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FUNCTIONS OF MEASURES AND A VARIATIONAL PROBLEM OF THE TYPE OF THE NONPARAMETRIC MINIMAL SURFACE

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Introduction

Let us define the functional

$$J(u, \Omega)=\int_{\Omega} f(u_{x_1}, \dots u_{x_N})\,\mathrm{d}x$$

on the space $W_1^1(\Omega)$, where f is a continuous, non-negative, convex function defined on E_N, for which there holds

$$f(x)\leq C(1+|x|), \quad x\in E_N.$$

Let us consider the following variational problem: given any function $u_0\in W_1^1(\Omega)$, to find the function $u\in u_0+\mathring{W}_1^1(\Omega)$ such that $J(u)=\inf\limits_{v\in u_0+\mathring{W}_1^1} J(v)$.

Since the ball in the space W_1^1 is not weakly compact, direct methods cannot usually be used here. However, it is possible to look for the minimum on a larger space of functions $W_\mu^1(\bar{\Omega})\supset W_1^1(\Omega)$, which does have a compact ball in a weak* topology (for the definition and properties of the space W_μ^1 the reader is referred to [7], the results from this work will be often used in this paper). There remains the problem to extend the functional J by any natural (and reasonable) way to the whole space W_μ^1 (resp. to the space $W_1^1+\mathring{W}_\mu^1$). Such a problem was investigated in [8], there are two posibilities of such extending:

$$F((u, \alpha), \bar{\Omega})=\inf\{\lim_{n\to\infty} J(u_n, \Omega);$$
$$u_n\rightharpoonup(u, \alpha) \text{ in } W_\mu^1, u_n\in W_1^1\}$$

for $(u, \alpha)\in W_\mu^1$ and

$$F_1((u, \alpha), \bar{\Omega})=\inf\{\underline{\lim}_{n\to\infty} J(u_n, \Omega);$$
$$u_n\rightharpoonup(u, \alpha) \text{ in } W_\mu^1, u_n\in(u, \alpha)+\mathring{W}_\mu^1, u_n\in W_1^1\}$$

for $(u, \alpha)\in W_1^1+\mathring{W}_\mu^1$.

It is possible to prove that $F_1 = F = J$ on W_1^1 and that F is weak* lower semicontinuous on W_μ^1 (resp. F_1 is weak* lower semicontinuous in $u_0 + \mathring{W}_\mu^1$ for all $u_0 \in W_1^1$) — see [8].

The functional F is of interest because it is the greatest (in the sense of values) extension of J on W_μ^1 which is weak* lower semicontinuous (the same is true for F_1 on $u_0 + \mathring{W}_\mu^1$, $u_0 \in W_1^1$).

Now (as in [8] for a more general case) we can find in the usual way the solution of our variational problem for the functionals F and F_1.

The handling with these functionals F, F_1 is difficult, for their definitions are very abstract. The aim of this work is to express the functional F analytically by means of a "function of measures" (see Sec. 1) and to investigate on this base the functional F and the corresponding variational problem. In Section 1 (§ 1 and § 2) we define the function of measures $\bar{f}(\alpha, \lambda)$, which is again measure, there is proved the weak lower semicontinuity of the measure $\bar{f}(\alpha, \lambda)$ with respect to α (in some sense), further, we prove there the possibility of integral representation

$$\bar{f}(\alpha, \lambda)(E) = \int_E \bar{f}\left(\frac{d\alpha}{dv}, \frac{d\lambda}{dv}\right) \mathrm{d}v, \quad E \subset \bar{\Omega}, \quad v = |\alpha| + \lambda$$

and other properties of a function of measures.

In section 2, § 3 there is shown the analytic expresion of the functional F (there λ denotes the Lebesque measure)

$$F((u, \alpha), \bar{\Omega}) = \bar{f}(\alpha, \lambda)(\bar{\Omega})$$

and other explicit expressions for F.

In § 4 there is proved the main result, $F = F_1$, from which, among others, two important consequences follow:

1) If $u \in W_1^1$ is the solution of our variational problem on the space W_1^1, then it is also the solution of the same variational problem in the extending formulation with the functional F on the space W_μ^1.
2) If $u \in W_\mu^1$ is the solution of the extending variational problem with the functional F on the space W_μ^1 and with the boundary condition $u' \in L_1(\partial\Omega)$, then the paradox situation $F((u, \alpha), \bar{\Omega}) < \inf_{u \in W_1^1} J(u, \Omega)$, the trace of

$$u|_{\partial\Omega} = u'$$

(u, α) is equal to u', cannot happen. It means that the variational problem with the functional F on the space W_μ^1 is a reasonable one in some sense.

By means of results from § 3 and § 4 we prove in § 5 the unicity of the solution of this variation problem and in § 6 the maximum principle.

Notation

f — a continuous function, which is non-negative and convex on E_N and for which there holds the growth condition

$$f(a) \leq C(1+|a|), \quad a \in E_N.$$

C — a constant depending only on the function f and

$$|a| = |a_1| + \ldots + |a_N|.$$

X — a compact set in E_N.

$L_\mu(X)$ — the space of all Borel σ-additive measures α, which are defined on X with norm $\|\alpha\|_{L_\mu(X)} = |\alpha|(X) < \infty$, where $|\alpha|$ is the total variation of α.

In the space $L_\mu(X)$ we shall define the weak convergence by

$$\alpha_n \rightharpoonup \alpha \text{ in } L_\mu(X) \text{ iff } \int_X \varphi d\alpha_n \to \int_X \varphi \, d\alpha \text{ for all } \varphi \in C(X)$$

$L_\mu^N(X) = [L_\mu(X)]^N$ — the space of N-tuples of measures $\alpha = (\alpha_1, \ldots, \alpha_N)$ with the norm $|\alpha|(X)$, $|\alpha| = |\alpha_1| + \ldots + |\alpha_N|$ and with the weak convergence defined as the weak convergence in each component.

λ — fixed non-negative measure from $L_\mu(X)$.

$\mathcal{B}$ — the family of all Borel subsets of E_N.

$$\mathcal{B}(X) = \{E \in \mathcal{B}\,;\ E \subseteq X\}.$$

$L_1(X, \nu)$ — the space of all Borel functions, which are integrable by the measure $\nu \in L_\mu(X)$, $\nu \geq 0$.

I. A function of measures

§ 1. Definition of the function of measures and its weak semicontinuity

Definition 1. For $a \in E_N$, $b > 0$ let us set

$$\bar{f}(a, b) = f\left(\frac{a}{b}\right) b,$$

$$\bar{f}(a, 0) = \lim_{b \to 0} f(a, b).$$

Remark 1. With regard to the convexity of f, the expression $\frac{f(ra) - f(0)}{r}$ is nondecreasing as $r \to \infty$ and hence $\lim_{r \to \infty} \frac{f(ra)}{r}$ exists. Thus, $\bar{f}(a, 0)$ is well-defined for each $a \in E_N$.

Theorem 1.

1) $\bar{f}(a, b) \leqslant C(|a| + |b|)$ *for all* $a \in E_N$, $b \geqslant 0$.

2) $\bar{f}(ka, kb) = k\bar{f}(a, b)$ *for all* $a \in E_N$, $b \geqslant 0$, $k \geqslant 0$, *i.e.* $\bar{f}(0, 0) = 0$.

3) *The function* $\bar{f}$ *is continuous on* $E_N \times \langle 0, \infty)$.

4) $\bar{f}\left(\sum_{i=1}^{\infty} a_i, \sum_{i=1}^{\infty} b_i\right) \leqslant \sum_{i=1}^{\infty} \bar{f}(a_i, b_i)$ *provided* $\sum_{i=1}^{\infty} a_i$, $\sum_{i=1}^{\infty} b_i$ *are convergent, where* $a_i \in E_N$, $b_i \geqslant 0$, $i = 1, 2, \ldots$

5) $|\bar{f}(a_1, b) - \bar{f}(a_2, b| \leqslant C|a_1 - a_2|$ *for all* $a_1, a_2 \in E_N$, $b \geqslant 0$.

Proof. Assertions 1) and 2) are evident. First we shall prove 4). Let $\varepsilon > 0$ be a positive number. Let us choose $\varepsilon_i > 0$ such that $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon$. There exists $\delta > 0$ such that for $0 < \eta < \delta$ there holds

$$\bar{f}\left(\sum_{i=1}^{\infty} a_i, \sum_{i=1}^{\infty} b_i\right) \leqslant \bar{f}\left(\sum_{i=1}^{\infty} a_i, \sum_{i=1}^{\infty} b_i + \eta\right) + \varepsilon.$$

There exist $\delta_i > 0$, $i = 1, 2, \ldots$ such that $\sum_{i=1}^{\infty} \delta_i < \delta$ and

$$\bar{f}(a_i, b_i + \delta_i) \leqslant \bar{f}(a_i, b_i) + \varepsilon_i \quad \text{for} \quad i = 1, 2, \ldots$$

From the convexity of f we conclude

$$\bar{f}\left(\sum_{i=1}^{\infty} a_i, \sum_{i=1}^{\infty} b_i\right) \leqslant \bar{f}\left(\sum_{i=1}^{\infty} a_i, \sum_{i=1}^{\infty} b_i + \sum_{i=1}^{\infty} \delta_i\right) + \varepsilon =$$

$$= f\left(\frac{\Sigma a_i}{\Sigma(a_i + \delta_i)}\right) \Sigma(b_i + \delta_i) + \varepsilon =$$

$$= f\left(\frac{b_1 + \delta_1}{\Sigma(a_i + \delta_i)} \cdot \frac{a_1}{b_1 + \delta_1} + \frac{b_2 + \delta_2}{\Sigma(b_i + \delta_i)} \cdot \frac{a_2}{b_2 + \delta_2} + \ldots\right) \Sigma(b_i + \delta_i) + \varepsilon \leqslant$$

$$\leqslant \left(\frac{b_1 + \delta_1}{\Sigma(b_i + \delta_i)} f\left(\frac{a_1}{b_1 + \delta_1}\right) + \frac{b_2 + \delta_2}{\Sigma(b_i + \delta_i)}\right) f\left(\frac{a_2}{b_2 + \delta_2}\right) + \ldots\right) \Sigma(b_i + \delta_i) + \varepsilon \leqslant$$

$$\leqslant \sum_{i=1}^{\infty} f\left(\frac{a_i}{b_i + \delta_i}\right)(b_i + \delta_i) + \varepsilon = \sum_{i=1}^{\infty} \bar{f}(a_i, b_i + \delta_i) + \varepsilon \leqslant$$

$$\leqslant \sum_{i=1}^{\infty} \bar{f}(a_i, b_i) + 2\varepsilon,$$

from which the assertion 4) follows.

Now we prove the assertion 3). If

$$a_n \to 0, \quad b_n \to b, \quad a, a_n \in E_N, \quad b, b_n \geqslant 0,$$

then

$$\bar{f}(a_n, b_n) = \bar{f}(a + a_n - a, b_n + 0) \leqslant \bar{f}(a, b_n) + \bar{f}(a_n - a, 0),$$

$$\bar{f}(a, b_n) = \bar{f}(a_n + a - a_n, b_n + 0) \leq \bar{f}(a_n, b_n) + \bar{f}(a - a_n, 0).$$

These inequalities imply

$$|\bar{f}(a_n, b_n) - \bar{f}(a, b_n)| \leq C|a - a_n|.$$

Using the continuity of f, we obtain $|\bar{f}(a, b_n) - \bar{f}(a, b)| \to 0$, from which the assertion 3) follows. The assertion 5) can be proved by reason of the assertion 1).

Definition 2. Let us set

$$\mathcal{R}(E) = \{\{E_i\}_{i=1}^{\infty};$$
$$E_i \cap E_j = \emptyset \text{ for each } i \neq j, \ \cup E_i = E, \ E_i \in \mathcal{B}\}$$

for each $E \in \mathcal{B}(X)$. Suppose $\alpha = (\alpha_1, \ldots, \alpha_N) \in L_\mu^N(X)$.

For $E \in \mathcal{B}(X)$ let us define

$$\bar{f}(\alpha, \lambda)(E) = \sup_{\{E_i\} \in \mathcal{R}(E)} \sum_{i=1}^{\infty} \bar{f}(\alpha(E_i), \lambda(E_i)),$$

Remark 2. The correctness of this definition follows from the consequence of Theorem 6. In definition 2 it is evidently sufficient to consider the supremum only on the finite decompositions of the set E.

Lemma 1. *Suppose $E \in \mathcal{B}(X)$, $\{E_i\}$, $\{F_j\} \in \mathcal{R}(E)$ and let us assume that the decomposition $\{F_j\}$ is more fine than $\{E_i\}$. Then*

$$\sum_{i=1}^{\infty} \bar{f}(\alpha(E_i), \lambda(E_i)) \leq \sum_{j=1}^{\infty} \bar{f}(\alpha(F_j), \lambda(F_j)).$$

Proof. From the assertion 4) of Theorem 1 we conclude

$$\bar{f}(\alpha(E_i), \lambda(E_i)) \leqq \sum_{F_j \subset E_i} \bar{f}(\alpha(F_j), \lambda(F_j)), \quad i = 1, 2, \ldots$$

Adding $i = 1, 2, \ldots$ we obtain Lemma 1.

Theorem 2.

1) $\bar{f}(\alpha, \lambda)(E) \leq C(|\alpha|(E) + \lambda(E))$ *for all* $E \in \mathcal{B}(X)$, *where* $|\alpha| = |\alpha_1| + \ldots + |\alpha_N|$.

2) $\bar{f}(k\alpha, k\lambda)(E) = k\bar{f}(\alpha, \lambda)(E)$ *for all* $k \geq 0$, $E \in \mathcal{B}(X)$.

3) $\bar{f}(\alpha, \lambda) \in L_\mu(X)$, $\bar{f}(\alpha, \lambda) \geq 0$.

4) *Suppose* $\alpha_1, \ldots, \alpha_k \in L_\mu^N(X)$, $t_1, \ldots, t_k \geq 0$, $t_1 + \ldots + t_k = 1$. *Then*

$$\bar{f}\left(\sum_{i=1}^{k} t_i \alpha_i, \lambda\right) \leq \sum_{i=1}^{k} t_i \bar{f}(\alpha_i, \lambda).$$

5) $|\bar{f}(\alpha_1, \lambda) - \bar{f}(\alpha_2, \lambda)| \leq C|\alpha_1 - \alpha_2|$ *for all* $\alpha_1, \alpha_2 \in L_\mu^N(X)$.

Proof. Assertions 1) and 2) follow from Theorem 1. Now we shall prove the σ-additivity of the set function $\bar{f}(\alpha, \lambda)$ on the ring $\mathcal{B}(X)$ of Borel subsets of X. Suppose $E \in \mathcal{B}(X)$, $\{E_i\}$, $\{A_i\} \in \mathcal{R}(E)$. Let us put $E_k^i = A_i \cap E_k$.

With respect to Lemma 1 we have

$$\sum_{i=1}^{\infty}\bar{f}(\alpha(A_i),\ \lambda(A_i))\leqslant\sum_{i,k=1}^{\infty}\bar{f}(\alpha(E_k^i),\lambda(E_k^i))\leqslant\sum_{k=1}^{\infty}\bar{f}(\alpha,\lambda)(E_k)$$

and thus $\bar{f}(\alpha,\ \lambda)(E)\leqslant\sum_{i=1}^{\infty}\bar{f}(\alpha,\lambda)(E_k)$.

Now we prove the reverse inequality. Let $\varepsilon>0$ be given. Let us take $\varepsilon_k>0$, $\sum_{k=1}^{\infty}\varepsilon_k<\varepsilon$. There exist the decompositions $\{E_k^i\}_{i=1}^{\infty}\in\mathcal{R}(E_k)$, $k=1,2,\ldots$ such that

$$\bar{f}(\alpha,\lambda)(E_k)\leqslant\sum_{i=1}^{\infty}\bar{f}(\alpha(E_k^i),\lambda(E_k^i))+\varepsilon_k,\quad k=1,2,\ldots$$

Then $\sum_{k=1}^{\infty}\bar{f}(\alpha,\lambda)(E_k)\leqslant$

$$\leqslant\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{\infty}\bar{f}(\alpha(E_k^i),\lambda(E_k^i)\Big)+\varepsilon_k\leqslant\bar{f}(\alpha,\lambda)(E)+\varepsilon.$$

Further, $\bar{f}(a,b)\geqslant 0$ implies $\bar{f}(\alpha,\lambda)\geqslant 0$.

Using Theorem 1 we prove the assertion 4). Suppose $E\in\mathcal{B}(X)$. Then

$$\bar{f}\Big(\sum_{l=1}^{k}t_l\alpha_l,\lambda\Big)(E)=\sup_{\{E_i\}\in\mathcal{R}(E)}\sum_{i=1}^{\infty}\bar{f}\Big(\sum_{l=1}^{k}t_l\alpha_l(E_i),\ \sum_{l=1}^{k}t_l\lambda(E_i)\Big)\leqslant$$

$$\leqslant\sup_{\{E_i\}\in\mathcal{R}(E)}\sum_{i=1}^{\infty}\sum_{l=1}^{k}t_l\bar{f}(\alpha_l(E_i),\lambda(E_i))\leqslant$$

$$\leqslant\sum_{l=1}^{k}t_l\sup_{\{E_i\}\in\mathcal{R}(E)}\sum_{i=1}^{\infty}\bar{f}(\alpha_l(E_i),\lambda(E_i))=\sum_{l=1}^{k}t_l\bar{f}(\alpha_l,\lambda)(E).$$

For the proof of the assertion 5) we suppose $E\in\mathcal{B}(X)$, $\{E_i\}\in\mathcal{R}(E)$.

With regard to Theorem 1 and the preceding assertion we conclude

$$|\bar{f}(\alpha_1(E_i),\lambda(E_i))-\bar{f}(\alpha_2(E_i),\lambda(E_i))|\leqslant$$
$$\leqslant C|\alpha_1(E_i)-\alpha_2(E_i)|\leqslant C|\alpha_1-\alpha_2|(E_i),\quad i=1,2,\ldots,$$

$$|\bar{f}(\alpha_1,\lambda)-\bar{f}(\alpha_2,\lambda)|(E)=$$
$$=\sup_{\{E_i\}\in\mathcal{R}(E)}\sum_{i=1}^{\infty}|\bar{f}(\alpha_1,\lambda)(E_i)-\bar{f}(\alpha_2,\lambda)(E_i)|\leqslant$$
$$\leqslant\sup_{\{E_i\}\in\mathcal{R}(E)}\sum_{i=1}^{\infty}C|\alpha_1-\alpha_2|(E_i)=C|\alpha_1-\alpha_2|(E).$$

Theorem 3. *Suppose* $\alpha=(\alpha_1,\ \ldots,\ \alpha_N)\in L_\mu^N(X)$ *and denote*

$$\sigma=\Big\{\{\omega_i\}_{i=1}^{\infty};\ \omega_i\in C(E_N),\ \omega_i\geqslant 0,\ \sum_{i=1}^{\infty}\omega_i=1\Big\}.$$

Then we have

$$\int_X \varphi \, d\bar{f}(\alpha,\lambda) = \sup_{\{\omega_i\}\in\sigma} \sum_{i=1}^{\infty} \bar{f}\left(\int_X \varphi\omega_i d\alpha, \int_X \varphi\omega_i d\lambda\right),$$

for each $\varphi \in C(X)$, $\varphi \geq 0$.

It is clear that it is sufficient to consider the supremum only on finite decompositions of the unit.

Remark 3. Especially for $\varphi \equiv 1$ we obtain an equivalent definition of the function of measures

$$\bar{f}(\alpha,\lambda)(E) = \sup_{\{\omega_i\}\in\sigma} \sum_{i=1}^{\infty} \bar{f}\left(\int_E \omega_i d\alpha, \int_E \omega_i \, d\lambda\right),$$

where E is an arbitrary compact $E \subset X$.

Proof. Suppose $\{\omega_1, \ldots, \omega_m, 0, \ldots\} \in \sigma$,

$$K = \max\,(\|\alpha_i\|_{L_\mu(X)}, \|\lambda\|_{L_\mu(X)}, \max_X |\varphi|).$$

Let $\varepsilon > 0$ be fixed. There exists a finite decomposition $\{E_1, \ldots, E_r, 0, \ldots\} \in \mathcal{R}(X)$ such that $\sup_{x\in E_j} \varphi(x)\omega_i(x) - \inf_{x\in E_j} \varphi(x)\omega_i(x) < \varepsilon$ for each i, j. Let us denote $a_{ij} = \inf_{x\in E_j} \varphi(x)\omega_i(x)$.

Then the assertions

$$\sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m} \inf_{E_j} \varphi\omega_i \leq \inf_{E_j} \sum_{i=1}^{m} \varphi\omega_i = \inf_{E_j} \varphi,$$

$$\left| \int_X \varphi\omega_i \, d\alpha - \sum_{j=1}^{r} a_{ij}\alpha(E_j) \right| \leq K\varepsilon \quad \text{hold.}$$

Let $\delta(\varepsilon)$ be the module of continuity of $\bar{f}$ on $\langle -K, K\rangle^N \times \langle 0, K\rangle$ (i.e. $\varrho((x_1, \lambda_1), (x_2, \lambda_2)) < \delta$ implies $\varrho(\bar{f}(x_1, \lambda_1), \bar{f}(x_2, \lambda_2)) < \varepsilon$ for all $x_1, x_2 \in \langle -K, K\rangle^N$, $\lambda_1, \lambda_2 \in \langle 0, K\rangle$).

Then we have

$$(1) \qquad \sum_{i=1}^{m} \bar{f}\left(\int_X \varphi\omega_i \, d\alpha, \int_X \varphi\omega_i \, d\lambda\right) \leq \sum_{i=1}^{m} \bar{f}\left(\sum_{j=1}^{r} a_{ij}\alpha(E_j), \sum_{j=1}^{r} a_{ij}\lambda(E_j)\right) +$$

$$+ m\delta(K\varepsilon) \leq \sum_{i,j} a_{ij}\bar{f}(\alpha(E_j), \lambda(E_j)) + m\delta(K\varepsilon) \leq$$

$$\leq \sum_j \inf_{E_j} \varphi \cdot \bar{f}(\alpha(E_j), \lambda(E_j)) + m\delta(K\varepsilon) \leq$$

$$\leq \sum_j \inf_{E_j} \varphi \cdot \bar{f}(\alpha, \lambda)(E_j) + m\delta(K\varepsilon) \leq \int_X \varphi \, d\bar{f}(\alpha, \lambda) + m\delta(K\varepsilon).$$

Now, we shall prove an inequality reverse to that of (1). There exists a decomposition $\{E_1, \ldots, E_m, \emptyset, \ldots\} \in \mathcal{R}(X)$ such that

$$\sup_{E_i} \varphi - \inf_{E_i} \varphi < \frac{\varepsilon}{3}, \quad i = 1, \ldots, m,$$

$$\int_X \varphi \, \mathrm{d}\bar{f}(\alpha, \lambda) < \sum_i \sup_{E_i} \varphi \bar{f}(\alpha(E_i), \lambda(E_i)) + \varepsilon,$$

since $\bar{f}(\alpha, \lambda) \in L_\mu(X)$.

Let us denote $a_i = \sup\limits_{E_i} \varphi + \varepsilon/3$. There measures α, λ are regular. There exist compacts $F_i \subset E_i$ such that

$$\int_X \varphi \, \mathrm{d}\bar{f}(\alpha, \lambda) < \sum_i a_i \bar{f}(\alpha(F_i), \lambda(F_i)) + 2\varepsilon.$$

Similarly, there exist disjoint open sets $G_i \supset F_i$ satisfying $a_i - \varepsilon < \varphi(x) < a_i$ for each $x \in G_i$, $i = 1, \ldots m$,

$$|\alpha|(G_i - F_i) < \frac{\varepsilon}{m}, \quad \lambda(G_i - F_i) < \frac{\varepsilon}{m}$$

and

(2) $$\int_X \varphi \, \mathrm{d}\bar{f}(\alpha, \lambda) < \sum_i a_i \bar{f}(\alpha(G_i), \lambda(G_i)) + 3\varepsilon.$$

There exist $\omega_i \in C(E_N)$ such that

$$\omega_i = 1 \quad \text{on} \quad F_i, \operatorname{supp} \omega_i \subset G_i, \quad 0 \leqslant \omega_i \leqslant 1.$$

Then we conclude

$$\left| a_i \alpha(G_i) - \int_X \varphi \omega_i \, \mathrm{d}\alpha \right| \leqslant \left| \int_{F_i} (a_i - \varphi) \, \mathrm{d}\alpha \right| +$$

$$+ \left| \int_{G_i - F_i} (a_i - \varphi \omega_i) \, \mathrm{d}\alpha \right| \leqslant \varepsilon |\alpha|(F_i) + (K + \varepsilon) \frac{\varepsilon}{m}$$

$$\left| a_i \lambda(G_i) - \int_X \varphi \omega_i \, \mathrm{d}\lambda \right| \leqslant \varepsilon \lambda(F_i) + (K + \varepsilon) \frac{\varepsilon}{m}$$

(3) $$a_i \lambda(G_i) - \int_X \varphi \omega_i \, \mathrm{d}\lambda \geqslant 0.$$

Ussing the assertion 4) from Theorem 1 and (3) we obtain

$$\sum_i a_i \bar{f}(\alpha(G_i), \lambda(G_i)) =$$

$$= \sum_i \bar{f} \left(\int_X \varphi\omega_i \, d\alpha + a_i\alpha(G_i) - \int_X \varphi\omega_i \, d\alpha, \int_X \varphi\omega_i \, d\lambda + \right.$$

$$\left. + a_i\lambda(G_i) - \int_X \varphi\omega_i \, d\lambda \right) \leqslant$$

$$\leqslant \sum_i \bar{f} \left(\int_X \varphi\omega_i \, d\alpha, \int_X \varphi\omega_i \, d\lambda \right) + \sum_i \bar{f}(a_i\alpha(G_i) -$$

$$- \int_X \varphi\omega_i \, d\alpha, a_i\lambda(G_i) - \int_X \varphi\omega_i \, d\lambda \,) \leqslant$$

$$\leqslant \sum_i \bar{f} \left(\int_X \varphi\omega_i \, d\alpha, \int_X \varphi\omega_i \, d\lambda \right) + \sum_{i=1}^{m} C\varepsilon(|\alpha|(F_i) + \lambda(F_i)) +$$

$$+ \frac{2(K+\varepsilon)}{m} \leqslant \sum_{i=1}^{m} \bar{f} \left(\int_X \varphi\omega_i \, d\alpha, \int_X \varphi\omega_i \, d\lambda \right) + C\varepsilon \cdot 4(K+\varepsilon).$$

Adding the function $1 - \sum_{i=1}^{m} \omega_i$ we shall complete the system of functions $\omega_1, \ldots, \omega_m$ to the decomposition of the unit. Using (2) we obtain the required inequality.

Theorem 4 *(Jensen's inequality). Suppose $\alpha \in L_\mu^N(X)$, $\varphi \in C(X)$, $\varphi \geqslant 0$. Then we have*

$$\bar{f} \left(\int_X \varphi \, d\alpha, \int_X \varphi \, d\lambda \right) \leqslant \int_X \varphi \, d\bar{f}(\alpha, \lambda \quad .$$

Proof. Jensen's inequality is a consequence of the previous Theorem if we consider the following decomposition of the unit

$$\{1, 0, 0 \ldots\} \in \sigma.$$

It is possible to prove Jensen's inequality directly without using Theorem 3. From definition 2 we see that $\bar{f}(\alpha(E), \lambda(E)) \leqslant \bar{f}(\alpha, \lambda)(E)$ for all $E \in \mathscr{B}(X)$. Then we proceed as in the proof of Theorem 3, where we estimate Riemann's integrals by Riemann's sums.

Theorem 5. *The mapping*

$$\alpha \in L_\mu^N(X) \to \bar{f}(\alpha, \lambda) \in L_\mu(X)$$

is weakly lower semicontinuous, i.e. if

$$\alpha_n, \alpha \in L_\mu^N(X), \alpha_n \rightharpoonup \alpha \quad in \quad L_\mu^N(X),$$

then

$$\int_X \varphi \, d\bar{f}(\alpha, \lambda) \leqslant \varliminf_{n \to \infty} \int_X \varphi \, d\bar{f}(\alpha_n, \lambda)$$

for each $\varphi \in C(X)$, $\varphi \geqslant 0$.

Remark 4. Especially for $\varphi \equiv 1$ we conclude that $\alpha_n \rightharpoonup \alpha$ in $L_\mu^N(X)$ implies $\bar{f}(\alpha, \lambda)(X) \leq \underline{\lim}\, \bar{f}(\alpha_n, \lambda)(X)$.

Proof. If $\varphi \in C(X)$, $\varphi \geq 0$, $\{\omega_1, \ldots, \omega_m, 0, \ldots\} \in \sigma$, then

$$\sum_i \bar{f}\left(\int_X \varphi\omega_i \, d\alpha, \int_X \varphi\omega_i \, d\lambda\right) = \sum_i \lim_{n\to\infty} \bar{f}\left(\int_X \varphi\omega_i \, d\alpha_n, \int_X \varphi\omega_i \, d\lambda\right) =$$

$$= \lim_{n\to\infty} \sum_i \bar{f}\left(\int_X \varphi\omega_i \, d\alpha_n, \int_X \varphi\omega_i \, d\lambda\right) \leq \underline{\lim}_{n\to\infty} \int_X \varphi \, d\bar{f}(\alpha_n, \lambda)$$

because of Theorem 3.

2. Equivalent definitions for the functions of measures

In accordance with Bourbaki [4] let us state.

Definition 3. Suppose $\alpha = (\alpha_1, \ldots, \alpha_N) \in L_\mu^N(X)$ and let $\nu \in L_\mu(X)$, $\nu \geq 0$ be such that the measures $\alpha_1, \ldots, \alpha_N, \lambda$ are absolutely continuous with respect to the measure ν (such measure ν exists, for example $\nu = |\alpha| + \lambda$). Let us denote by

$$\frac{d\alpha_1}{d\nu}, \ldots, \frac{d\alpha_N}{d\nu}, \quad \frac{d\lambda}{d\nu} \in L_1(X, \nu)$$

the densities of the measures $\alpha_1, \ldots, \alpha_N, \lambda$ with respect to the measure ν. This notation will be used in the following. For $E \in \mathcal{B}(X)$ in [4] is defined

$$\bar{f}^*(\alpha, \lambda)(E) = \int_E \bar{f}\left(\frac{d\alpha_1}{d\nu}, \ldots, \frac{d\alpha_N}{d\nu}, \frac{d\lambda}{d\nu}\right) d\nu$$

or equivalently

$$\int_X \varphi \, d\bar{f}^*(\alpha, \lambda) = \int_X \varphi \bar{f}\left(\frac{d\alpha}{d\nu}, \frac{d\lambda}{d\nu}\right) d\nu$$

for all $\varphi \in C(X)$.

Remark 5. In Bourbaki [4] a composed function of measure is defined in a somewhat more general way. He considers a continuous, non-negative, positively homogeneous function

$$g(x_1, \ldots, x_N), \; x \in E_N \; (g: E_N \to R)$$

satisfying

$$|g(x_1, \ldots, x_N)| \leq C(|x_1| + \ldots + |x_N|).$$

Suppose $\alpha_1, \ldots, \alpha_N \in L_\mu(X)$. Let us take a non-negative Borel measure ν such that $\alpha_1, \ldots, \alpha_N$ are absolutely continuous with respect to ν. Then they define

$$g(\alpha_1, \ldots, \alpha_N)(E) = \int_E g\left(\frac{d\alpha_1}{d\nu}, \ldots, \frac{d\alpha_N}{d\nu}\right) d\nu, \quad E \in \mathcal{B}(X)$$

and it is proved in [4] that the above integral has a sense and that the defined measure is independent of the choice of the measure ν.

The main result of this paragraph is the following

Theorem 6. *Suppose*

$$\alpha=(\alpha_1, \ldots, \alpha_N)\in L_\mu^N(X).$$

Then

$$\bar{f}(\alpha, \lambda)=\bar{f}^*(\alpha, \lambda) \quad in \quad L_\mu(X).$$

Consequence. If the measures $\alpha_1, \ldots, \alpha_N$ are absolutely continuous with respect to λ, then for $\nu=\lambda$ we deduce

$$\int_X \varphi \, \mathrm{d}\bar{f}(\alpha, \lambda)=\int_X \varphi f\left(\frac{\mathrm{d}\alpha_1}{\mathrm{d}\lambda}, \ldots, \frac{\mathrm{d}\alpha_N}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda, \quad \varphi\in C(X),$$

i.e.

$$\frac{\mathrm{d}\bar{f}(\alpha, \lambda)}{\mathrm{d}\lambda}=f\left(\frac{\mathrm{d}\alpha_1}{\mathrm{d}\lambda}, \ldots, \frac{\mathrm{d}\alpha_N}{\mathrm{d}\lambda}\right) \text{ in } L_1(X, \lambda).$$

Thus in this case the definition of the function of measures coincides with the definition of the composed function.

Remark 6. Suppose that $\alpha_i=\alpha_i^r+\alpha_i^s$, $i=1, \ldots, N$ are decompositions of the measures $\alpha_1, \ldots, \alpha_N$, where α_i^r, α_i^s are absolutely continuous and singular parts of α_i with respect to the measure λ.

There exists $F_0\in\mathscr{B}(X)$ such that

$$|\alpha_i^s|(X-E_0)=0 \quad \text{for each} \quad i=1, \ldots, N, \quad \lambda(E_0)=0.$$

From the preceding Theorems and Definitions we conclude

$$\begin{aligned}\bar{f}(\alpha, \lambda)(X)&=\bar{f}(\alpha, \lambda)(X-E_0)+\bar{f}(\alpha, \lambda)(E_0)=\\&=\bar{f}(\alpha^r, \lambda)(X-E_0)+\bar{f}(\alpha^s, \lambda)(E_0)=\\&=\bar{f}(\alpha^r, \lambda)(X)+\bar{f}(\alpha^s, \lambda)(X)\end{aligned}$$

i.e.

$$\bar{f}(\alpha, \lambda)(X)=\int_X f\left(\frac{\mathrm{d}\alpha^r}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda+\int_X \bar{f}\left(\frac{\mathrm{d}\alpha^s}{\mathrm{d}|\alpha^s|}, 0\right)\mathrm{d}|\alpha^s|. \tag{4}$$

Proof of Theorem 6. It is sufficient to prove that $\bar{f}(\alpha, \lambda)(Y) = f^*(\alpha, \lambda)(Y)$, where Y is an arbitrary compact set, $Y\subseteq X$. Suppose $\{E_i\}\in\mathscr{R}(Y)$.

Owing to Jensen's inequality (see [2])

$$\begin{aligned}\sum_i\bar{f}(\alpha(E_i), \lambda(E_i))&=\sum_i\bar{f}\left(\int_{E_i}\frac{\mathrm{d}\alpha}{\mathrm{d}\nu}\mathrm{d}\nu, \int_{E_i}\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\mathrm{d}\nu\right)\leqslant\\&\leqslant\sum_i\int_{E_i}\bar{f}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\nu}, \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\right)\mathrm{d}\nu=\int_Y\bar{f}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\nu}, \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\right)\mathrm{d}\nu\end{aligned}$$

and hence $0\leqslant\bar{f}(\alpha, \lambda)(Y)\leqslant\bar{f}^*(\alpha, \lambda)(Y)$.

By reason of this inequality we deduce that the measure $\bar{f}(\alpha, \lambda)$ is absolutely continuous with respect to the measure $\bar{f}^*(\alpha, \lambda)$. With regard to the definition of $\bar{f}^*(\alpha, \lambda)$ we have that the measure $\bar{f}^*(\alpha, \lambda)$ is absolutely continuous with respect to the measure ν. Let us set

$$h = \frac{d\bar{f}(\alpha, \lambda)}{d\nu} \in L_1(X, \nu).$$

The above inequality implies that

$$0 \leq h \leq \bar{f}\left(\frac{d\alpha}{d\nu}, \frac{d\lambda}{d\nu}\right), \quad \nu\text{ — a.e. on } X.$$

Now let us assume that $h < \bar{f}\left(\frac{d\alpha}{d\nu}, \frac{d\lambda}{d\nu}\right)$ on a set of a positive measure ν. Then there exist $\varepsilon > 0$ and $E_0 \in \mathscr{B}(X)$ satisfying

$$\nu(E_0) > 0,$$

$$h < \bar{f}\left(\frac{d\alpha}{d\nu}, \frac{d\lambda}{d\nu}\right) - \varepsilon \quad \nu\text{ — a.e. on } E_0.$$

With respect to Luzin's Theorem (see [3]) there exists $E_1 \in \mathscr{B}(X)$ such that $E_1 \subset E_0$, $\nu(E_1) > 0$ and the functions $\frac{d\alpha_1}{d\nu}, \ldots, \frac{d\alpha_N}{d\nu}, \frac{d\lambda}{d\nu}$ are continuous in E_1.

With respect to the regularity of the measure ν we can take a closed subset $E_2 \subset E_1$ with $\nu(E_2) > 0$.

There exists a point $x_0 \in E_2$ such that $\nu(F_n) > 0$ for $F_n = E_2 \cap \left\{|x - x_0| \leq \frac{1}{n}\right\}$ (see Remark 7).

With regard to the continuity of the functions $\frac{d\alpha_1}{d\nu}, \ldots, \frac{d\alpha_N}{d\nu}, \frac{d\lambda}{d\nu}$ on the compact E_2 and owing to the continuity of $\bar{f}$, we conclude

$$(5) \quad \begin{gathered} \frac{1}{\nu(F_n)} \int_{F_n} \frac{d\alpha_i}{d\nu} d\nu \to \frac{d\alpha_i}{d\nu}(x_0), \frac{1}{\nu(F_n)} \int_{F_n} \frac{d\lambda}{d\nu} \to \frac{d\lambda}{d\nu}(x_0), n \to \infty \\ \frac{1}{\nu(F_n)} \int_{F_n} \bar{f}\left(\frac{d\alpha}{d\nu}, \frac{d\lambda}{d\nu}\right) d\nu \xrightarrow[n\to\infty]{} \bar{f}\left(\frac{d\alpha}{d\nu}(x_0), \frac{d\lambda}{d\nu}(x_0)\right). \end{gathered}$$

From the definition of the measure $\bar{f}(\alpha, \lambda)$ we obtain

$$\bar{f}\left(\int_{F_n} \frac{d\alpha}{d\nu} d\nu, \int_{F_n} \frac{d\lambda}{d\nu} d\nu\right) \leq \bar{f}(\alpha, \lambda)(F_n) =$$

$$= \int_{F_n} h \, d\nu \leq \int_{F_n} \bar{f}\left(\frac{d\alpha}{d\nu}, \frac{d\lambda}{d\nu}\right) d\nu - \varepsilon \nu(F_n).$$

We divide this inequality by $\nu(F_n)$ and apply the homogenity and continuity of the function $\bar{f}$. Then by the limiting process we deduce

$$\bar{f}\left(\frac{d\alpha}{d\nu}(x_0), \frac{d\lambda}{d\nu}(x_0)\right) \leq \bar{f}\left(\frac{d\alpha}{d\nu}(x_0), \frac{d\lambda}{d\nu}(x_0)\right) - \varepsilon,$$

which is a contradiction.

Thus $h = \bar{f}\left(\frac{d\alpha}{d\nu}, \frac{d\lambda}{d\nu}\right)$ in $L_1(X, \nu)$ and hence

$$\bar{f}(\alpha, \lambda) = \bar{f}^*(\alpha, \lambda).$$

Remark 7. For completness we shall prove the following assertion. Let $E \subset X$ be a compact and suppose

$$\nu \in L_\mu(X), \quad \nu(E) > 0, \quad \nu \geq 0.$$

Let us denote $B(x, r) = \{y \in E_N;\ |x - y| \leq r\}$.

Then there exists a point $x_0 \in E$ such that $\nu(F_n) > 0$ for $F_n = E \cap B\left(x_0, \frac{1}{n}\right)$, $n = 1, 2, \ldots$

We put $M_n = \left\{x \in E;\ \nu\left(E \cap B\left(x, \frac{1}{n}\right)\right) > 0\right\}$.

From $\nu(E) > 0$ we deduce that $M_n \neq \emptyset$ for $n = 1, 2, \ldots$

We can easily verify the inclusion $M_n \supset \bar{M}_{n+1}$, $n = 1, 2, \ldots$

There exists $x_0 \in \bigcap_{n=1}^{\infty} \bar{M}_n$ and hence $x_0 \in \bigcap_{n=1}^{\infty} M_n$.

We shall prove some further properties of the measure $\bar{f}(\alpha, \lambda)$. From now on throughout we shall assume this section that λ is the Lebesque measure in E_n. We shall use the canonical imbedding $L_1(X, \lambda) \subset L_\mu(X)$ defined by (see [7])

$$u \in L_1(X, \lambda) \to \alpha \in L_\mu(X),$$

$$\alpha(E) = \int_E u \, d\lambda \quad \text{for all} \quad E \in \mathscr{B}(X).$$

Theorem 7. *Suppose $E \in \mathscr{B}(X)$, $\lambda(E) > 0$, then*

$$\bar{f}(\alpha, \lambda)(E) = \sup_{\{E_i\} \in \mathscr{R}(E)} \sum_{i=1} \bar{f}(\alpha(E_i), \lambda(E_i)).$$

$$\lambda(E_i) > 0, \quad i = 1, 2, \ldots$$

Proof. Let us denote $K = \max(|\alpha|(E), \lambda(E))$ and let $\varepsilon_0 > 0$ be fixed. Let us take $\varepsilon_i > 0$, $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon_0$. Owing to the uniform continuity of the function $\bar{f}$ on $\langle -K, K\rangle^N \times \langle 0, K\rangle$ there exist $\delta_i > 0$, $i = 0, 1, \ldots$ with $\sum_{i=1}^{\infty} \delta_i < \delta_0$ such that for $a_1, a_2 \in E_N$, $b_1, b_2 > 0$ we obtain

(6) if $|a_1-a_2|+|b_1-b_2|\leq\delta_i$, then $|\bar{f}(a_1,b_1)-\bar{f}(a_2,b_2)|\leq\varepsilon_i$, $i=0,1,\ldots$

There exists a decomposition $\{E_i\}_{i=0}^{\infty}\in\mathcal{R}(E)$ for which

$$\sum_{i=0}^{\infty}\bar{f}(\alpha(E_i),\lambda(E_i))\geq\bar{f}(\alpha,\lambda)(E)-\varepsilon_0.$$

In accordance with Lemma 1 we can assume that the decomposition $\{E_i\}_{i=0}^{\infty}$ is sufficiently fine and (after suitable relabelling) satisfies

(7) $$|\alpha|(E_0)<\delta_0,\quad \lambda(E_0)<\delta_0,\quad \lambda(E_0)>0.$$

By induction we find a sequence of disjoint Borel sets $F_n\subset E_0$, $n=1,2,\ldots$, satisfying

(8) $$\lambda(F_n)>0,\quad \lambda(F_n)<\delta_n,\quad |\alpha|(F_n)<\delta_n,\quad n=1,2,\ldots$$

It is sufficient to take into account that λ is the Lebesque measure $\alpha_1,\ldots,\alpha_N$ are σ-additive measures and to use Remark 7. From (6), (7), (8) we conclude

$$\sum_{i=1}^{\infty}\bar{f}(\alpha(E_i),\lambda(E_i))\geq\bar{f}(\alpha,\lambda)(E)-2\varepsilon_0,$$

$$\sum_{i=1}^{\infty}\bar{f}(\alpha(E_i\cup F_i),\quad \lambda(E_i\cup F_i))\geq\bar{f}(\alpha,\lambda)(E)-2\varepsilon_0-\sum_{i=1}^{\infty}\varepsilon_i.$$

Finally it suffices to add

$$\bar{f}\left(\alpha\left(E_0-\bigcup_{i=1}^{\infty}F_i\right),\lambda\left(E_0-\bigcup_{i=1}^{\infty}F_i\right)\right)\geq 0$$

to the left-hand side of the above inequality.

Theorem 8. *Suppose $\lambda(X)>0$, $\alpha\in L_\mu^N(X)$. Then there exist function $u_n=(u_n^1,\ldots,u_n^N)\in L_1^N(X,\lambda)$, $n=1,2,\ldots$ such that $u_n\rightharpoonup\alpha$ in $L_\mu^N(X)$, $\bar{f}(\alpha,\lambda)(X)=\lim_{n\to\infty}\int_X f(u_n)\,d\lambda(x)$.*

Remark 8. Taking into account the Remark 4 and the consequence of Theorem 6, we obtain a further equivalent definition of the measure $\bar{f}(\alpha,\lambda)$ if $\lambda(X)>0$:

$$\bar{f}(\alpha,\lambda)(X)=\inf\varliminf_{n\to\infty}\int_X f(u_n(x))\,d\lambda(x),$$

where the infimum is taken over all the sequences $\{u_n\}_{n=1}^{\infty}$ satisfying $u_1,u_2,\ldots\in L_1^N(X,\lambda)$, $u_n\rightharpoonup\alpha$ in $L_\mu^N(X)$.

Proof. From Theorem 7 and Lemma 1 it follows that there exist decompositions $\{E_i^n\}_{i=1}^{\infty}\in\mathcal{R}(X)$, $n=1,2,\ldots$ satisfying

(9) $$\lambda(E_i^n)>0, \quad \operatorname{diam}(E_i^n)\leq\frac{1}{n} \quad \text{for each} \quad i, n=1,2,\ldots,$$

(10) $$\sum_{i=1}^{\infty}\bar{f}(\alpha(E_i^n), \lambda(E_i^n))\geq\bar{f}(\alpha,\lambda)(X)-\frac{1}{n}.$$

For each $n=1, 2, \ldots$ let us denote

$$u_n(x)=\frac{\alpha(E_i^n)}{\lambda(E_i^n)} \quad \text{for} \quad x\in E_i^n, \quad i=1,2,\ldots$$

These vector functions belong to $L_1^N(X,\lambda)$, because

$$\int_X |u_n(x)|\, d\lambda(x)=\sum_{i=1}^{\infty}\int_{E_i}\frac{|\alpha(E_i^n|}{\lambda(E_i^n)}\,dx\leq$$

$$\leq\sum_{i=1}^{\infty}|\alpha|(E_i^n)\leq|\alpha|(X)<\infty.$$

With respect to the definition of f and from (10) we deduce

$$\int_X f(u_n(x))\,dx=\sum_{i=1}^{\infty}\int_{E_i^n} f\left(\frac{\alpha(E_i^n)}{\lambda(E_i^n)}\right)dx=$$

$$=\sum_i\bar{f}(\alpha(E_i^n),\lambda(E_i^n))\to\bar{f}(\alpha,\lambda)(X).$$

Now we prove that $u_n\rightharpoonup\alpha$ in $L_\mu^N(X)$. Suppose $\varphi\in C(X)$. For $n=1,2,\ldots$ let us set

$$\varphi_n(x)=\int_{E_i^n}\frac{\varphi(y)}{\lambda(E_i^n)}\,d\lambda(y) \quad \text{for} \quad x\in E_i^n, \quad i=1,2,\ldots$$

From the uniform continuity of φ on X and from (9) we obtain $\varphi_n\to\varphi$ in $C(X)$ and hence

$$\int_X \varphi u_n\,d\lambda=\sum_i\int_{E_i^n}\varphi\,\frac{\alpha(E_i^n)}{\lambda(E_i^n)}\,d\lambda=$$

$$=\sum_i\int_{E_i^n}\frac{\varphi}{\lambda(E_i^n)}\,d\lambda\cdot\alpha(E_i^n)=\int_X\varphi_n\,d\alpha\to\int_X\varphi\,d\alpha.$$

II. Application of the function of measures in the calculus of variation

We shall consider a bounded domain $\Omega\subset E_N$ with the boundary $\partial\Omega$ of the class C^1 (see [7], [8]). We recapitulate for the reader the definition and some basic properties of the space $W_\mu^1(\bar{\Omega})$ (for details see [7]).

$W^1_\mu(\bar\Omega)$ is the space of all $(N+1)$-tuples $(u, \alpha_1, \dots, \alpha_N)$ for which

i) $u \in L_1(\Omega)$, $\alpha_1, \dots, \alpha_N \in L_\mu(\bar\Omega)$,

ii) there exists a measure $\beta \in L_\mu(\partial\Omega)$ such that

$$\int_{\partial\Omega} \varphi v_i \,\mathrm{d}\beta = \int_{\Omega_1} u\varphi_{x_i}\,\mathrm{d}x + \int_\Omega \varphi\,\mathrm{d}\alpha_i, \quad i = 1, \dots, N$$

holds for all $\varphi \in C^1(\bar\Omega)$, where $v \equiv (v_1, \dots, v_N)$ is the normal exterior of $\partial\Omega$.

The measure β, which is uniquely determined by (u, α_i), will be called the trace of the element (u, α_i). The norm in $W^1_\mu(\bar\Omega)$ is defined by

$$\|(u, \alpha_i)\| W^1_\mu = \|u\|_{L_1(\Omega)} + \sum_{i=1}^N |\alpha_i|(\bar\Omega).$$

By $\mathring{W}^1_\mu(\bar\Omega)$ we denote the subspace of all elements of $W^1_\mu(\bar\Omega)$ with the trace $\beta = 0$. The measure

$$\alpha_v \in L_\mu(\partial\Omega),\ \alpha_v = \sum_{i=1}^N v_i\alpha_i|_{\partial\Omega}$$

is called the side of the element $(u, \alpha_i) \in W^1_\mu(\bar\Omega)$, where the obvious definition of the measure $v_i\alpha_i|_{\partial\Omega}$ ($v_i \in C(\partial\Omega)$, $\alpha_i|_{\partial\Omega}$ is the restriction of α_i on $\partial\Omega$) has been used.

The measure $\beta^0 = \beta - \alpha_v$ is called the inner trace of (u, α_i). It is proved in [7] that $\beta^0 \in L_1(\partial\Omega)$. For each $(u, \alpha_i) \in W^1_\mu(\bar\Omega)$ there exists $\{u_n\}_{n=1}^\infty$, $u_n \in W^1_1(\Omega)$ such that

$$\int_\Omega u_n\varphi\,\mathrm{d}x \to \int_\Omega u\varphi\,\mathrm{d}x, \quad \int_\Omega u_{n,x_i}\varphi\,\mathrm{d}x \to \int_\Omega \varphi\,\mathrm{d}\alpha_i$$
$$(i = 1, \dots, N)$$

for all $\varphi \in C(\bar\Omega)$, i.e., $W^1_\mu(\bar\Omega)$ is the completion of $W^1_1(\Omega)$ in this convergence (weak* convergence). The ball in $W^1_\mu(\bar\Omega)$ is compact with respect to this weak* convergence (contrary to the space $W^1_1(\Omega)$).

§3. $F((u, \alpha), \bar\Omega) = \bar f(\alpha, \lambda)(\bar\Omega)$

The main result of this paragraph is Theorem 9. Then we present some consequences of this Theorem.

Theorem 9. *For* $(u, \alpha) \in W^1_\mu(\bar\Omega)$

$$F((u, \alpha), \bar\Omega) = \bar f(\alpha, \lambda)(\bar\Omega).$$

Proof. We recall that in [8] it is proved that $F = J$ on the space $W^1_1(\Omega)$. The consequence of Theorem 6 implies that

$$\bar f(\alpha, \lambda)(\Omega) = J(u, \Omega) \quad \text{for} \quad (u, \alpha) \in W^1_1(\Omega).$$

From Remark 4 on the semicontinuty we deduce that for $(u_n, \alpha_n), (u, \alpha) \in W^1_\mu$ such that $(u_n, \alpha_n) \rightharpoonup (u, \alpha)$ in W^1_μ $\bar{f}(\alpha, \lambda)(\bar{\Omega}) \leqslant \varliminf_{n\to\infty} \bar{f}(\alpha_n, \lambda)(\bar{\Omega})$ holds, i.e. the functional $\bar{f}(\cdot, \lambda)(\bar{\Omega})$ is weakly lower semicontinuous in W^1_μ and hence $\bar{f}(\alpha, \lambda)(\bar{\Omega}) \leqslant F((u, \alpha), \bar{\Omega})$ for all $(u, \alpha) \in W^1_\mu(\bar{\Omega})$.

The Proof will be divided into three parts, in which we shall prove the reverse inequality

$$\bar{f}(\alpha, \lambda)(\bar{\Omega}) \geqslant F((u, \alpha), \bar{\Omega}), \tag{11}$$

1) for function from $W^1_1 + \mathring{W}^1_\mu = \{v + (u, \alpha);\ v \in W^1_1,\ (u, \alpha) \in \mathring{W}^1_\mu\}$,
2) for functions $(u, \alpha) \in W^1_\mu(\bar{\Omega})$ with a non-negative (a non-positive) side $\alpha_\nu \in L_\mu(\partial\Omega)$
3) for an arbitrary function from W^1_μ.

For the proof of 1) let us consider $(u, \alpha) \in W^1_1 + \mathring{W}^1_\mu$. The proof is similar to the proof of Theorem 13 in [7]. Firstly, we extend the function (u, α) from $\bar{\Omega}$ to the bounded domain $\bar{\Omega}^* \supset \bar{\Omega}$.

There exists $(u^*, \alpha^*) \in W^1_\mu(\Omega^*)$ satisfying (see [7])

$$u^* = u \quad \text{on} \quad \Omega, \quad \alpha^* = \alpha \quad \text{on} \quad \Omega, \quad \alpha^* = 2\alpha \quad \text{on} \quad \partial\Omega \tag{12}$$

and

$$u^*|_{\Omega^* - \bar{\Omega}} \in W^1_1(\Omega^* - \bar{\Omega}).$$

Let there be

$$\omega_h(x) = \begin{cases} \exp(|x|^2/(|x|^2 - h^2)) & \text{for} \quad |x| < h \\ 0 & \text{for} \quad |x| \geqslant h \end{cases} \quad \text{and} \quad K^h(x) = \frac{R}{h^N}\,\omega_h(x), \quad \text{where} \quad R = \int_{|x|<1} \omega_1(x)\,dx$$

We denote

$$u_h(x) = \int_{\Omega^*} K^h(x - y)\,u^*(y)\,dy, \quad x \in \Omega. \tag{13}$$

The following assertions are valid (see [7])

$$u_{hx_i}(x) = \int_{\Omega^*} K^h(x - y)\,d\alpha_i^*(y), \quad x \in \Omega, \tag{14}$$

$$u_h \rightharpoonup (u, \alpha) \quad \text{in} \quad W^1_\mu(\bar{\Omega}), \tag{15}$$

$$\int_\Omega K^h(x - y)\,dx \to \frac{1}{2} \quad \text{uniformly for} \quad y \in \partial\Omega. \tag{16}$$

From (14) and owing to Jensen's inequality (Theorem 4) we obtain

$$J(u_h, \Omega) = \int_\Omega f\left(\int_{\Omega^*} K^h(x-y)\,\mathrm{d}\alpha^*(y)\right)\mathrm{d}x =$$

$$= \int_\Omega \bar{f}\left(\int_{\Omega^*} K^h(x-y)\,\mathrm{d}\alpha^*(y), \int_{\Omega^*} K^h(x-y)\,\mathrm{d}y\right)\mathrm{d}x \leqslant$$

$$\leqslant \int_\Omega \int_{\Omega^*} K^h(x-y)\,\mathrm{d}\bar{f}(\alpha^*, \lambda)(y)\,\mathrm{d}x =$$

$$= \iint\limits_{\substack{x\in\Omega\\ y\in\Omega}} \cdots + \iint\limits_{\substack{y\in\Omega\\ y\in\partial\Omega}} \cdots + \iint\limits_{\substack{x\in\Omega\\ y\in S_h^*}} \cdots$$

where $S_h^* = \{x \in \Omega^* - \bar{\Omega}\,;\ \mathrm{dist}\,(x, \partial\Omega) < h\}$.

For the estimation of the first and second integral we use (12), (13) and (16)

$$\iint\limits_{\substack{x\in\Omega\\ y\in\Omega}} \cdots \leqslant \int_\Omega \mathrm{d}\bar{f}(\alpha, \lambda) = \bar{f}(\alpha, \lambda)(\Omega),$$

$$\iint\limits_{\substack{x\in\Omega\\ y\in\partial\Omega}} \cdots = \iint\limits_{\substack{x\in\Omega\\ y\in\partial\Omega}} K^h(x-y)\,\mathrm{d}\bar{f}(2\alpha, 0)(y)\,\mathrm{d}x \xrightarrow[h\to 0]{} \frac{1}{2}\int_{\partial\Omega} \mathrm{d}\bar{f}(2\alpha, 0) =$$

$$= \bar{f}(\alpha, 0)(\partial\Omega) = \bar{f}(\alpha, \lambda)(\partial\Omega),$$

since $\lambda(\partial\Omega) = 0$.

Since $\bigcap\limits_{h>0} S_h^* = \emptyset$ we conclude

$$\iint\limits_{\substack{x\in\Omega\\ y\in S_h^*}} K^h(x-y)\,\mathrm{d}\bar{f}(\alpha^*, \lambda)(y)\,\mathrm{d}x \leqslant \bar{f}(\alpha^*, \lambda)(S_h^*) \to 0$$

as $h \to 0$.

Thus we obtain $\bar{f}(\alpha, \lambda)(\bar{\Omega}) \geqslant \varlimsup\limits_{h\to 0} J(u_h, \Omega)$.

On the other hand, we conclude from (15) $\varliminf\limits_{h\to 0} J(u_h, \Omega) \geqslant F((u, \alpha), \bar{\Omega})$ and hence

$$\text{(17)} \qquad \lim_{h\to 0} J(u_h, \Omega) = \bar{f}(\alpha, \Omega) = \bar{f}(\alpha, \lambda)(\bar{\Omega}) = F((u, \alpha), \bar{\Omega}).$$

Now we prove 2). Let $(u, \alpha) \in W_\mu^1$ possess the side $\alpha_\nu \geqslant 0$ (see [7]). By the method of regularization such measures $\alpha_{\nu h} \in L_\mu(\partial\Omega)$, $h>0$ can be found that are absolutely continuous with respect to the Hausdorff measure dS on $\partial\Omega$ and satisfy

$$\alpha_{\nu h} \geqslant 0, \quad \alpha_{\nu h} \underset{h\to 0}{\longrightarrow} \alpha_\nu \quad \text{in} \quad L_\mu(\partial\Omega).$$

The existence of such measures follows from Lemma 1 in [7]. In addition to the above it is proved in [7] that the side α_ν satisfies

$$\alpha_i|_{\partial\Omega} = \nu_i\alpha_\nu, \quad i=1, \ldots, N, \tag{18}$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is the exterior normal to $\partial\Omega$. Thus, let us set

$$\alpha_{ih} = \alpha_i \text{ on } \Omega, \quad \alpha_{ih} = \nu_i\alpha_{\nu h} \text{ on } \partial\Omega, \quad i=1, \ldots, N. \tag{19}$$

In [7] (see proof of Theorem 14) it is proved that

$$(u, \alpha_h) \in W_1^1 + \mathring{W}_\mu^1, \quad (u, \alpha_h) \rightharpoonup (u, \alpha) \text{ in } W_\mu^1 \tag{20}$$

and that the side of (u, α_h) is $\alpha_{\nu h}$.

Now we shall use the first part of the proof for the functions $(u, \alpha_h) \in W_1^1 + \mathring{W}_\mu^1$, $h>0$. Our next aim is to prove

$$\bar{f}(\alpha, \lambda)(\bar{\Omega}) = \lim_{h\to 0} \bar{f}(\alpha_h, \lambda)(\bar{\Omega}), \tag{21}$$

$$F((u, \alpha), \bar{\Omega}) \leqslant \varliminf_{h\to 0} F((u, \alpha_h), \bar{\Omega}). \tag{22}$$

These inequalities imply the desired inequality (11).

Ising Theorem 6, (18), (19) and the fact that $\alpha_\nu \geqslant 0$, $\alpha_{\nu h} \geqslant 0$, $\lambda(\partial\Omega) = 0$ we obtain

$$\bar{f}(\alpha, \lambda)(\partial\Omega) = \int_{\partial\Omega} \bar{f}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\alpha_\nu}, 0\right) \mathrm{d}\alpha_\nu = \int_{\partial\Omega} \bar{f}(\nu, 0)\, \mathrm{d}\alpha_\nu,$$

$$\bar{f}(\alpha_h, \lambda)(\partial\Omega) = \int_{\partial\Omega} \bar{f}\left(\frac{\mathrm{d}\alpha_h}{\mathrm{d}\alpha_{\nu h}}, 0\right) \mathrm{d}\alpha_{\nu h} = \int_{\partial\Omega} \bar{f}(\nu, 0)\, \mathrm{d}\alpha_{\nu h}.$$

With regard to (20) and using $\bar{f}(\nu, 0) \in C(\partial\Omega)$ we deduce (21). The assertion (22) is proved in the more general form

$$\text{if } \hat{u} \in W_\mu^1, \quad \hat{u}_n \in W_1^1 + \mathring{W}_\mu^1, \quad \hat{u}_n \rightharpoonup \hat{u} \text{ in } W_\mu^1, \quad \text{then} \tag{23}$$

$$F(\hat{u}, \bar{\Omega}) \leqslant \varliminf_{n\to\infty} F(\hat{u}_n, \bar{\Omega}).$$

For the proof we use the same method as in the proof of Theorem 1 in [8]. Owing to (15) and (17), there exist $u_{nk} \in W_1^1$, $n, k = 1, 2, \ldots$, such that $u_{nk} \rightharpoonup \hat{u}_n$ in W_μ^1, $J(u_{nk}, \Omega) \to F(\hat{u}_n, \bar{\Omega})$ as $k\to\infty$. With respect to the Theorem 13 in [7], these sequences satisfy $\|u_{nk}\|_{W_1^1} \to \|\hat{u}_n\|_{W_\mu^1}$.

From $u_n \rightharpoonup \hat{u}$ in W^1_μ it follows $\sup_n \|\hat{u}_n\|_{W_\mu^1} < \infty$. Thus there exist $R > 0$ and a sequence of positive integers $\{k_n\}$ such that $\|u_{nk}\|_{W_1^1} \leq R$ for all n and $k \geq k_n$ and $\|\hat{u}_n\|_{W_\mu^1} \leq R$ for all n, $\|\hat{u}\|_{W_\mu^1} \leq R$.

With regard to Lemma 2 in [8], the weak topology in the ball $\{\hat{v} \in W^1_\mu; \|\hat{v}\|_{W_\mu^1} \leq R\}$ can be metrized by some metric ϱ. Then, for each index n, there exists an index $l(n)$ such that for $w_n = u_{n,l(n)}$ there is satisfied

$$\varrho(w_n - \hat{u}_n, 0) < \frac{1}{n}, J(w_n, \Omega) \leq F(\hat{u}_n, \bar{\Omega}) + \frac{1}{n}, \quad n = 1, \dots$$

Hence and from

$$\varrho(w_n - \hat{u}, 0) \leq \varrho(\hat{u}_n - \hat{u}, 0) + \varrho(w_n - \hat{u}_n, 0)$$

we conclude that $w_n \rightharpoonup \hat{u}$ as $n \to \infty$.

With respect to the definition of the functional F we obtain

$$F(\hat{u}, \bar{\Omega}) \leq \varliminf_{n\to\infty} J(w_n, \Omega) \leq \lim \left(F(\hat{u}_n, \bar{\Omega}) + \frac{1}{n}\right) = \varliminf_{n\to\infty} F(\hat{u}_n, \bar{\Omega})$$

and hence the relation (23) is proved.

Finally we prove the assertion 3) using the assertion 2). We assume that $(u, \alpha) \in W^1_\mu$ possesses the side $\alpha_\nu \in L_\mu(\partial\Omega)$. There exists a Hahn decomposition $\partial\Omega = \Gamma^+ \cup \Gamma^-$, $\Gamma^+ \cap \Gamma^- = \emptyset$, Γ^+, $\Gamma^- \in \mathscr{B}$ such that $\alpha_\nu^+ = \alpha_\nu$, $\alpha_\nu^- = 0$ on Γ^+, $\alpha_\nu^+ = 0$, $\alpha_\nu^- = -\alpha_\nu$ on Γ^- and $\alpha_\nu = \alpha_\nu^+ - \alpha_\nu^-$, α_ν^+, $\alpha_\nu^- \geq 0$.

Let us set $\alpha_i^1 = \alpha_i^2 = \alpha_i$ on Ω,

$$\alpha_i^1 = 2\nu_i\alpha_\nu^+, \quad \alpha_i^2 = -2\nu_i\alpha_\nu^- \quad \text{on} \quad \partial\Omega, \quad i = 1, \dots, N.$$

With respect to Theorem 14 in [7], the functions (u, α^1) and (u, α^2) belong to the space W^1_μ and moreover (u, α^1) possesses the side $2\alpha_\nu^+$ and (u, α^2) possesses the side $-2\alpha_\nu^-$. Evidently $(u, \alpha) = \frac{1}{2}(u, \alpha^1) + \frac{1}{2}(u, \alpha^2)$ is valid. The convexity of the functional J implies the convexity of the functional F and hence

$$F((u, \alpha), \bar{\Omega}) \leq \tfrac{1}{2} F((u, \alpha^1), \bar{\Omega}) + \tfrac{1}{2} F((u, \alpha^2), \bar{\Omega}). \tag{24}$$

Using Theorem 6 and the homogeneity of the function f, we obtain

$$\begin{aligned}\bar{f}(\alpha, \lambda)(\bar{\Omega}) &= \bar{f}(\alpha, \lambda)(\Omega) + \bar{f}(\alpha, \lambda)(\Gamma^+) + \bar{f}(\alpha, \lambda)(\Gamma^-) = \\ &= \bar{f}(\alpha, \lambda)(\Omega) + \bar{f}(\nu\alpha_\nu^+, 0)(\partial\Omega) + \bar{f}(-\nu\alpha_\nu^-, 0)(\partial\Omega) = \\ &= \bar{f}(\alpha, \lambda)(\Omega) + \tfrac{1}{2}\bar{f}(\alpha^1, 0)(\partial\Omega) + \tfrac{1}{2}\bar{f}(\alpha^2, 0)(\partial\Omega) = \\ &= \tfrac{1}{2} f(\alpha^1, \lambda)(\bar{\Omega}) + \tfrac{1}{2}\bar{f}(\alpha^2, \lambda)(\bar{\Omega}).\end{aligned}$$

From (24) and owing to the proved assertion 2, we deduce the required inequality (11).

Remark 9. From Theorem 9 it follows that

$$F((u,\alpha),\bar{\Omega})=\bar{f}(\alpha,\lambda)(\Omega)+\bar{f}(\alpha,0)(\partial\Omega), \tag{25}$$

where $(u,\alpha)\in W^1_\mu(\bar{\Omega})$.

The functional $\bar{f}(\alpha,\lambda)(\Omega)$ is closely related to the function $\bar{F}(u,\Omega)$, which is defined by Serrin in [5]:

$$\bar{F}(u,\Omega)=\inf\{\varliminf_{n\to\infty} J(u_n,\Omega_n);\quad u_n\in L_{1,\mathrm{loc}}(\Omega)\cap C^1(\Omega_n),$$
$$u_n\to u \quad \text{in} \quad L_{1,\mathrm{loc}}(\Omega),\ \Omega_n\nearrow\Omega\}.$$

Let us set $\bar{\alpha}=\alpha$ on Ω, $\bar{\alpha}=0$ on $\partial\Omega$.

Then with respect to [7], $(u,\bar{\alpha})\in W^1_\mu$ and evidently

$$\bar{f}(\alpha,\lambda)(\Omega)=\bar{f}(\bar{\alpha},\lambda)(\bar{\Omega})=F((u,\bar{\alpha}),\bar{\Omega}).$$

The side of the function $(u,\bar{\alpha})$ is equal to zero and for each such function it is proved in [8] that

$$F((u,\bar{\alpha}),\bar{\Omega})=\bar{f}(u,\Omega).$$

J. Serrin proved in [5] the relation

$$\bar{F}(u,\Omega)=\lim_{h\to 0} J(u_h,\Omega_h),$$

where

$$u_h(x)=\int_\Omega K^h(x-y)u(y)\,\mathrm{d}y,\ \Omega_h=\{x\in\Omega;\ \mathrm{dist}\,(x,\partial\Omega)>h\}.$$

From the preceding we conclude

$$\bar{f}(\alpha,\lambda)=\bar{F}(u,\Omega)=\lim_{h\to 0} J(u_h,\Omega_h).$$

Now let (u,α) possess the side $\alpha_\nu\in L_\mu(\partial\Omega)$. We use the Hahn decomposition $\alpha_\nu=\alpha_\nu^+-\alpha_\nu^-$, $\partial\Omega=\Gamma^+\cup\Gamma^-$ (see the proof 3) in Theorem 9). Let us set sign $\alpha_\nu=1$ on Γ^+ and sign $\alpha_\nu=-1$ on Γ^-.

Using Theorem 6 we can write

$$\bar{f}(\alpha,0)(\partial\Omega)=\int_{\partial\Omega}\bar{f}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}|\alpha_\nu|},0\right)\mathrm{d}|\alpha_\nu|=$$

$$=\int_{\Gamma^+}\bar{f}(\nu,0)\,\mathrm{d}|\alpha_\nu|+\int_{\Gamma^-}\bar{f}(-\nu,0)\,\mathrm{d}|\alpha_\nu|=\int_{\partial\Omega}\bar{f}(\nu\ \mathrm{sign}\ \alpha_\nu,0)\,\mathrm{d}|\alpha_\nu|,$$

for we have $\dfrac{\mathrm{d}\alpha|_{\partial\Omega}}{\mathrm{d}\alpha_\nu}=\nu$, which is a consequence of $\alpha_i|_{\partial\Omega}=\nu_i\alpha_\nu$ (see [7]).

Remark 10. Let us especially consider

$$f(a_1, \ldots, a_N) = \sqrt{1 + a_1^2 + \ldots + a_N^2}.$$

In this case $J(u, \Omega)$ denotes the functional of area,

$$\bar{f}(a, b) = \sqrt{a_1^2 + \ldots + a_N^2 + b^2}, \quad a \in E_N, \quad b \geqslant 0.$$

As a consequence of Remark 9 we obtain

$$\bar{f}(\alpha, 0)(\partial\Omega) = \int_{\partial\Omega} \sqrt{\sum_{i=1}^{N} (\nu_i \operatorname{sign} \alpha_\nu)^2}\, \mathrm{d}|\alpha_\nu| = \int_{\partial\Omega} \mathrm{d}|\alpha_\nu|.$$

To make the application of Theorem 9 clear we refer to the example in [8]. In that example we deduce

$$F((u, \bar{\alpha}), \bar{\Omega}) = \bar{F}(u, \Omega) + \int_{\partial\Omega} \mathrm{d}|\alpha_\nu| = 1 + \int_0^1 |g(x)|\, \mathrm{d}x_1.$$

Remark 11. From Theorems 9 and 3 we conclude that the functional F is lower weakly semicontinuous in the space W_μ^1.

In [8] this semicontinuity was proved under more general conditions but coerciveness of the functional $J(u, \Omega)$ was supposed. In our special case the semicontinuity was proved without assumption of coerciveness.

§ 4. $F = F_1$

The purpose of this paragraph is to prove the equality $F = F_1$. Then we present some important consequences of this result.

Theorem 10. *If*

$$(u, \alpha) \in W_1^1(\Omega) + \mathring{W}_\mu^1(\bar{\Omega}),$$

then

$$F((u, \alpha), \bar{\Omega}) = F_1((u, \alpha), \bar{\Omega}).$$

Evidently, the inequality $F_1 \geqslant F$ is valid (see the definitions in the introduction). It suffices to prove the reverse inequality. In the proof we use the regularized functions defined in § 3 by the formulas (12), (13). Owing to (15) and (17), the functions u_h satisfy

$$u_h \rightharpoonup (u, \alpha) \text{ in } W_\mu^1, \quad J(u_h, \Omega) \to F((u, \alpha), \bar{\Omega}) \text{ as } h \to 0.$$

The proof Theorem 10 is based on the following theorem.

Theorem 11. *Let $u_h' \in L_1(\partial\Omega)$ be the trace of the function $u_h \in W_1^1$ from (13) and let $u' \in L_1(\partial\Omega)$ be the trace of the function $(u, \alpha) \in W_1^1 + \mathring{W}_\mu^1$. Then $u_h' \to u'$ as $h \to 0$ in the norm of the space $L_1(\partial\Omega)$.*

Proof. Assertion (15) implies only $u_h' \rightharpoonup u'$ in $L_\mu(\partial\Omega)$ (see [7]). Let us denote $\bar{\alpha} = \alpha$ on Ω, $\bar{\alpha} = 0$ on $\partial\Omega$ and $\alpha' = \alpha - \bar{\alpha}$. In [7] it is proved that $(u, \bar{\alpha})$, $(0, \alpha') \in W_\mu^1$ and the trace of the function $(u, \bar{\alpha})$ belongs to the space $L_1(\partial\Omega)$. From the assumption $(u, \alpha) \in W_1^1 + \mathring{W}_\mu^1$ we deduce that the trace of the function $(0, \alpha')$ belongs to $L_1(\partial\Omega)$, too. Evidently $(u, \alpha) = (u, \bar{\alpha}) + (0, \alpha')$ is satisfied. Now we shall choose a function $\tilde{u} \in W_1^1(\Omega)$ possessing the same trace on $\partial\Omega$ as the function $(u, \bar{\alpha})$ (see [6]).

We can write the following decomposition

$$(u, \alpha) = \tilde{u} + (0, \alpha') + [(u, \bar{\alpha}) - \tilde{u}]$$

for all $(u, \alpha) \in W_1^1 + \mathring{W}_\mu^1$, hence it is clearly sufficient to prove Theorem 11 only for functions of the following three types:

1) $(u, \alpha) \in W_1^1(\Omega)$,
2) $(u, \alpha) \in W_1^1 + \mathring{W}_\mu^1$, $u = 0$ on Ω,
3) $(u, \alpha) \in W_\mu^1$ with the side and the trace equal to zero.

1) In this case the extension (u^*, α^*) of (u, α) can be constructed so that $(u^*, \alpha^*) \in W_1^1(\Omega^*)$ (see [1]). By (12) we define u_h. It is known that in this case $u_h \to (u, \alpha)$ in the norm of the space $W_1^1(\Omega)$ and hence (see [1]) their traces satisfy $u_h' \to u'$ in $L_1(\partial\Omega)$.
2) In this case the extension (u^*, α^*) satisfies

$$u^* = 0 \text{ on } \Omega,\ u^*|_{\Omega^* - \bar{\Omega}} \in W_1^1(\Omega^* - \bar{\Omega})$$

and the function $u^*|_{\Omega^* - \bar{\Omega}}$ possesses the trace $2u'$ on $\partial\Omega$ (where u' is the trace of the function $(0, \alpha)$). Let $\varepsilon > 0$ be fixed. Let us choose the function $\varphi \in C(\bar{\Omega}^* - \Omega)$ such that

$$\|u^*|_{\Omega^* - \bar{\Omega}} - \varphi\|_{W_1^1(\Omega^* - \bar{\Omega})} < \varepsilon. \tag{26}$$

In [7] it is proved (see the relation (57)) that

$$\int_{\Omega^* - \Omega} K^h(x - y)\varphi(y)\, dy \to \frac{1}{2}\varphi(x) \text{ as } h \to 0 \tag{27}$$

in the norm of the space $L_1(\partial\Omega)$.

From (26) we conclude that $\|\varphi|_{\partial\Omega} - 2u'\|_{L_1(\partial\Omega)} \leq C \cdot \varepsilon$. With regard to (26), (27) we obtain

$$\overline{\lim_{h \to 0}} \int_{\partial\Omega} |u_h'(x) - u'(x)|\, dS(x) =$$

$$= \overline{\lim_{h\to 0}} \int_{\partial\Omega} \left| \int_{\Omega^*} K^h(x-y)u^*(y)\,\mathrm{d}y - u'(x) \right| \mathrm{d}S(x) \leq$$

$$\leq \overline{\lim_{h\to 0}} \int_{\partial\Omega} \left| \int_{\Omega^*-\Omega} K^h(x-y)\varphi(y)\,\mathrm{d}y - u'(x) \right| \mathrm{d}S(x) +$$

$$+ \overline{\lim_{h\to 0}} \int_{\Omega} \int_{\Omega^*-\Omega} K^h(x-y)|\varphi(y)-u^*(y)|\,\mathrm{d}y\,\mathrm{d}S(x) \leq C\cdot\varepsilon.$$

The theorem on imbedding from $W_1^1(\Omega)\to L_1(\partial\Omega)$ has been used. For the proof of the case 3) we use the following inequalities

(28) $$\|u\|_{L_1(\partial\Omega)} \leq C\left(\frac{1}{h}\|u\|_{L_1(S_h)} + \|u\|_{\dot{W}_1{}^1(S_h)}\right) \quad \text{for } u\in W_1^1(\Omega)$$

and

(29) $$\|\hat{u}\|_{L_1(S_h)} \leq C\cdot h\cdot\|\hat{u}\|_{\dot{W}_\mu{}^1(S_h)} \quad \text{for } \hat{u}\in \mathring{W}_\mu^1(\Omega), \quad \text{where}$$

$$\|u\|_{\dot{W}_1{}^1} = \sum_{i=1}^{N} \|u_{x_i}\|_{L_1}, \quad S_h = \{x\in\bar{\Omega}\,;\ \mathrm{dist}\,(x,\partial\Omega)<h\}$$

and C is independent of u and (h being sufficiently small). For the completness we suggest the proof of these inequalities. The boundary $\partial\Omega\in C^1$ can be covered by the finite number of the cubes $K_1, \dots, K_R$. Let us consider the corresponding decomposition $\gamma_1, \dots, \gamma_R$ of the unit with respect to these cubes (see [1]). Now it is sufficient to prove (28), (29) for the function $u\cdot\gamma_r$ with the support in K_r, $r=1, \dots, R$. Then we carry out a linear transformation of coordinates, so that it remains to prove (28) and (29) for $u\in W_1^1(K\cap\bar{\Omega})$ with the support in $(K\cap\Omega)\cup(K\cap\partial\Omega)$. The set $\partial\Omega\cap K$ can be described by $x_N = a(x')\in C^1$, $x'=(x_1, \dots, x_{N-1})$. For a smooth u we obtain

$$u(x', a(x')) = u(x', a(x')-s) + \int_{a(x)-s}^{a(x')} \frac{\partial u(x', \xi_N)}{\partial x_N}\,\mathrm{d}\xi_N,$$

$h>s>0$ and hence

$$|u(x', a(x'))| \leq |u(x', a(x')-s)| + \int_{a(x')-h}^{a(x')} \left|\frac{\partial u}{\partial x_N}\right| \mathrm{d}\xi_N$$

from which we deduce

$$h\cdot\|u\|_{L_1(\partial\Omega\cap K)} \leq C(\|u\|_{L_1(S_h)} + h\cdot\|u\|_{\dot{W}_1{}^1(S_h)})$$

for $u\in W_1^1(\Omega\cap K)$, which implies (28).

If $u(x', a(x'))=0$ then

$$|u(x', a(x')-s)| \leqslant \int_{a(x')-h}^{a(x')} \left|\frac{\partial u}{\partial x_N}\right| d\xi_N \quad \text{for} \quad h>s>0$$

and hence

$$\|u\|_{L_1(S_h)} \leqslant c \cdot h \|u\|_{\mathring{W}_1{}^1(S_h)} \quad \text{for} \quad u \in \mathring{W}_1^1(\Omega \cap K).$$

Thus, (29) is proved for $u \in \mathring{W}_1^1(\Omega)$. Now we prove (29) for $\hat{u} \in \mathring{W}_\mu^1(\bar{\Omega})$. For this purpose we use Theorem 4 from [7]. With respect to this theorem for $u \in \mathring{W}_\mu^1(\bar{\Omega})$ there exists $u_n \in \mathring{W}_1^1(\Omega)$, $n=1, 2, \ldots$, such that $u_n \rightharpoonup (u, \alpha)$ in W_μ^1 and

$$\|u_{nx_i}\|_{L_1(\Omega)} \leqslant C \|\alpha_i\|_{L_\mu(\bar{\Omega})} \quad \text{for} \quad i=1, \ldots, N,$$

where the constant C is independent of n. Using semicontinuity of the norm with respect of the w^*-convergence, we obtain

$$\|u\|_{L_1(S_h)} \leqslant \underline{\lim} \|u_n\|_{L_1(S_h)} \leqslant C \cdot h \|u\|_{\mathring{W}_\mu{}^1(S_h)}$$

for $u \in \mathring{W}_\mu^1$. Now let us extend u to $\Omega^* \supset \bar{\Omega}$ by zero and let us consider u_h from (12), (13).

Evidently, for u_h we have

$$\|u_h\|_{L_1(S_h)} \leqslant \|u\|_{L_1(S_{2h})}, \ \|u_h\|_{\mathring{W}_1{}^1(S_h)} \leqslant \|u\|_{\mathring{W}_\mu{}^1(S_{2h})}.$$

From (28) and (29) we deduce

$$\|u_h\|_{L_1(\partial\Omega)} \leqslant C\left(\frac{1}{h}\|u_h\|_{L_1(S_h)} + \|u_h\|_{\mathring{W}_1{}^1(S_h)}\right) \leqslant$$

$$\leqslant C\left(\frac{1}{h}\|u\|_{L_1(S_{2h})} + \|u\|_{\mathring{W}_\mu{}^1(S_{2h})}\right) \leqslant$$

$$\leqslant C\left(\frac{2h}{h}\|u\|_{\mathring{W}_\mu{}^1(S_{2h})} + \|u\|_{\mathring{W}_\mu{}^1(S_{2h})}\right) \leqslant C\|u\|_{\mathring{W}_\mu{}^1(S_{2h})}.$$

With respect to the fact that $(u, \alpha) \in \mathring{W}_\mu^1(\bar{\Omega})$ with $\alpha = 0$ on $\partial\Omega$, we deduce $\alpha_i = 0$ on $\partial\Omega$, $i=1, \ldots, N$ (see [7]) and hence

$$\|u\|_{\mathring{W}_\mu{}^1(S_{2h})} \to 0 \quad \text{as} \quad h \to 0$$

for functions of the third type. Thus, Theorem 11 is proved.

P r o o f o f T h e o r e m 10. Let us consider the function $(u, \alpha) \in W_1^1 + \mathring{W}_\mu^1$ and $u_h \in W_1^1$, $h>0$ its regularization from (13). Let us denote by u', $u_h' \in L_1(\partial\Omega)$ the traces of these functions. With regard to (15), (17) and Theorem 11 the following relations are satisfied

$$u_h \rightharpoonup (u, \alpha) \text{ in } W_\mu^1(\bar{\Omega}), \quad u_h' \to u' \text{ in } L_1(\partial\Omega)$$
$$J(u_h, \Omega) \to F(u, \bar{\Omega}) \text{ as } h \to 0.$$

Let us denote $\Omega_h = \{x \in \Omega ; \text{dist}(x, \partial\Omega) > h\}$, $S_h = \Omega - \bar{\Omega}_h$. In [1] there is proved the existence of the functions $v_h \in W_1^1$ possessing the traces $v_h' = u' - u_h'$ on $\partial\Omega$ and satisfying

$$\|v_h\|_{W_1^1} \leq C\|u' - u_h'\|_{L_1(\partial\Omega)} \to 0 \quad \text{as} \quad h \to 0, \tag{30}$$

where the constant C is independent of h.

It can be easily seen that $u_h + v_h \rightharpoonup (u, \alpha)$ in $W_\mu^1(\bar{\Omega})$ and $u_h' + v_h' = u'$ on $\partial\Omega$. Owing to the assertion 5 of Theorem 1 we obtain

$$|f(a_1) - f(a_2)| \leq C|a_1 - a_2|, \quad a_1, a_2 \in E_N. \tag{31}$$

Thus, from (30), (31) and from the definition of F_1 we conclude

$$F_1((u, \alpha), \bar{\Omega}) \leq \varliminf_{h\to 0} J(u_h + v_h, \Omega) \leq$$

$$\leq \varliminf_{h\to 0} J(u_h, \Omega) + \varliminf_{h\to 0} \int_\Omega [f(\nabla u_h + \nabla v_h) - f(\nabla u_h)]\, dx \leq$$

$$\leq F((u, \alpha), \bar{\Omega}) + \varlimsup_{h\to 0} C \int_\Omega |\nabla v_h|\, dx \leq F((u, \alpha), \bar{\Omega}),$$

and the proof is complete.

Remark 12. Let us assume $u_0 \in W_1^1$.

1) The functional F_1 evidently satisfies

$$\inf_{\hat{u} \in u_0 + \mathring{W}_\mu^1} F_1(\hat{u}, \bar{\Omega}) = \inf_{u \in u_0 + \mathring{W}_1^1} J(u, \Omega).$$

Theorem 10 implies that this equality is valid if we substitute F instead F_1.

2) If $u \in u_0 + \mathring{W}_1^1$ is the solution of the boundary value problem

$$J(u, \Omega) = \inf_{v \in u_0 + \mathring{W}_1^1} J(v, \Omega),$$

then u is also the solution of the boundary value problem

$$J(u, \Omega) = \inf_{v \in u_0 + \mathring{W}_\mu^1} F(v, \bar{\Omega}).$$

3) The functional F_1 is weakly lower semicontinuous on the space $W_1^1 + \mathring{W}_\mu^1$ (see the Remark 11). In [8] the semicontinuity of F_1 has been proved only on $u_0 + \mathring{W}_1^1$.

Int the next theorem a classical inequality from [9] will be generalized and strengthened.

Theorem 12. *Suppose that the functions $\hat{u}_1 = (u_1, \alpha_1)$, $\hat{u}_2 = (u_2, \alpha_2) \in W_\mu^1$ possess the traces $\beta_1, \beta_2 \in L_\mu(\partial\Omega)$. If $\hat{u}_1$ is a solution of the boundary value problem*

$$F(\hat{u}_1, \bar{\Omega}) = \inf_{\hat{v} \in \hat{u} + \mathring{W}_\mu^1} F(\hat{v}, \bar{\Omega}), \quad \textit{then}$$

(32) $$F(u_1, \bar{\Omega}) \leqslant F(\hat{u}_2, \bar{\Omega}) + \int_{\partial\Omega} \bar{f}(\nu \text{ sign } (\beta_1 - \beta_2), 0) \, \mathrm{d}|\beta_1 - \beta_2|$$

is valid (see Remark 9).

If $\hat{u}_2$ is also a solution of the corresponding boudary value problem, then

(33) $$|F(\hat{u}_1, \bar{\Omega}) - F(u_2, \bar{\Omega})| \leqslant \max \left(\int_{\partial\Omega} \bar{f}(\nu \text{ sign } (\beta_1 - \beta_2), 0) \, \mathrm{d}|\beta_1 - \beta_2|, \right.$$
$$\left. \int_{\partial\Omega} \bar{f}(\nu \text{ sign } (\beta_2 - \beta_1, 0) \, \mathrm{d}|\beta_1 - \beta_2| \right) \leqslant C \int_{\partial\Omega} \mathrm{d}|\beta_1 - \beta_2|.$$

If, particularly $f(a) = \sqrt{1 + |a|^2}$, then

(34) $$|F(\hat{u}_1, \bar{\Omega}) - F(\hat{u}_2, \bar{\Omega})| \leqslant \int_{\partial\Omega} \mathrm{d}|\beta_1 - \beta_2|.$$

Remark 13. Let us assume that $u_1, u_2 \in W_1^1$ solve the boundary value problem in the sense of Remark 12. If $f(a) = \sqrt{1 + |a|^2}$, then Remark 12 and the relation (34) imply

(35) $$|J(u_1, \Omega) - J(u_2, \Omega)| \leqslant \int_{\partial\Omega} |u_1' - u_2'| \, \mathrm{d}S,$$

where $u_1', u_2' \in L_1(\partial\Omega)$ are the traces of the functions u_1, u_2. If $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ solves the equation for the minimal surfaces, then we find out easily (owing to the mentioned inequality from [9]) that $u \in W_1^1(\Omega)$ and that u solves the variational boundary value problem in W_1^1. Then the estimate from [9] is a consequence of (35) if $u_2 = \text{const}$.

Proof. Let us set $\tilde{\alpha}_i = \nu_i(\beta_1 - \beta_2)$ on $\partial\Omega$, $\tilde{\alpha}_i = 0$ on Ω (see [7]). Then the function $(0, \tilde{\alpha}) \in W_\mu^1$ possesses the trace $\beta_1 - \beta_2$ (see [7]) and hence $(u^2, \alpha^2 + \tilde{\alpha}) \in W_\mu^1$ possesses the trace β_1. Owing to Theorem 9 we obtain

$$F(\hat{u}_1, \bar{\Omega}) \leqslant F((u^2, \alpha^2 + \tilde{\alpha}), \bar{\Omega}) =$$
$$= \bar{f}(\alpha^2, \lambda)(\Omega) + \bar{f}(\alpha^2 + \tilde{\alpha}, 0)(\partial\Omega).$$

With regard to the assertion 2 and 4 from Theorem 2 we conclude

$$\bar{f}(\alpha^2 + \tilde{\alpha}, 0)(\partial\Omega)(\partial\Omega) = 2\bar{f}(\tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\tilde{\alpha}, 0)(\partial\Omega) \leqslant$$
$$\leqslant \bar{f}(\alpha^2, 0)(\partial\Omega) + \bar{f}(\tilde{\alpha}, 0)(\partial\Omega).$$

Using Remark 9, we deduce

$$F(\hat{u}_1, \bar{\Omega}) \leqslant \bar{f}(\alpha^2, \lambda)(\Omega) + \bar{f}(\alpha^2, 0)(\partial\Omega) + \bar{f}(\tilde{\alpha}, 0)(\partial\Omega) =$$
$$= F(\hat{u}_2, \bar{\Omega}) + \int_{\partial\Omega} \bar{f}(\nu \text{ sign } (\beta_1 - \beta_2), 0) \, \mathrm{d}|\beta_1 - \beta_2|,$$

since the function $(0, \tilde{\alpha})$ possesses the side $\beta_1 - \beta_2$ (see [7]). The inequality (33) can be obtained from (32) exchanging $\hat{u}_1$ and $\hat{u}_2$. Owing to the Remark 10, the inequality (34) is a consequence of (33).

By reason of Theorem 10 we deduce a remarkable theorem for the function from W^1_μ, which strengthens essentially

Theorem 4) ii) and Theorem 13 from [7].

Theorem 13. *If* (u, α) $W^1_1 + \mathring{W}^1_\mu$ *then there exist functions* $u_h \in W^1_1$, $h>0$ *such that* $u_h - (u, \alpha) \in \mathring{W}^1_\mu$, $u_h \rightharpoonup (u, \alpha)$ *in* W^1_μ

$$\|u_h\|_{L_1(\Omega)} \to \|u\|_{L_1(\Omega)} \text{ and } \|u_{hx_i}\|_{L_1(\Omega)} \to \|\alpha_i\|_{L_\mu(\bar{\Omega})} \text{ as } h \to 0,$$

where $i = 1, 2, \ldots, N$.

Proof. Let us set $f(a_1, \ldots, a_N) = |a_1| + \ldots + |a_N|$. Evidently, $\bar{f}(a, b) = f(a)$, where $a \in E_N$, $b \geq 0$. With respect to Definition 1 and Theorem 9 we conclude

$$F((u, \alpha), \bar{\Omega}) = \bar{f}(\alpha, \lambda)(\bar{\Omega}) = \sup_{\{E_i\} \in \mathscr{R}(\bar{\Omega})} \sum_{i=1}^{\infty} f(\alpha(E_i)) = |\alpha|(\bar{\Omega}).$$

With regard to Theorem 10, there exist functions

$$u_h \in W^1_1, \quad u_h \in (u, \alpha) + \mathring{W}^1_\mu, \quad h>0 \text{ such that}$$
$$u_h \rightharpoonup (u, \alpha) \text{ in } W^1_\mu, \sum_{i=1}^{N} \|u_{hx_i}\|_{L_1(\Omega)} \to \sum_{i=1}^{N} \|\alpha_i\|_{L_\mu(\bar{\Omega})}$$

as $h \to 0$, $u_h \rightharpoonup (u, \alpha)$ implies that $\|\alpha_i\|_{L_\mu(\bar{\Omega})} \leq \varliminf_{h \to 0} \|u_{hx_i}\|_{L_1(\Omega)}$, $i = 1, \ldots, N$. Thus, we deduce $\|u_{hx_i}\|_{L_1(\Omega)} \to \|\alpha_i\|_{L_\mu(\bar{\Omega})}$ as $h \to 0$ for $i = 1, \ldots, N$. Owing to the theorems on imbedding (see [7]), we conclude from $u_h \rightharpoonup (u, \alpha)$ that $u_h \rightharpoonup u$ in $L_1(\Omega)$, i.e. $\|u_h\|_{L_1(\Omega)} \to \|u\|_{L_1(\Omega)}$.

5. Unicity

J. Serrin proved in [5] (part I.4 and I.5) a unicity result and some further results for the functional $\bar{F}(u, \Omega)$ (see Remark 9). In this paragraph we present an analogous result for the functional $F((u, \alpha), \bar{\Omega})$ under somewhat more general assumptions than those in [5]. Methods of proofs are similar to those in [5], but using our result of the preceding paragraphs the proofs are simplified. Part of the results in this section can be proved with the help of Serrin's results in [5]. For this purpose a function $(u, \alpha) \in W^1_\mu(\bar{\Omega})$ must be extended by a function from $W^1_1(\Omega^* - \bar{\Omega})$ to a larger domain Ω^* and then we can use the equality $f = \bar{F}$ on Ω^* (see Remark 9). This equality was proved in [8] for the function $u \in W^1_\mu(\bar{\Omega})$ possessing the side $\alpha_\nu = 0$ on $\partial\Omega$.

Let us denote by α^r, α^s the regular and singular parts of the measure $\alpha \in L^N_\mu(\bar{\Omega})$ with respect to the Lebesque measure λ. From Remark 6 we obtain

$$(36)\qquad F((u,\alpha),\bar{\Omega}) = \bar{f}(\alpha^r,\lambda)(\Omega) + \bar{f}(\alpha^s,0)(\bar{\Omega}) = \\ = \int_\Omega f\left(\frac{d\alpha^r}{d\lambda}\right) d\lambda + \int_\Omega \bar{f}\left(\frac{d\alpha^s}{d|\alpha^s|}, 0\right) d|\alpha^s|.$$

Thus, from (36) we conclude that

$$(37)\qquad \frac{d\bar{f}(\alpha,\lambda)^r}{d\lambda} = f\left(\frac{d\alpha^r}{d\lambda}\right), \left(\frac{d\bar{f}(\alpha,\lambda)^s}{d|\alpha^s|}\right) = \bar{f}\left(\frac{d\alpha^s}{d|\alpha^s|}, 0\right).$$

The function f is supposed to be continuous, non-negative, convex and satisfying $f(a) \leq C(1+|a|)$.

Analogously as in [5] let us set

$$(38)\qquad J(u,\Omega) = J((u,\alpha),\Omega) = \int_\Omega f\left(\frac{d\alpha^r}{d\lambda}\right) d\lambda$$

for $(u,\alpha) \in W^1_\mu(\bar{\Omega})$ (the measure α^r is uniquely determined by the function u).

Theorem 14.

1) *The functional F is convex on $W^1_\mu(\bar{\Omega})$.*
2) *$J(u,\Omega) \leq F((u,\alpha),\bar{\Omega})$ for all $(u,\alpha) \in W^1_\mu(\bar{\Omega})$.*
3) *Let the function f satisfy*

$$(39)\qquad f(a) \geq C_1|a| - C_2, \quad \text{where} \quad a \in E_N,\ C_1 > 0.$$

Suppose $(u,\alpha) \in W^1_\mu(\bar{\Omega})$. Then $J(u,\Omega) = F((u,\alpha),\bar{\Omega})$ if and only if $(u,\alpha) \in W^1_1$ (i.e. $\alpha = \alpha^r$).

4) *Let us assume that f is strictly convex. Suppose $\hat{u}_1 = (u_1,\alpha_1)$, $\hat{u}_2 = (u_2,\alpha_2)$. If for some $t \in (0,1)$ there is satisfied*

$$(40)\qquad F(t\hat{u}_1 + (1-t)\hat{u}_2, \bar{\Omega}) = tF(\hat{u}_1,\bar{\Omega}) + (1-t)F(\hat{u}_2,\bar{\Omega}),$$

then $\alpha_1^r = \alpha_2^r$.

Proof. Assertion 1) is a consequence of the definition of F and of the convexity of the functional J.

2) From (36) and from (38) we conclude

$$F((u,\alpha),\bar{\Omega}) = \bar{f}(\alpha,\lambda)(\bar{\Omega}) \geq \bar{f}(\alpha^r,\lambda)(\Omega) = J(u,\Omega).$$

3) By reason of (39) we obtain $\bar{f}(a,0) \geq C_1|a|$.

Owing to (36) we deduce

$$F((u,\alpha),\bar{\Omega}) = J(u,\Omega) + \int_\Omega \bar{f}\left(\frac{d\alpha^s}{d|\alpha^s|}, 0\right) d|\alpha^s|.$$

If $\alpha^s \neq 0$, then the integral in the equality is evidently positive.

4) Let us denote $u_t=(u_t,\alpha_t)=t\hat{u}_1+(1-t)\hat{u}_2$ for $t\in(0,1)$.

Using Theorem 1, we obtain

$$\bar{f}(\alpha_t^r,\lambda)(\Omega)\leq t\bar{f}(\alpha_1^r,\lambda)(\Omega)+(1-t)\bar{f}(\alpha_2^r,\lambda)(\Omega), \tag{41}$$

$$\bar{f}(\alpha_t^s,0)(\bar{\Omega})\leq t\bar{f}(\alpha_1^s,0)(\bar{\Omega})+(1-t)\bar{f}(\alpha_2^s,0)(\bar{\Omega}). \tag{42}$$

Adding (41) and (42) we obtain (40) and hence in (41) and (42) the equalities are valid. Then, from (41), we deduce

$$\int_\Omega f\left(\frac{\mathrm{d}\alpha_t^r}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda=t\int_\Omega f\left(\frac{\mathrm{d}\alpha_1^r}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda+(1-t)\int_\Omega f\left(\frac{\mathrm{d}\alpha_2^r}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda.$$

Thus, the strict convexity of the function f implies

$$\frac{\mathrm{d}\alpha_1^r}{\mathrm{d}\lambda}=\frac{\mathrm{d}\alpha_2^r}{\mathrm{d}\lambda}\quad\text{a.e. in }\Omega.$$

Theorem 15. *Let us assume that f is strictly convex and satisfies (39).*

1) *If $\hat{u}_1=(u_1,\alpha_1)$ and $\hat{u}_2=(u_2,\alpha_2)$ are two solutions of the same variational problem in W_μ^1, i.e.,*

$$F(\hat{u}_1,\bar{\Omega})=F(\hat{u}_2,\bar{\Omega})=\inf_{\hat{u}\in\hat{u}_1+\dot{W}_\mu^1}F(\hat{u},\bar{\Omega}), \tag{43}$$

then $\alpha_1^r=\alpha_2^r$.

2) *If $u_1\in W_1^1$ is the solution of the variational problem*

$$J(u_1,\Omega)=\inf_{u\in u_1+\dot{W}_1^1}J(u,\Omega),$$

then for all $\hat{u}_2\in u_1+\dot{W}_\mu^1$, $\hat{u}_2\neq u_1$ $F(\hat{u}_2,\bar{\Omega})>J(u_1,\Omega)$ is valid.

Proof. 1) With regard to the convexity of the functional F and from (43) we conclude

$$F(t\hat{u}_1+(1-t)\hat{u}_2)=tF(\hat{u}_1)+(1-t)F(\hat{u}_2)\quad\text{for all}\quad t\in(0,1).$$

Thus, it is sufficient to use the assertion 4) from the preceding theorem.

2) With respect to Remark 12, u_1 is also a solution of the boundary value problem in W_μ^1. If $F(\hat{u}_2,\bar{\Omega})=J(u_1,\Omega)$ were satisfied, then owing to the proved assertion 1) we would deduce $\alpha_1^r=\alpha_2^r$ and hence $J(\hat{u}_2,\Omega)=J(u_1,\Omega)=F(u_2,\bar{\Omega})$. By reason of the assertion 3) from Theorem 14 we conclude $\hat{u}_2\in W_1^1$ and thus

$$u_{1x_i}=u_{2x_i}\quad\text{a.e. in }\Omega,\text{ for}\quad i=1,2,\ldots,N.$$

u_1, $\hat{u}_2$ possess the same trace and hence $u_1=u_2$.

Remark 14. Only partial unicity has been proved. This is due to the fact that the function $\bar{f}$ is never strictly convex, because of the equality

$$\bar{f}(ka,kb)=k\bar{f}(a,b),\quad k\geq 0.$$

With regard to Remark 6, the functional F satisfies

$$F((u, \alpha), \bar{\Omega}) = \int_{\Omega} f\left(\frac{\mathrm{d}\alpha^r}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda + \int_{\Omega} \bar{f}\left(\frac{\mathrm{d}\alpha^s}{\mathrm{d}|\alpha^s|}, 0\right)\mathrm{d}|\alpha^s|.$$

If $\alpha^s \neq 0$, then non-strictly convexity can be presented in the second integral. Now we present an example, where the functional F is not strictly convex on the set $u_0 + \mathring{W}^1_\mu$.

Example. Let us consider $f(a) = \sqrt{1 + |a|^2}$, $\Omega = \{x \in E_2, |x| < 1\}$. Let us define $\beta \in L_\mu(\partial\Omega)$ by the prescription

$$\beta = 0 \quad \text{on} \quad \{(x_1, x_2) \in \partial\Omega;\ x_1 \leq 0\},$$
$$\beta = dS \quad \text{on} \quad \{(x_1, x_2) \in \partial\Omega;\ x_1 > 0\},$$

where dS is a one-dimensional Lebesque measure on $\partial\Omega$. There exist functions (u_1, α_1), $(u_2, \alpha_2) \in W^1_\mu(\bar{\Omega})$ with the trace β and satisfying $u_1 = 0$, $u_2 = 1$ on Ω (see [7]). These functions are uniquely determined.

Their inner traces satisfy (see [7]) $\beta^0_1 = 0$, $\beta^0_2 = dS$. The sides of these functions satisfy (see [7]) $\alpha_{1\nu} = \beta - \beta^0_1$, $\alpha_{2\nu} = \beta - \beta^0_2$. Remark 10 implies

$$F((u_1, \alpha_1), \bar{\Omega}) = \int_{\Omega} \mathrm{d}\lambda + \int_{\partial\Omega} \mathrm{d}|\alpha_{1\nu}| = 2\pi$$

and

$$F((u_2, \alpha_2), \bar{\Omega}) = 2\pi.$$

Let us set

$$(u_t, \alpha_t) = t(u_1, \alpha_1) + (1-t)(u_2, \alpha_2)$$

for $0 < t < 1$.

This function satisfies

$$u_t = 1 - t \quad \text{on } \Omega, \quad \alpha_{t\nu} = t\alpha_{1\nu} + (1-t)\alpha_{2\nu}.$$

From this we obtain

$$F((u_2, \alpha_2), \bar{\Omega}) = \int_{\Omega} \mathrm{d}\lambda + \int_{\partial\Omega} \mathrm{d}|\alpha_{t\nu}| = 2\pi.$$

Thus, the functional F is not strictly convex on the set $u_0 + \mathring{W}^1_\mu$, where $u_0 \in W^1_1$ is the function with the trace

$$\frac{\mathrm{d}\beta}{\mathrm{d}S} \in L_1(\partial\Omega).$$

6. The principle of the maximum

The classical principle of the maximum asserts that if we have $u_1 \leq u_2$ on $\partial\Omega$ two solutions u_1, u_2 of the equation for the minimal surface, then $u_1 \leq u_2$ on $\bar{\Omega}$.

We prove this principle of the maximum in a somewhat weakened form for the solution of the boundary value problem for the functional F, on the space $W^1_\mu(\bar{\Omega})$. For this purpose we use the results from § 4 and § 5.

Definition 4. Let us consider (u_1, α_1), $(u_2, \alpha_2) \in W^1_\mu$ with the traces β_1, $\beta_2 \in L_\mu(\partial\Omega)$. We say that $(u_1, \alpha_1) \leq (u_2, \alpha_2)$ iff $u_1 \leq u_2$ in $L_1(\Omega)$ and $\beta_1 \leq_\beta \beta_2$ in $L_\mu(\partial\Omega)$.

Theorem 16. *Let $\hat{u}_1$, resp. $\hat{u}_2 \in W^1_\mu$, be the two solutions of the boundary value problem in W^1_μ with the boundary concition u'_1, resp. $u'_2 \in L_1(\partial\Omega)$. Let us assume that $u'_1 \leq u'_2$ a.e. in $\partial\Omega$. Then there exists a solution $\hat{v} \in W^1_\mu$ of the boundary value problem with the boundary contition u'_2 and satisfying $\hat{u}_1 \leq \hat{v}$.*

The same assertion for the revers inequality is valid.

Proof. The equality $F = F_1$ implies the existence of the functions u^1_n, $u^2_n \in W^1_1$ such that $\hat{u}^1_n \rightharpoonup \hat{u}_1$, $u^2_n \rightharpoonup u_2$ in W^1_μ and

$$J(u^1_n, \Omega) \leq F(\hat{u}_1, \bar{\Omega}) + \frac{1}{n}, \quad u^1_n|_{\partial\Omega} = u'_1,$$

$$J(u^2_n, \Omega) \leq F(\hat{u}_2, \bar{\Omega}) + \frac{1}{n}, \quad u^2_n|_{\partial\Omega} = u'_2,$$

(where $u^i_n|_{\partial\Omega}$ is the trace of u^i_n on $\partial\Omega$, for $i = 1, 2$).

Let us set $v_n = \max(u^1_n, u^2_n)$, $w_n = \min(u^1_n, u^2_n)$. Evidently $v_n|_{\partial\Omega} = u'_2$ and $w_n|_{\partial\Omega} = u'_1$.

Now let n be fixed. There exists a decomposition $\Omega = E_1 \cup E_2$, where E_1, E_2 are measurable and $u^1_n \geq u^2_n$ on E_1, $u^1_n < u^2_n$ on E_2.

From the assumptions we deduce

$$J(w_n, \Omega) = \int_{E_1} f(\nabla u^2_n)\, dx + \int_{E_2} f(\nabla u^1_n)\, dx \geq J(u^1_n, \Omega) - \frac{1}{n},$$

i.e.

$$\int_{E_1} f(\nabla u^2_n)\, dx \geq \int_{E_1} f(\nabla u^1_n)\, dx - \frac{1}{n}.$$

Thus, we conclude

$$J(v_n, \Omega) = \int_{E_1} f(\nabla u^1_n)\, dx + \int_{E_2} f(\nabla u^2_n)\, dx \leq \int_{E_1} f(\nabla u^2_n)\, dx +$$

$$+ \int_{E_2} f(\nabla u^2_n)\, dx + \frac{1}{n} \leq J(u^2_n, \Omega) + \frac{1}{n} \leq F(\hat{u}_2, \bar{\Omega}) + \frac{2}{n}.$$

Owing to this inequality, $\{v_n\}$ is a minimizing sequence for the boundary value problem with the boundary condition u_2. The norms $\|v_n\|_{W_1{}^1(\Omega)}$ are bounded, because $\|v_n\|_{W_1{}^1} \leq \|u^1_n\|_{W_1{}^1} + \|u^2_n\|_{W_1{}^1}$. The ball in the space W^1_μ is weakly compact (see [7]). Thus, there exists a subsequence $\{v_{n_k}\}$ and $v \in W^1_\mu$ such that $v_{nk} \rightharpoonup v$. Thus, $v_{n_k}|_{\partial\Omega}$ are weakly convergent in $L_\mu(\partial\Omega)$ to the trace of the function $v \in W^1_\mu$, i.e., v possesses the trace u'_2. The function $\hat{v}$ solves the variational problem with the boundary condition u'_2, since

$$F(\hat{v}) \leq \lim_{k\to\infty} J(v_{n_k}) \leq F(\hat{u}_2).$$

From $u^1_{n_k} \rightharpoonup \hat{u}_1$ and from $v_{n_k} \rightharpoonup \hat{v}$ as $k \to \infty$ we conclude (see [7]) that $u^1_{n_k} \to u_1$ and $v_{n_k} \to v$ in $L_1(\Omega)$ and hence $u_1 \leq v$ a.e. in Ω, because $u^1_{n_k} \leq v_{n_k}$ a.e. in Ω. Thus we conclude that $\hat{u}_1 \leq \hat{v}$. For the proof of the reverse inequality we use w_n instead of v_n.

If one of the solution of the variational problem belongs to the space W^1_1, then Theorem 16 can be strengthened.

Theorem 17. *Let us suppose that f is strictly convex and satisfies (39). Let $u_1 \in W^1_1$, resp. $\hat{u}_2 \in W^1_\mu$, be the two solutions of the variational problem in W^1_μ, with the boundary condition u'_1, resp. u'_2, where u'_1, $u'_2 \in L_1(\partial\Omega)$.*

If $u'_1 \leq u'_2$ a.e. in $\partial\Omega$, then $u_1 \leq \hat{u}_2$.

Proof. From the preceding Theorem we deduce that there exists $\hat{v} \in W^1_\mu$ solving the variational problem with the boundary condition u'_1 and satisfying $\hat{v} \leq \hat{u}_2$. With regard to Theorem 15, 2) on unicity we conclude that $u_1 = \hat{v}$.

Remark 15. 1) In Theorem 17 it is sufficient to assume that u_1 is the solution of the variational problem in W^1_1, because of the Remark 12, § 4, it is also the solution of the same problem in W^1_μ.

2) Let us set $u_1 = K$ (constant). Evidently, u_1 is the weak solution of the corresponding Euler equation and hence the minimum of the functional F on the set $u_1 + \mathring{W}^1_1$.

With respect to Remark 15 and Theorem 15 it is also the minimum on the set $u_1 + \mathring{W}^1_\mu$. Thus, if $u'_2 \leq K$ a.e. on $\partial\Omega$, then $\hat{u}_2 \leq K$ in W^1_μ, where $\hat{u}_2$ is the solution of the variational problem with the boundary condition u'_2.

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ФУНКЦИИ МЕР И ВАРИАЦИОННЫЕ ЗАДАЧИ ТИПА МИНИМАЛЬНЫХ ПОВЕРХНОСТЕЙ

Йозеф Качур, Ержи Соучек

Резюме

В настоящей работе авторы продолжают предыдущую работу касающуюся прямых вариационных методов в нерефлексивных пространствах. В этой работе построена и рассмотрена функция мер при помощи которой возможно подходящим образом анализировать решение вариационных задач типа минимальных поверхностей.