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ON THE OSCILLATION OF SOLUTIONS OF $y^{(2n)} + By' + A_1(t)y = 0$, $B < 0$

JURAJ MAMRILLA

In this paper we consider the differential equation

$$y^{(2n)} + By' + A_1(t)y = 0, \tag{1}$$

where $B = \text{constant} < 0$, $A_1(t) \in C < t_0, \infty)$. Using the substitution

$$t = \varphi(x) = -\sqrt[2n-1]{\frac{2n}{B}}\, x \tag{2}$$

we obtain the equation

$$y^{(2n)} - 2ny' + A(x)y = 0, \tag{3}$$

where $A(x) \in C < \bar{\varphi}(t_0), \infty)$, $x_0 = \bar{\varphi}(t_0)$. Equation (3) and the system of equations

$$\begin{aligned} y^{(2n-2)} + 2y^{(2n-3)} + 3y^{(2n-4)} + \ldots + (2n-1)y &= ze^{x} \\ z'' + [A(x) - (2n-1)]e^{-x}y &= 0 \end{aligned} \tag{4}$$

are mutually equivalent.

First the existence of oscillatory solutions of equation (3) will be investigated. A solution of equation (3) is called oscillatory on $\langle x_0, \infty)$ if it has at least one zero point on any interval $\langle \bar{x}, \infty)$, $\bar{x} > x_0$ and it is called nonoscillatory in the reverse case, i.e. if there exists an interval $\langle \bar{x}, \infty)$, $\bar{x} > x_0$ such that the solution $y(x)$ of (3) is not vanishing in any point of this interval.

The oscillatory properties of solutions of equation (1) for $B = 0$ were investigated in many papers by various authors. For illustration we quote only some of them — [1], [4], [5], [6]. The case $B > 0$ was investigated in the author's paper [2].

In the present paper the following lemmas and theorem are proved:

Lemma 1. *Let the coefficients a_j for $j = 1, 2, \ldots, 2m$ of the differential equation*

$$u^{(2m)} + a_1 u^{(2m-1)} + \ldots + a_{2m}u = z(x) \tag{5}$$

be (real) constants such that the characteristic equation

$$k^{2m}+a_1k^{2m-1}+\ldots+a_{2m}=0 \tag{6}$$

has not any real roots and it possesses at most one couple of conjugate pure imaginary roots Let $z(x)$, $x>x_0$ be a concave positive function. Then there exists a solution of the equation (5) such that

$$u(x)=\frac{z(x)}{a_{2m}}+o(1),\ u'(x)=o(1) \tag{7_1}$$

(*if* $\lim_{x\to\infty} z'(x)=0$) *or*

$$u(x)=\frac{z(x)}{a_{2m}}-\frac{a}{a_{2m}^2}+o(1),\ u'(x)=\frac{a}{a_{2m}}+o(1) \tag{7_2}$$

(*if* $\lim_{x\to\infty} z'(x)=a>0$).

Lemma 2. *Equation*

$$k^{2n-2}+2k^{2n-3}+3k^{2n-4}+\ldots+(2n-2)k+(2n-1)=0 \tag{8}$$

has neither real nor pure imaginary roots and at most one couple of roots is in the form $1\pm\alpha i$, $\alpha\in R$, $\alpha\neq 0$, i.e. both the number $1+\alpha_1 i$ and $1+\alpha_2 i$ for $0<|\alpha_1|<|\alpha_2|$, α_1, $\alpha_2\in R$ cannot be roots of equation (8).

Lemma 3. *Equation*

$$k^{2n-4}+2k^{2n-6}+\ldots+(n-2)k^2+(n-1)=0 \tag{9}$$

has neither real nor pure imaginary roots for n — even, but it has one couple of conjugate pure imaginary roots for n — odd and at most one couple of conjugate roots is in the form $-1\pm\alpha i$, $\alpha\in R$, $\alpha\neq 0$, i.e. both the number $-1+\alpha_1 i$ and $1+\alpha_2 i$ for $0<|\alpha_1|<|\alpha_2|$, α_1, $\alpha_2\in R$ cannot be roots of equation (9).

Theorem. *Let $v''+A_\varepsilon\Psi(x)v=0$ be an oscillatory equation (i.e. its every solution is oscillatory) such that $\Psi(x)=A(x)-(2n-1)>0$ and $A_\varepsilon=\dfrac{1}{n(2n-1)}-\varepsilon>0$, where ε is small arbitrary positive constant. Then every solution of the differential equation (3) is oscillatory on $\langle x_0, \infty)$.*

Remark 1. *If $\Psi(x)=A(x)-(2n-1)\equiv 0$, then equation (3) has the fundamental system of solutions; two of which are nonoscillatory ($y_1=e^x$, $y_2=xe^x$) and the remaining $(n-2)$ solutions are oscillatory.*

Remark 2. This theorem generalizes a sufficient condition of the oscillation of all solutions of equation (3) from paper [3] (Theorem 2). The generalization concerns both the order equation and the coefficient $A(x)$.

Proof of Lemma 1. This lemma is proved in [2] (Lemma 1).

Proof of Lemma 2. Directly from the characteristic equation (8) it follows that the number $k=1$ is not its root. Modifying the left-hand side of (8) we obtain

$$\frac{k^{2n}-2kn+2n-1}{(k-1)^2}=0, \quad k\neq 1. \tag{8$_1$}$$

Hence we see that equation (8_1) has neither real ($k\neq 1$) nor pure imaginary roots. In order to prove that there does not exist more than one number $\alpha>0$ such that $1\pm\alpha i$ are roots of equation $k^{2n}-2kn+(2n-1)=0$ for $k=1+\alpha i$ instead of (8). Then

$$(1+\alpha i)^{2n}=1+2n\alpha i$$

and for the modulus of these complex numbers we get

$$(1+\alpha^2)^n=\sqrt{1+4n^2\alpha^2} \quad \text{and} \quad (1+\alpha^2)^{2n}=1+4n^2\alpha^2.$$

Consider the functions

$$f(\alpha)=(1+\alpha^2)^{2n}-(1+4n^2\alpha^2) \quad \text{and}$$
$$g(\beta)=(1+\beta)^{2n}-(1+4n^2\beta) \quad \text{with} \quad \alpha^2=\beta.$$

Since the function $g(\beta)$ is convex and $g(0)=0$, $g'(0)=2n-4n^2<0$, the function $g(\beta)$ possesses at most one root besides the one $\beta=0$. The number $\beta=0$, i.e. $\alpha=0$ or $k=1$ is not a root of equation (8), which we showed above.

Proof of Lemma 3. Proof of this lemma is analogous to that of Lemma 2. From characteristic equation (9) we have that the numbers $k\in R$ are not its roots. Modifying the left-hand side of equation (9) we get

$$\frac{k^{2n}-nk^2+n-1}{(k^2-1)^2}=0, \quad k^2\neq 1. \tag{9$_1$}$$

Then if n is an even number, equation (9) has no pure imaginary roots and for n — odd it has one couple of conjugate pure imaginary roots. For the proof of the second part of Lemma 3, we use the equation $k^{2n}-nk^2+(n-1)=0$ for $k=-1+\alpha i$, i.e. the equation

$$(-1+\alpha i)^{2n}=1-n\alpha^2-2n\alpha i.$$

Hence for the modulus of these numbers we obtain

$$(1+\alpha^2)^{2n}=1-2\alpha^2n+4\alpha^2n^2+\alpha^4n^2.$$

Consider the function

$$f(\alpha)=(1+\alpha^2)^{2n}-(1-2\alpha^2n+4\alpha^2n^2+\alpha^4n^2)$$

or after the substitution $\alpha^2=\beta$

$$g(\beta)=(1+\beta)^{2n}-(1-2n\beta+4n^2\beta+n^2\beta^2)$$

Function $g(\beta)$ is convex. Since $g''(\beta)=2n(2n-1)(1+\beta)^{2n-2}-2n^2>0$ for $n>1$ and $g(0)=0$, $g'(0)=4n-4n^2<0$, the function $g(\beta)$ has at most one zero point besides the point $\beta=0$. The number $\beta=0$, i.e. $\alpha=0$ or $k=-1$ is not the root of equation (9), which we shoved at the beginning. This concludes the proof of Lemma 3.

Proof of Theorem. Assume that equation (3) has a nonoscillatory solution $y=\bar{y}(x)$. Using the substitution $e^{-x}y=u$ it is evident that to the solution $\bar{y}(x)>0$ there corresponds $\bar{u}(x)>0$, $x>x_1$. We can write the system of equations (4) (equivalent to equation (3)) in the form

$$u^{(2n-2)}+\left[\binom{2n-2}{1}+2\right]u^{(2n-3)}+\left[\binom{2n-2}{2}+2\binom{2n-3}{1}+3\right]\cdot \qquad (10_1)$$

$$\cdot u^{(2n-4)}+\ldots+[1+\ldots+(2n-1)]u=z(x)$$

$$z''+[A(x)-(2n-1)]u=0. \qquad (10_2)$$

From the assumption of this Theorem we have that the function

$$z(x)=\bar{u}^{(2n-2)}+\left[\binom{2n-2}{1}+2\right]\bar{u}^{(2n-3)}+\ldots+[1+2+\ldots+(2n-1)]u \qquad (10_3)$$

is concave for $x>x_1$ and it is either $z(x)\geqq k>0$ or $z(x)<0$ for $x>x_2\geqq x$. Investigate now the solutions of equation (10_1) with the right-hand side defined by (10_3). It is evident that $u=\bar{u}(x)$ is the solution of (10_1).

First of all consider the case $z(x)\geqq k>0$, $x>x_2$. From the characteristic equation corresponding to the first equation (4) and by Lemma 2 we get that the characteristic equation corresponding to (10_1) has only complex roots or one couple conjugate pure imaginary roots. In view of Lemma 1 for $m=n-1>0$ we can write equation (10_1) in the form

$$\begin{aligned} u=u_1, \quad & u_1''+c_1u_1'+d_1u_1=u_2,\\ & u_2''+c_2u_2'+d_2u_2=u_3,\\ & \cdots\cdots\cdots\cdots\cdots\cdots \qquad (11)\\ & u_{n-2}''+c_{n-2}u_{n-2}'+d_{n-2}u_{n-2}=u_{n-1},\\ & u_{n-1}''+c_{n-1}u_{n-1}'+d_{n-1}u_{n-1}=z(x), \end{aligned}$$

where c_j, d_j are (real) constants such that $D_j=d_j-\left(\frac{c_j}{2}\right)^2>0$ for $j=1, 2, \ldots, n-1$. If the characteristic equation corresponding to (10_1) has pure imaginary roots then we write the corresponding linear differential equation as the last one in (11). Thus we have guaranteed its solution in the form (7_1) or (7_2). Take (7_1) (the following consideration holds also for (7_2)) and put it into (10_2). As a result we obtain

$$z'' + [A(x) - (2n-1)]\left[\frac{1}{n(2n-1)} + \frac{o(1)}{z(x)}\right] z = 0. \qquad (12)$$

Hence and by the assumptions of the Theorem it follows that equation (12) is oscillatory. This gives the contradiction with the inequality $z(x) \geqq k > 0$, $x > x_2$, which implies the nonexistence of solution $y = \bar{y}(x) > 0$ of equation (3).

Now consider the case of $z(x)$ being a negative concave function for $x > x_2$. In this case it is possible to write the first equation (4) as follows

$$\{e^x[y^{(2n-4)} + 2y^{(2n-6)} + 3y^{(2n-8)} + \ldots + (n-2)y'' + (n-1)y]\}'' + ne^x y = e^{2x} z. \qquad (13)$$

Since $z(x) < 0$, considering only $y(x) > 0$ for $x > x_2$, we obtain that the function

$$z_1(x) = e^x[y^{(2n-4)} + 2y^{(2n-6)} + \ldots + (n-2)y'' + (n-1)y] \qquad (14)$$

is concave. We can understand relation (14) as a differential equation with constant coefficients and the right-hand side $z_1(x)e^{-x}$, where $z_1(x)$ is a concave function.

Let $z_1(x) \geqq k_1 > 0$, $x > x_3$. According to Lemma 3 the corresponding characteristic equation

$$k^{2n-4} + 2k^{2n-6} + \ldots + (n-2)k^2 + (n-1) = 0$$

has complex roots (if n is an odd number it has also one couple conjugate pure imaginary roots) but at most one couple roots $-1 \pm \alpha i$, $\alpha \in R$. Using the substitution $e^x y = u$ in (14) we obtain

$$u^{(2n-4)} + (2n-4)u^{(2n-5)} + \ldots + [1 + 2 + \ldots + (n-1)]u = z_1(x). \qquad (15)$$

With regard to the substitution $e^x y = u$ and to the assertion of Lemma 3 and to Lemma 1 we have

$$u(x) = \frac{2z_1(x)}{n(n-1)} + o(1), \quad u'(x) = o(1) \qquad (16)$$

(we took the "smaller" solution of (7_1)). Putting (16) in $e^x y = u$ we obtain

$$y(x) > s_1 e^{-x}, \quad s_1 = \text{konst} > 0, \quad x > x_4 > x_3. \qquad (17)$$

Using (17) and (13) we get $\lim_{x \to \infty} z_1'(x) = -\infty$, which is a contradiction to the inequality $z_1(x) \geqq k_1 > 0$, $x > x_3$. Now it is sufficient to investigate the case of $z_1(x) < 0$ ($z_1(x)$-concave) $x > x_3$. From (14) we obtain

$$[y^{(2n-6)} + 2y^{(2n-8)} + \ldots + (n-2)y]'' + (n-1)y = z_1(x)e^{-x}. \qquad (18)$$

Then the function

$$z_2(x)=y^{(2n-6)}+2y^{(2n-8)}+\ldots+(n-2)y \tag{19}$$

is concave and it is either $z_2(x)\geqq k_2>0$ or $z_2(x)<0$ for $x\geqq x_4\geqq x_3$.

Relation (19) can be understood again as a differential equation with a concave right-hand side $z_2(x)$, $x>x_4$. It is evident that the characteristic equation corresponding to (19) has only complex roots (if for n — even there is one couple conjugate pure imaginary roots; this characteristic equation is an equation of type (9) with the diference that here we have $(n-1)$ instead of n). If $z_2(x)\geqq k_2>0$, $x\geqq x_5$, then we can use Lemma 1 and the obtained solution (7_1) (it is sufficient to confine ourselves to the solution (7_1), which we pointed out several times above) is put into (18) (precisely into the expression $(n-1)y$), whence $\lim_{x\to\infty} z_2(x)=-\infty$. This gives a contradiction with $z_2(x)\geqq k_2>0$, $x\geqq x_5$. We must again investigate whether the inequality $z_2=z_2(x)<0$ ($z_2(x)$-concave) for $x>x_5$ implies $y(x)>0$ for $x>x_1$. In this case we write (19) in the form

$$[y^{(2n-8)}+2y^{(2n-10)}+\ldots+(n-3)y]''+(n-2)y=z_2(x).$$

Repeating the same consideration as above we obtain always a contradiction for concave functions $z_3(x)$, $z_4(x)$, ..., $z_{2n-5}(x)$. It remains to investigate equation

$$y''+2y=z_{2n-6}(x) \tag{20}$$

for the concave function $z_{2n-6}(x)$. If $z_{2n-6}(x)<0$ for $x\geqq x_N$ (x_N is a sufficiently large number), then from (20) it is obvious that $y(x)$ cannot be positive.

Now consider the case $z_{2n-6}(x)\geqq k_{2n-6}>0$, $x\geqq x_N$. In this case we proceed as follows: Dividing (20) by x and integrating over $\langle x_N, x\rangle$ we obtain

$$\frac{y'}{x}+\frac{y}{x^2}+\int_{x_N}^{x}\left(\frac{2}{t^3}+\frac{2}{t}\right)y\,\mathrm{d}t=K+\int_{x_N}^{x}\frac{z_{2n-6}(t)}{t}\,\mathrm{d}t \tag{21}$$

where $K=(x^{-1}y'+x^{-2})(x_N)$. With regard to that $\int^{\infty} t^{-1}z_{2n-6}(t)\,\mathrm{d}t=\infty$ and $\lim_{x\to\infty}\int^{x}(2t^{-3}+2t^{-1})y(t)\,\mathrm{d}t$ exist, one obtains either $\int^{\infty}(2t^{-3}+2t^{-1})y\,\mathrm{d}t=\infty$ or $\int^{\infty}(2t^{-3}+2t^{-1})y\,\mathrm{d}t<\infty$. If the last case holds, then $\lim_{x\to\infty}(x^{-1}y'+x^{-2}y)=+\infty$. In other words from (21) it follows

$$y'+x^{-1}y=xg(x), \tag{22}$$

where

$$g(x)=K+\int_{x_N}^{x}t^{-1}z_{2n-6}\,\mathrm{d}t-\int_{x_N}^{x}(2t^{-3}+2t^{-1})y\,\mathrm{d}t$$

and

$$\lim_{x\to\infty} g(x) = \infty.$$

The solution of equation (22) has the form

$$y(x) = \left[\int_{x_N}^{x} t^2 g(t)\,\mathrm{d}t + y(x_N)\right] x^{-1},$$

for which $\lim_{x\to\infty} \frac{y(x)}{x^2} = \infty$. Then $y \geqq k_0 x^2$ ($k_0 = \text{const} > 0$). Applying the last inequality we get that the integral $\int^{x} (2t^{-3} + 2t^{-1}) y(t)\,\mathrm{d}t$ is divergent as $x \to \infty$. This gives a contradiction to the assumption $\lim_{x\to\infty} \int^{x} (2t^{-3} + 2t^{-1}) y\,\mathrm{d}t < \infty$. Then the integral $\int^{\infty} (2t^{-3} + 2t^{-1}) y\,\mathrm{d}t = \infty$ and hence also the integral $\int^{\infty} y(t)\,\mathrm{d}t = \infty$. It means that if $z_{2n-6}(x) > k_{2n-6} > 0$, there exists the solution $y(x)$ of (20) (which is also the solution of equation (3) and we assume that it is positive) with the property $\int^{\infty} y(t)\,\mathrm{d}t = \infty$.

Now we easily derive a contradiction. Indeed, from (18) the same procedure by which we got equation (20) gives

$$[y'' + 2y]'' + 3y = z_{2n-5}, \tag{23}$$

where $z_{2n-5}(x)$ is a concave negative function for $x > x_{N-1}$. Just on the basis of the proved property $\int^{\infty} y(t)\,\mathrm{d}t = \infty$ it follows that $\lim_{x\to\infty} z'_{2n-6}(x) = -\infty$, which leads to the contradiction with $z_{2n-6}(x) \geqq k_{2n-6} > 0$, $x \geqq x_N$.

This ends the proof of the Theorem.

REFERENCES

[1] КОНДРАТЬЕВ, В. А.: О колеблемости решений уравнения $y^{(n)}+p(x)y=0$, Труды Моск. мат. общ.-ва 10 (1961), 419–436.

[2] MAMRILLA, J.: О колеблемости решений уравнения $y^{(2n)}+By'+A(t)y=0$, $B>0$ (To appear in Acta Univ. Comen. — Mathematica XXXV 1980).

[3] MAMRILLA, J.: On the Oscillation of Solution of $y^{(4)}+A_5y'+A_6(t)y+g(t, y)=0$, Bolletino U.M.I. (4) 4 (1971) 68—75.

[4] ŠVEC, M.: Sur les dispersions des intégrales de l'équation $y^{(4)}+Q(x)y=0$, Čech. mat. ž., T.5 (80) 1955.

[5] ŠVEC, M : Sur une propriété des integrales de l'équation $y^{(n)}+Q(x)y=0$, $n=3, 4$, Čech. mat. ž., T. 7 (82) 1957.

[6] ŠVEC, M.: Eine Eigenwertaufgabe der Differentialgleichung $y^{(n)}+Q(x, \lambda)y=0$, Čech. mat. ž., T. 6 (81) 1956.

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О КОЛЕБЛЕМОСТИ РЕШЕНИЙ УРАВНЕНИЯ $y^{(2n)}+By+A(t)y=0$, $B<0$

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Резюме

В работе доказана теорема: все решения дифференциального уравнения (3) колеблются, если тем же свойством обладает уравнение $v''+A_\varepsilon\Psi(x)v=0$, где

$$\Psi(x)=A(x)-(2n-1)\geq 0,\quad A_\varepsilon=\frac{1}{n(2n-1)}-\varepsilon>0,$$

причем ε произвольная положительная постоянная.