some positive operator, say B. By the weak *-lower semicontinuity of τ, we have

$$\tau(D_{N_s}) = \tau(\alpha f(W(\xi)) \ge \tau(B)\,. \tag{12}$$

On the other hand, the sequence of functions (g_N) converges uniformly on the spectrum of $W(\alpha\xi)$ to the function f. Thus $g_N(W(\alpha\xi) \to f(W(\alpha\xi))$ in the uniform topology, so, by (11),

$$\int_c^d f(\lambda)\, e(\mathrm{d}\lambda) = f(W(\alpha\xi)) \le B\,. \tag{13}$$

Consequently, we get the formula

$$\tau\big(f(W(\alpha\xi)\big) \le \tau(B) \le \tau\big(\alpha f\big(W(\xi)\big)\big)\,, \tag{14}$$

still under the assumption that $f(s) \ge 0$, for $s \in I$.

Part II.

Let us assume now that $f(s) < 0$ for some $s \in I$. In this case, the set of zero's of f is one point set (if not empty). Let $f(s_0) = 0$, and assume for a moment that $\tau\big(|f(W(\alpha\xi))|\big) < \infty$. Then, for every Borel subset $Z \subset [c,d]$ separated from s_0 (i.e., with $\mathrm{dist}(s_0, Z) > 0$), we have $\tau\big(e(Z)\big) < \infty$. Moreover, if $\tau\big(e(\{s_0\})\big) < \infty$, then the trace τ restricted to the von Neumann algebra $\big(W(\alpha\xi)\big)''$ is semifinite. In the case $\tau\big(e(\{s_0\})\big) = \infty$, we can fix an increasing net $\mathbb{K}$ of projections q in $\mathbb{A}$ such that $q \le e\big(\{s_0\}\big)$, $\tau(q) < \infty$, and $\lim_{\mathbb{K}} q = e\big(\{s_0\}\big)$. Let $\mathbb{B}$ be a von Neumann subalgebra generated by $W(\alpha\xi)$ and $\mathbb{K}$. Then $\tau|_{\mathbb{B}}$ is semifinite, and there exists a τ-preserving normal faithful conditional expectation $\mathbb{E}^{\mathbb{B}}$ of $\mathbb{A}$ onto $\mathbb{B}$, and we have $\mathbb{E}^{\mathbb{B}}\alpha f\big(W(\xi)\big) \ge f\big(W(\alpha\xi)\big)$. Consequently,

$$\tau\big(\alpha f\big(W(\xi)\big)\big) \ge \tau\big(f\big(W(\alpha\xi)\big)\big)$$

under the assumption that the both sides of this inequality are finite.

The cases $\tau\big(\alpha f\big(W(\xi)\big)\big) = +\infty$ and $\tau\big(f\big(\alpha\big(W(\xi)\big)\big)\big) = -\infty$ are trivial. Let us consider the case $\tau\big(\alpha f\big(W(\xi)\big)\big) = -\infty$ (which means that $\tau\big(\alpha f\big(W(\xi)\big)_+\big) < \infty$ and $\tau\big(\alpha f\big(W(\xi)\big)_-\big) = +\infty$). Keeping the notation of the first part of the proof, we can start from formula (11) and use the fact that

$$\int_c^d g_N(\lambda)\, e(\mathrm{d}\lambda) \to f\big(W(\alpha\xi)\big)$$

in the uniform operator topology as $N \to \infty$, say

$$\left\| \int_c^d g_N(\lambda)\, e(\mathrm{d}\lambda) - f\big(W(\alpha\xi)\big) \right\| = \delta_N \to 0\,.$$

Putting $\omega_N = \max(2\varepsilon_N, \delta_N)$, we obtain

$$f\big(W(\alpha\xi)\big) \le \mathbb{E}^{\mathbb{B}_N} f\big(W(\xi)\big) + 2\omega_N\,.$$

Modifying slightly the definition of the sequence of the algebras $\mathbb{B}_N$, we can assume that it is increasing.

Take an increasing net (p_s) of projections such that $\tau(p_s) < \infty$, $p_s \in \mathbb{B}_{N_s}$ ($D_{N_s} \to B$ weakly) and $\omega_{N_s}\tau(p_s) \underset{s}{\to} 0$ (clearly, such (p_s) exists). Then we have

$$\tau\big(p_s \mathbb{E}^{\mathbb{B}_{N_s}} \alpha f\big(W(\xi)\big)\big) \ge \tau\big(p_s f\big(W(\alpha\xi)\big)\big) + 2\omega_{N_s}\tau(p_s)\,.$$

Since $p_s \in \mathbb{B}_{N_s}$, we obtain

$$\tau\big(p_s \alpha f(W(\xi))\big) \geq \tau\big(p_s f(W(\alpha\xi))\big) + \sigma_s\,, \qquad \text{with} \quad \sigma_s \to 0\,.$$

On the other hand, by the normality of τ, we have

$$\tau\big(\alpha f(W(\xi))\big) = -\lim_s \tau\big(p_s \alpha f(W(\xi))_-\big)\,.$$

Consequently,

$$\begin{aligned} +\infty = \lim_s \tau\big(p_s \alpha f(W(\xi))_-\big) &= -\lim_s \tau\big(p_s \alpha f(W(\xi))\big) \\ &\leq \underline{\lim}_s \tau\big(p_s(-f(W(\alpha\xi))_+ + f(W(\alpha\xi))_-)\big) \\ &\leq \lim_s \tau\big(p_s f(W(\alpha\xi))_-\big)\,, \end{aligned}$$

so $\tau\big(f(W(\alpha\xi))_-\big) = +\infty$, which means that $\tau\big(f(W(\alpha\xi))\big) = -\infty$. It remains to consider the case $\tau\big(f(W(\alpha\xi))\big) = +\infty$. Going back to formula (11), we can construct the operator B as before (as the weak limit of D_{N_s}). The only difference is that now B is not necessarily positive. We take the positive part B_+ of B, and by (13), we get the inequality

$$f(W(\alpha\xi)) \leq B_+ = \lim_s \mathbb{E}^{\mathbb{B}_{N_s}} \alpha f(W(\xi))_+\,.$$

By the weak *-lower semicontinuity of τ, we obtain

$$\tau(B_+) \leq \tau\big(\alpha f(W(\xi))_+\big)\,.$$

Taking an increasing net of projections (p_s) with $\tau(p_s) < \infty$ and $\lim_s p_s = \mathbf{1}$, we obtain

$$\begin{aligned} +\infty = \lim_s \tau\big(p_s f(W(\alpha\xi))_+\big) &= \lim_s \tau\big(p_s f(W(\alpha\xi))\big) \\ &\leq \lim_s \tau\big(p_s B_+\big) \leq \tau\big(\alpha f(W(\xi))_+\big)\,, \end{aligned}$$

which concludes the proof. □

Remark. A very special case of formula (6) can be found in [10].

REFERENCES

[1] ANDO, T.: *Topics on Operator Inequalities.* Lecture Notes, Hokkaido University, Sapporo, 1978.

[2] ASH, R. B.: *Real Analysis and Probability*, Academic Press, New York-San Francisco, 1972.

[3] BEREZIN, F. A.: *Convex operator functions*, Mat. Sb. **88** (1972), 268–276. (Russian)

[4] BROWN, L. G.—KOSAKI, H.: *Jensen's inequality in semifinite von Neumann algebras*, J. Operator Theory **15** (1986), 205–225.

[5] DAVIS, C.: *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc. **8** (1957), 42–44.

[6] DAVIS, C.: *Notions generalizing convexity for functions defined on spaces of matrices*, Proc. Amer. Math. Soc. Symposia (Convexity) **7** (1963), 187–201.

[7] HANSEN, F.: *An operator inequality*, Math. Ann. **246** (1980), 249–250.

[8] HANSEN, F.—PEDERSEN, G. K.: *Jensen's inequality for operators and Löwner's theorem*, Math. Ann. **258** (1982), 229–241.

[9] JENSEN, I. L. W. V.: *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. **30** (1906), 175–193.

[10] JAJTE, R.: *Jensen's inequality for Schwarz maps in von Neumann algebras*, Probab. Math. Statist. **12** (1991), 251–254.

[11] LINDBLAD, G.: *Expectations and entropy inequalities for finite quantum systems*, Comm. Math. Phys. **39** (1974), 111–119.

[12] PETZ, D.: *Jensen's inequality for positive contractions on operator algebras*, Proc. Amer. Math. Soc. **99** (1987), 273–277.

[13] SIMON, B.: *Trace Ideals and Their Applications*, Cambridge Univ. Press, Cambridge, 1979.

[14] STINESPRING, W. F.: *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1965), 211–216.

[15] TAKESAKI, M.: *Operator Algebras* I, Springer-Verlag, Berlin-New York, 1979.

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Institute of Mathematics
University of Łódź
ul. Banacha 22
PL-90-238 Łódź
POLAND