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POPRODUCT DECOMPOSITION OF A LATTICE

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In [1] it was proved that in the classes of lattices satisfying the condition (J) any two free decompositions of a given lattice have a common refinement. In the present paper we generalize this result to the case of the poproduct of lattices. The poproduct of lattices was defined in [2].

Let K be an equational class of lattices. The following condition (J) was stated in [1].

(J) If L is a free K-product of the lattices $(L_i,\ i\in I)$, A_i is a sublattice of L_i for $i\in I$ and A is the sublattice of L generated by $\cup(A_i,\ i\in I)$, then A is a free K-product of $(A_i,\ i\in I)$.

Lemma. *An equational class of lattices satisfies the condition* (J) *if and only if it satisfies the following condition* (J')

(J') *If L is a K-poproduct of lattices $(L_p,\ p\in P)$, A_p is a sublattice of L_p for $p\in P$ and A is the sublattice of L generated by $\cup(A_p, p\in P)$, then A is a K-poproduct of $(A_p,\ p\in P)$.*

Proof. Clearly (J') implies (J). We shall show that (J) implies (J'). Let P be a partially ordered set and for each $p\in P$ let L_p be a lattice. Denote by $L=P_K(L_p;\ p\in P)$ a K-poproduct of lattices $(L_p,\ p\in P)$ and by F their free K-product. Then there exists a congruence relation Θ such that $L=F/\Theta$. Denote by $[\cup(A_p;\ p\in P)]_L$ the sublattice of L generated by the set $\cup(A_p;\ p\in P)$ and by $[\cup(A_p;\ p\in P)]_F$ the sublattice of F generated by the set $\cup(A_p;\ p\in P)$. Then there holds

$$A=[\cup(A_p;\ p\in P)]_L=[\cup(A_p;\ p\in P)]_F/\Theta=F/\Theta=P_K(A_p;\ p\in P).$$

The lemma is proved.

For an element a from the poproduct, the covers $a_{(p)}$, $a^{(p)}$ were defined in [2]. Instead of $a_{(p)}$, $a^{(p)}$ we shall write a_{L_p}, a^{L_p}. In [2] also ideals $T_p(a)$, $T^p(a)\subseteqq L_p$ were defined. Instead of $T_p(a)$, $T^p(a)$ we shall write $T_{L_p}(a)$, $T^{L_p}(a)$.

We shall introduce some other notions. Let R, S be partially ordered sets. Let $(A_r,\ r\in R)$, $(B_s,\ s\in S)$ be systems of pairwise disjoint lattices. Let $L=P_K(A_r;\ r\in R)=P_K(B_s;\ s\in S)$. Let the set $R\times S$ be partially ordered as follows:

$$\langle r_1, s_1\rangle\leqq\langle r_2, s_2\rangle \quad \text{if and only if} \quad r_1\leqq r_2 \quad \text{and} \quad s_1\leqq s_2.$$

If p is a lattice polynomial symbol, then we denote by $\bar{p}$ a polynomial symbol arising from p in such a way that the symbols $\wedge$, $\vee$ will be replaced by $\triangle$, ∇, respectively ($\triangle$, ∇ are the operations in the lattice of ideals (see [2])).

If M, N are two subsets of the K-poproduct L, then $M < N$ denotes that for the ideals $(M\rangle$, $(N\rangle$ there holds $(M\rangle \subseteqq (N\rangle$. Especially, $M \leqq N$ denotes that $m \leqq n$ for each pair $m \in M$, $n \in N$.

If $(L_r, r \in R)$ is a system of pairwise disjoint lattices such that some of them can also be empty, then under $P_K(L_r\,;\, r \in R)$ we shall understand $P_K(L_r\,;\, r \in R')$, where $R' \subseteqq R$ is the maximal subset of R such that for $r \in R'$, $L_r \neq \emptyset$.

Theorem 1. *Let K be a nontrivial equational class of lattices satisfying the condition (J'). Let $L \in K$. Any two representations of L as a K-poproduct have a common refinement.*

Theorem 1 will be proved in the following form:

Theorem 1'. *Let K be a nontrivial equational class of lattices satisfying the condition (J'). Let $L \in K$. Let*

$$L = P_K(A_R\,;\, r \in R) = P_K(B_s\,;\, s \in S).$$

Then

$$L = P_K(A_r \cap B_s\,;\, \langle r, s\rangle \in R \times S).$$

Moreover, for $r \in R$,

$$A_r = P_K(A_r \cap B_s\,;\, s \in S)$$

and, for $s \in S$,

$$B_s = P_K(A_r \cap B_s\,;\, r \in R).$$

Proof. Let $L = P_K(A_r\,;\, r \in R) = P_K(B_s\,;\, S \in S)$. We shall show that

$$(*) \qquad \text{if } a \in A_r, \text{ then } a_{B_s} \in A_r \cap B_s \cup \{0\} \cup \{1\}.$$

Let $a \in A_r$ and let a_{B_s} be proper, i.e. $\neq 0$, $\neq 1$. Since L is generated by the set $\cup(B_s\,;\, s \in S)$, a can be written in the form

$$(1) \qquad a = p(b_{s_1,1}, \ldots, b_{s_1,n_1}, \ldots, b_{s_k,1}, \ldots, b_{s_k,n_k}),$$

where p is a $(n_1 + \ldots + n_k)$-ary polynomial, $s_1, \ldots, s_k \in S$ and $b_{s_h,m} \in B_{s_h}$ for $h = 1, \ldots, k$; $1 \leqq m \leqq n_h$. Now (1) implies

$$(2) \qquad (a\rangle_{\text{in } A_r} = T_{A_r}(a) = \bar{p}(T_{A_r}(b_{s_1,1}), \ldots, T_{A_r}(b_{s_k,n_k})).$$

Without loss of generality we can assume that $s = s_1$, then from (1) it follows that

$$(3) \qquad \begin{aligned} (a_{B_s}\rangle_{\text{in } B_s} &= T_{B_s}(a) = \bar{p}(T_{B_s}(b_{s_1,1}), \ldots, T_{B_s}(b_{s_1,n_1}), \\ &T_{B_s}(b_{s_2,1}), \ldots, T_{B_s}(b_{s_k,n_k})) = \bar{p}((b_{s_1,1}\rangle_{\text{in } B_s}, \ldots, \\ &(b_{s_1,n_1}\rangle_{\text{in } B_s}, T_{B_s}(b_{s_2,1}), \ldots, T_{B_s}(b_{s_k n_k})), \end{aligned}$$

because $b_{s_1,1}, \ldots, b_{s_1,n_1} \in B_s$. Consider now in (3) $T_{B_s}(b_{s_t,m})$ for $s_t \neq s$ $(= s_1)$:

1/ if $s \nleqq s_i$ in S, hence if $B_s \nleqq B_{s_i}$, then $T_{B_s}(b_{s_i,m}) = \emptyset$;
2/ if $s \leqq s_i$ in S, hence if $B_s \leqq B_{s_i}$, then $T_{B_s}(b_{s_i,m}) = B_s$, (in both cases $b_{s_i,m} \notin B_s$).
If $M \subseteqq L$, denote $T_{B_s}(M) = \cup(T_{B_s}(m);\ m \in M)$.

Since $T_{B_s}(b)$ is an isotone function of its argument b, there holds

$$(a_{B_s}\rangle_{\text{in } B_s} = T_{B_s}(a) = T_{B_s}((a\rangle_{\text{in } A_r}) = T_{B_s}(T_{A_r}(a))$$

and from (2) we get

$$(a_{B_s}\rangle_{\text{in } B_s} = T_{B_s}(T_{A_r}(a)) = \bar{p}(T_{B_s}(T_{A_r}(b_{s1,1})), \ldots, T_{B_s}(T_{A_r}(b_{sk,nk}))). \tag{4}$$

In (4) there holds

a) if $s \nleqq s_i$ in S, hence if $B_s \nleqq B_{s_i}$, then $B_s \nless T_{A_r}(b_{s_i,m})$ (because if it were $B_s < T_{A_r}(b_{s_i,m})$, from $T_{A_r}(b_{s_i,m}) < B_{s_i}$ (see [2], Lemma 1.1) we would get $B_s < B_{s_i}$, a contradiction). Now $B_s \nless T_{A_r}(b_{s_i,m})$ implies that $T_{B_s}(T_{A_r}(b_{s_i,m})) = \emptyset$ (because if it were nonempty, there would exist $b \in T_{B_s}(T_{A_r}(b_{s_i,m}))$ and there would be $b \leqq b_{s_i,m}$, $b \in B_s$, a contradiction with $B_s \nleqq B_{s_i}$;
b) if $s \leqq s_i$ in S, hence $B_s \leqq B_{s_i}$, then from $T_{A_r}(b_{s_i,m}) < (b_{s_i,m}\rangle_{\text{in } B_{si}}$ it follows that $T_{B_s}(T_{A_r}(b_{s_i,m})) < T_{B_s}((b_{s_i,m}\rangle) = T_{B_s}(b_{s_i,m})$.

Now the following inequalities hold:

$$\begin{array}{ll} \text{for} & s_i = s_1: (b_{s1,m}\rangle_{\text{in } B_s} > T_{A_r}(b_{s1,m}) > T_{B_s}(T_{A_r}(b_{s1,m})); \\ \text{for} & s_i \neq s_1: T_{B_s}(b_{s_i,m}) > X_{s_i,m} > T_{B_s}(T_{A_r}(b_{s_i,m})), \end{array} \tag{5}$$

where $X_{s_i,m}$ will be suitably defined as follows:

1/ if $T_{B_s}(b_{s_i,m}) = \emptyset$ (it was in the case 1/ $s \nleqq s_i$ after the inequality (3)), then also $T_{B_s}(T_{A_r}(b_{s_i,m})) = \emptyset$, because $T_{B_s}(T_{A_r}(b_{s_i,m})) < T_{B_s}(b_{s_i,m})$ (because $T_{A_r}(b_{s_i,m}) < (b_{s_i,m}\rangle_{\text{in } B_{si}}$) and we put $X_{s_i,m} = \emptyset$;
2/ if $T_{B_s}(b_{s_i,m}) = B_s$ (it was the case 2/$s \leqq s_i$ after the inequality (3)), then clearly $T_{B_s}(T_{A_r}(b_{s_i,m})) < B_s$ (because $T_{B_s}(T_{A_r}(b_{s_i,m})) \subseteqq B_s$) and we put $X_{s_i,m} = B_s$.

Now from (3) and (4) using (5) we get

$$\begin{aligned} (a_{B_s}\rangle_{\text{in } B_s} &= \bar{p}((b_{s1,1}\rangle_{\text{in } B_s}, \ldots, (b_{s1,n1}\rangle_{\text{in } B_s}, T_{B_s}(b_{s2,1}), \ldots, T_{B_s}(b_{sk,nk})) > \\ &> \bar{p}(T_{A_r}(b_{s1,1}), \ldots, T_{A_r}(b_{s1,n1}), X_{s2,1}, \ldots, X_{sk,nk}) > \\ &> \bar{p}(T_{B_s}(T_{A_r}(b_{s1,1}), \ldots, T_{B_s}(T_{A_r}(B_{sk,nk}))) = T_{B_s}(T_{A_r}(a)) = \\ &= T_{B_s}((a\rangle_{\text{in } A_r}) = (a_{B_s}\rangle_{\text{in } B_s}. \end{aligned} \tag{6}$$

From (6) it follows that

$$(a_{B_s}\rangle_{\text{in } B_s} = \bar{p}(T_{A_r}(b_{s1,1}), \ldots, T_{A_r}(b_{s1,n1}), X_{s2,1}, \ldots, X_{sk,nk}), \tag{7}$$

where $X_{s_i,m}$ is either $\emptyset$ or B_s.

By the definition of lower covers ([2]) there exists a polynomial q such that

$$a_{B_s} = q((b_{l_1})_{A_r}, \ldots, (b_{l_p})_{A_r}), \quad \text{where} \quad b_{l_1}, \ldots, b_{l_p},\ p \leqq n_1 \tag{8}$$

are those from among $b_{s_1,1}, \ldots, b_{s_1,n_1}$, for which there exist their lower covers in the lattice A_r.

Since $b_{t_n} \in A_r$ for $n = 1, \ldots, p$ by (8), we have also $a_{B_s} \in A_r$ and because by the definition of a_{B_s} there holds $a_{B_s} \in B_s$, we have also $a_{B_s} \in A_r \cap B_s$.

Now (*) is proved.

Since $a = a_{A_r}$, from (2) it follows by the definition of the lower covers ([2]) that there exists a polynomial w such that

$$a = w((b_{f_1})_{A_r}, \ldots, (b_{f_m})_{A_r}), \tag{9}$$

where $b_{f_i} \in B_{f_i}$ for $i = 1, \ldots, m$; $m = n_1 + \ldots + n_k$ and b_{f_i}, $i = 1, \ldots, n$ are those from among the $b_{s_1,1}, \ldots, b_{s_k,n_k}$ for which there exists $(b_{f_i})_{A_r}$.

By (*), $(b_{f_i})_{A_r} \in B_{f_i} \cap A_r$ for $i = 1, \ldots, m$ holds.

Now by (9) there is $a \in [\cup(A_r \cap B_s\,;\, s \in S)]_L$ for $a \in A_r$. Hence A_r is generated by the set $\cup(A_r \cap B_s\,;\, s \in S)$. By the property (J′), because $(A_r \cap B_s,\ s \in S)$ are sublattices of A_r, there holds $A_r = P_K(A_r \cap B_s\,;\, s \in S)$. Then by the "associativity" of the poproduct, [2], Lemma 4.2, $L = P_K(A_r \cap B_s\,;\, \langle r, s\rangle \in R \times S)$. Theorem 1′ is proved.

REFERENCES

[1] GRÄTZER, G.—SICHLER, J.: Free decomposition of a lattice. Canad. J. Math., 27, 1975, 276—285.

[2] LADZIANSKA, Z.: Poproduct of lattices. Math. Slovaca 32, 1982, 3—22.

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ПОПРОДУКТОВОЕ РАЗЛОЖЕНИЕ СТРУКТУРЫ

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Резюме

В работе обобщается теорема об общем подразделении всяких двух представлений структуры как свободного произведения структур на случай попродукта.