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REMARKS ON Q-INDEPENDENCE OF STONE ALGEBRAS

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ABSTRACT. We investigate Q-independent subsets in Stone algebras for some specified families Q of mappings (e.g.: M, S, S_0, A_1, G and I) using the well-known triple representation of Stone algebras.

1. Introduction

In 1958 E. Marczewski (see [12]) introduced a general notion of independence, also called *algebraic independence* (or *Marczewski independence*), which contains, as special cases, the majority of independence notions used in various branches of mathematics. There are several interesting results relating to this notion of independence. However, the important scheme of M-independence is not broad enough to cover stochastic independence, independence in projective spaces, and some other notions. This is why some notions weaker than M-independence have been developed. The notion of independence with respect to a family Q of mappings (defined on subsets of A) into A, Q-independence for short, is a common way of defining almost all known notions of independence.

Stone algebras first attracted interest when they were characterized by G. Grätzer and E. T. Schmidt as the solution of a Stone problem: they are bounded distributive lattices for which the set of prime ideals satisfies the

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† Passed away on September 25, 2005.

property that each prime ideal contains a unique minimal prime ideal (see [11]). Stone algebras have arisen in various applications, including conditional event algebras and rough sets.

2. Preliminaries

For a fixed algebra $\mathfrak{A} = (A; \mathbb{F})$ we denote by $\mathbb{T}^{(n)}(\mathfrak{A})$ $(n = 1, 2, \dots)$ the class of all *n-ary term operations* containing n-ary projections, and closed under compositions.

$\mathbb{T}^{(0)}(\mathfrak{A})$ denotes the set of all (algebraic) constant operations of the algebra $\mathfrak{A}$. It is convenient to identify a constant operation with its value.

Let $\mathfrak{A} = (A; \mathbb{F})$ be an algebra. A nonempty set $X \subseteq A$ is called *M-independent* if for any finite system $a_1, \dots, a_n \in X$ of different elements and for any $f, g \in \mathbb{T}^{(n)}(\mathfrak{A})$ the equality $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$ implies $f = g$ in A.

Now, denote by $M(A)$ the family of all mappings $p\colon X \to A$ from every nonempty subset $X \subseteq A$ to A, and by $H(A)$ the set of all mappings $p\colon X \to A$ $(X \subseteq A)$ which possess an extension to a homomorphism $\overline{p}\colon \langle X\rangle_{\mathfrak{A}} \to A$ $(\overline{p}|_X = p)$.

A nonempty set $X \subseteq A$ is said to be *independent with respect to the family* $Q \subseteq M(A)$ (Q-independent, or $X \in \operatorname{Ind}(\mathfrak{A}, Q)$, for short) if $Q \cap A^X \subseteq H(A)$, see [15] and [5], or equivalently

$$\begin{gathered}(\forall n \in \mathbb{N},\ n \le \operatorname{card}(X))\,(\forall f, g \in \mathbb{T}^{(n)}(\mathfrak{A}))\,(\forall p \in A^X)\,(\forall a_1, \dots, a_n \in X)\\ \big(f(a_1, \dots, a_n) = g(a_1, \dots, a_n) \implies f\big(p(a_1), \dots, p(a_n)\big) = g\big(p(a_1), \dots, p(a_n)\big)\big).\end{gathered}$$

If we put $Q = M = \bigcup\{A^X : X \subseteq A\}$, we get M-independence. For $Q = S = \bigcup\big\{\langle X\rangle_{\mathfrak{A}}^X : X \subseteq A\big\}$, we obtain *$S$-independence* (*local independence* introduced by J. Schmidt in [17]). If $Q = S_0 = \bigcup\{X^X : X \subseteq A\}$, we have *$S_0$-independence* (*weak independence* in the sense of S. Świerczkowski, [18]). Another notion of independence may be obtained by putting $Q = A_1 = \{f|_X : f \in \mathbb{T}^{(1)}(\mathfrak{A}),\ X \subseteq A\}$ (introduced by K. Głazek, [5]). For $Q = G = \bigcup\{p|_X : p \in A^A$ is diminishing, $X \subseteq A\}$ we get *G-independence* (*weak independence* in the sense of G. Grätzer, [9]), where the mapping p is called *diminishing* if $(\forall f, g \in \mathbb{T}^{(1)}(\mathfrak{A}))(\forall a \in A)\big(f(a) = g(a) \implies f(p(a)) = g(p(a))\big)$. For $Q = I = \bigcup\{p : p \in A^X$ injective, $X \subseteq A\}$, we get *I-independence* (introduced as *R-independence* by K. Głazek, [5]). Let us recall that:

$$\begin{aligned}\operatorname{Ind}(\mathfrak{A}, M) &\subseteq \operatorname{Ind}(\mathfrak{A}, Q) \qquad \text{for all} \quad Q \subseteq M,\\ \operatorname{Ind}(\mathfrak{A}, S) \subseteq \operatorname{Ind}(\mathfrak{A}, S_0) &\qquad \text{and} \qquad \operatorname{Ind}(\mathfrak{A}, S) \subseteq \operatorname{Ind}(\mathfrak{A}, A_1).\end{aligned} \tag{$*$}$$

A family J of sets is *hereditary* if

$$(\forall X \in J)(\forall Y \subseteq X)(Y \in J)\,.$$

The family $\mathrm{Ind}(\mathfrak{A}, Q)$ is hereditary for $Q = M, S, S_0, G, A_1$ and I.

Another kind of independence, the so-called t-independence, was introduce by J. Płonka (see [16], [8]). Namely, a set $X \subseteq A$ is called *t-independent* in algebra $\mathfrak{A} = (A; \mathbb{F})$ if for any finite system of different elements $a_1, \dots, a_n \in X$ and for any n-ary term operation f which is not a projection, we have $f(a_1, \dots, a_n) \neq a_i$ for all $i = 1, \dots, n$. We denote by $t\text{-}\mathrm{Ind}_{\mathfrak{A}}$ the family of all t-independent sets of the algebra $\mathfrak{A}$.

The remaining notions and notations used are rather standard, and for them the reader is referred to [2] and [3].

A *Stone algebra* is an algebra $\mathfrak{L} = (L; \vee, \wedge, {}^*, \mathbf{0}, \mathbf{1})$ of type $(2,2,1,0,0)$ such that $(L; \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a distributive lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$, * is a unary operation on L such that

$$a \wedge x = \mathbf{0} \iff x \leq a^*\,,$$

and the following *Stone identity* holds:

$$x^* \vee x^{**} = \mathbf{1}\,.$$

We assume that the reader is familiar with the basic properties of Stone algebras, as presented in [1] or [10].

Two significant subsets of a Stone algebra $\mathfrak{L}$ are the set $D(L) = \{x \in L :\ x^* = \mathbf{0}\}$ of *dense* elements and the *skeleton* $S(L) = \{x \in L :\ x^{**} = x\}$. The algebra $\mathfrak{S}(\mathfrak{L}) = \big(S(L); \vee, \wedge, {}^*, \mathbf{0}, \mathbf{1}\big)$ is a subalgebra of $\mathfrak{L}$, which is a Boolean algebra, and $\mathfrak{D}(\mathfrak{L}) = \big(D(L); \vee, \wedge, \mathbf{1}\big)$ is a distributive lattice with the greatest element $\mathbf{1}$, it is a filter of $\mathfrak{L}$. Let $\mathfrak{F}\big(D(L)\big)$ denote the family of all filters of $D(L)$. The relationship between elements of $S(L)$ and $D(L)$ is expressed by the homomorphism $\varphi_L\colon S(L) \to \mathfrak{F}\big(D(L)\big)$ defined by $\varphi_L(a) = \big\{x \in D(L) :\ x \geq a^*\big\}$ (see [4]).

On every Stone algebra $\mathfrak{L}$ we can define the so-called *Glivenko congruence* θ, by $(x,y) \in \theta$ iff $x^* = y^*$. It is well known that cokernel of θ, i.e. $[\mathbf{1}]_\theta$, is the dense filter $D(L)$; moreover L/θ is a Boolean algebra isomorphic to $S(L)$. Each θ-class contains exactly one element of $S(L)$ which is the greatest element in this class. For $a \in S(L)$, set $[a]_\theta = F_a = \{x \in L :\ x^{**} = a\}$. Then $\mathfrak{F}_a = (F_a; \vee, \wedge, a)$ is a distributive lattice with the greatest element a. It is a sublattice of the distributive lattice $\mathfrak{L}_D = (L; \vee, \wedge)$. Define a mapping $\phi\colon L \to D(L)$ by $\phi(x) = x \vee x^*$. Then $\phi\big|_{F_a}$ is a lattice-isomorphism from F_a onto $\varphi_L(a)$ for every $a \in S(L)$.

3. Independence of one and two element sets in Stone algebras

It is well known that $\mathbf{0}$ and $\mathbf{1}$ are the only algebraic constants and x, x^*, x^{**}, $x \vee x^*$ are the only non-constant unary algebraic operations in Stone algebras. Therefore:

Lemma 1. *All unary term equations in Stone algebras can be reduced to the following four types: $x = \mathbf{0}$, $x = \mathbf{1}$, $x^* = \mathbf{0}$, and $x^{**} = x$.*

Indeed, it is easy to calculate that the following four equations are equivalent:

$$x = \mathbf{0}\,, \qquad x^* = \mathbf{1}\,, \qquad x^{**} = \mathbf{0}\,, \qquad x^* = x \vee x^*\,;$$

and the next two:

$$x = \mathbf{1}, \qquad x^{**} = x \vee x^*\,,$$

are also equivalent. The following equivalent equations:

$$x^* = \mathbf{0}\,, \quad x^{**} = \mathbf{1}\,, \quad x = x \vee x^*$$

$$\text{(and, similarly,} \qquad x^{**} = x \qquad \text{and} \qquad \mathbf{1} = x \vee x^*\,)$$

are satisfied only by $x \in D(L)$ (by $x \in S(L)$, respectively).

From Lemma 1, we have immediately:

Corollary 1. *Let $(L; \vee, \wedge, {}^*, \mathbf{0}, \mathbf{1})$ be a Stone algebra. A mapping $p\colon X \to L$ (where $X \subseteq L$) is diminishing iff it preserves constants and the sets $D(L)$ and $S(L)$.*

Since all families of the considered independent sets are hereditary, we will first investigate Q-independence of one-element sets. Recall (see [5]) that:

$$\{a\} \in \operatorname{Ind}(\mathfrak{A}, S) \iff \{a\} \in \operatorname{Ind}(\mathfrak{A}, A_1)\,,$$
$$\{a\} \in \operatorname{Ind}(\mathfrak{A}, I) \iff \{a\} \in \operatorname{Ind}(\mathfrak{A}, M)\,,$$

and every one-element set belongs to the families $\operatorname{Ind}(\mathfrak{A}, S_0)$ and $\operatorname{Ind}(\mathfrak{A}, G)$ in every algebra $\mathfrak{A}$. Now, we can prove:

Proposition 1. *Let $\mathfrak{L} = (L; \vee, \wedge, {}^*, \mathbf{0}, \mathbf{1})$ be a Stone algebra and $a \in L$. Then*

(i) $\{a\} \in \operatorname{Ind}(\mathfrak{L}, M)$ *iff* $a \notin D(L) \cup S(L)$,
(ii) $\{a\} \in \operatorname{Ind}(\mathfrak{L}, S)$ *iff* $a \notin D(L) \cup \{\mathbf{0}\}$,
(iii) $\{a\} \in t\text{-}Ind_{\mathfrak{L}}$ *iff* $a \notin D(L) \cup S(L)$.

Proof.

(i): Let $\{a\} \in \mathrm{Ind}(\mathfrak{L}, M)$. Suppose, on the contrary, that $a \in D(L)$. Consider the following unary term operations: $f(x) = x^*$ and $g(x) = \mathbf{0}$. We thus get $f(a) = a^* = \mathbf{0} = g(a)$ and $f(\mathbf{0}) = \mathbf{1}$, which implies $f \neq g$ in $\mathfrak{L}$, a contradiction. Therefore, $a \notin D(L)$. In the same manner, we observe that $a \notin S(L)$.

Now, let $a \notin D(L) \cup S(L)$. By Lemma 1, $f(a) \neq g(a)$ for all different $f, g \in \mathbb{T}^{(1)}(\mathfrak{L})$. Consequently, $\{a\} \in \mathrm{Ind}(\mathfrak{L}, M)$.

(ii): Suppose that $\{a\} \in \mathrm{Ind}(\mathfrak{L}, S)$ and $a \in D(L)$. Consider the operations f, g defined above and a mapping $p\colon \{a\} \to \langle a \rangle_{\mathfrak{L}}$, given by $p(x) = x^*$. So $f(a) = g(a)$, which implies $f\big(p(a)\big) = f(\mathbf{0}) = \mathbf{1} \neq \mathbf{0} = g\big(p(a)\big)$, a contradiction. Similarly, we can prove that $a \neq \mathbf{0}$.

Now, let $a \notin D(L) \cup \{\mathbf{0}\}$. If, additionally, $a \notin S(L)$, then $\{a\} \in \mathrm{Ind}(\mathfrak{L}, M) \subseteq \mathrm{Ind}(\mathfrak{L}, S)$ by Proposition 1 (i). It is enough to consider $a \in S(L) \setminus \{\mathbf{0}, \mathbf{1}\}$. Assume that $f(a) = g(a)$ for some $f, g \in \mathbb{T}^{(1)}(\mathfrak{L})$, and $p\colon \{a\} \to \langle a \rangle_{\mathfrak{L}}$. Since $\mathfrak{S}(\mathfrak{L})$ is a subalgebra of $\mathfrak{L}$, $p(a) \in \langle a \rangle_{\mathfrak{L}} \subseteq S(L)$. Therefore, $f\big(p(a)\big) = g\big(p(a)\big)$ by Lemma 1. In consequence, $\{a\} \in \mathrm{Ind}(\mathfrak{L}, S)$.

(iii): Assume that $\{a\} \in t\text{-}\mathrm{Ind}_{\mathfrak{L}}$ and $a \in S(L)$. Thus $a^{**} = a$, hence we get $a \vee a^{**} = a$. Consider the term operation $f(x) = x \vee x^{**}$. Then $f(a) = a$ and $f(b) = b^{**} \neq b$ for $b \notin S(L)$, which is a contradiction. Similarly, we can prove that $a \notin D(L)$.

Now, suppose that $a \notin D(L) \cup S(L)$. Lemma 1 now yields $f(a) \neq a$, which implies $\{a\} \in t\text{-}\mathrm{Ind}_{\mathfrak{L}}$. □

For the next proposition, we define the following binary term operation:

$$r(x_1, x_2) = (x_1^* \wedge x_2) \vee (x_1 \wedge x_2^*)$$

in the algebra $\mathfrak{L}$. It is clear that the operation r is a symmetric difference of the Boolean algebra $\mathfrak{S}(\mathfrak{L})$.

PROPOSITION 2. *Let $\mathfrak{L} = (L; \vee, \wedge, {}^*, \mathbf{0}, \mathbf{1})$ be a Stone algebra and b_1, b_2 be different elements of F_a for some $a \in S(L)$. Then $\{b_1, b_2\} \notin t\text{-}\mathrm{Ind}_{\mathfrak{L}} \cup \mathrm{Ind}(\mathfrak{L}, Q)$, where $Q = M, S, I$. If, additionally, $a \neq \mathbf{1}$, then $\{b_1, b_2\} \notin \mathrm{Ind}(\mathfrak{L}, G)$.*

Proof. Suppose that $b_1, b_2 \in F_a$ for some $a \in S(L)$. Then $r(b_1, b_2) \in F_{r(a,a)} = F_{\mathbf{0}} = \{\mathbf{0}\}$, so $r(b_1, b_2) = \mathbf{0}$. Consider two binary term operations $r(x_1, x_2)$ and $f(x_1, x_2) = \mathbf{0}$ and a mapping $p \in S \cup I$ such that $p(b_1) = \mathbf{1}$, $p(b_2) = \mathbf{0}$. Then $r(b_1, b_2) = f(b_1, b_2)$ and $r\big(p(b_1), p(b_2)\big) = r(\mathbf{1}, \mathbf{0}) = \mathbf{1} \neq \mathbf{0} = f\big(p(b_1), p(b_2)\big)$. So $\{b_1, b_2\} \notin \mathrm{Ind}(\mathfrak{L}, S) \cup \mathrm{Ind}(\mathfrak{L}, I)$. By $(*)$ the same conclusion can be drawn for M-independence.

To prove that $\{b_1, b_2\} \notin t\text{-}\mathrm{Ind}_{\mathfrak{L}}$, consider $g(x_1, x_2) = x_1 \vee (x_1^* \wedge x_2) \vee (x_1 \wedge x_2^*)$. Then $g(b_1, b_2) = b_1$ and $g(\mathbf{0}, \mathbf{1}) = \mathbf{1}$.

Now, assume that $a \neq \mathbf{1}$. Of course, $a \neq \mathbf{0}$. Hence, p is diminishing by Corollary 1, and consequently $\{b_1, b_2\} \notin \mathrm{Ind}(\mathfrak{L}, G)$. □

4. Independent subsets of F_a

Recall the description of term operations in distributive lattices (see [14] and [13]). Let $\mathfrak{L}_D = (L; \vee, \wedge)$ be a distributive lattice. For every n-ary term operation f of $\mathfrak{L}_D$ there exists exactly one family $P \subseteq 2^{\{1,\dots,n\}}$ of subsets incomparable by the set inclusion such that

$$f(x_1,\dots,x_n) := \tilde{f}_P(x_1,\dots,x_n) = \bigvee_{S\in P} \bigwedge_{j\in S} x_j\,. \tag{$**$}$$

G. Szász and E. Marczewski (see [19] and [14]) proved the following theorem:

THEOREM 1. *Let $(L; \vee, \wedge)$ be a distributive lattice. Then $X \subseteq L$ is M-independent if and only if $c_1 \wedge \dots \wedge c_m \not\leq d_1 \vee \dots \vee d_n$ for each sequence $c_1,\dots,c_m,d_1,\dots,d_n$ of different elements of X.*

Taking into account this fact, we have:

THEOREM 2. *Let $\mathfrak{L}_D = (L; \vee, \wedge)$ be an arbitrary distributive lattice (without constants). Then*

$$\mathrm{Ind}(\mathfrak{L}_D, M) = \mathrm{Ind}(\mathfrak{L}_D, Q)\,, \qquad \textit{where} \quad Q = S, S_0, I, G\,.$$

P r o o f. By $(*)$ it is enough to prove the inclusion $\supseteq$. For this purpose, suppose that $X \in \mathrm{Ind}(\mathfrak{L}_D, S_0)$ and $X \notin \mathrm{Ind}(\mathfrak{L}_D, M)$. Theorem 1 gives $c_1 \wedge \dots \wedge c_m \leq d_1 \vee \dots \vee d_n$ for some $c_1,\dots,c_m,d_1,\dots,d_n \in X$. We thus get $(c_1 \wedge \dots \wedge c_m) \wedge (d_1 \vee \dots \vee d_n) = c_1 \wedge \dots \wedge c_m$. Consider the following term operations:

$$f(x_1,\dots,x_{m+n}) = (x_1 \wedge \dots \wedge x_m) \wedge (x_{m+1} \vee \dots \vee x_{n+m})\,,$$
$$g(x_1,\dots,x_{m+n}) = x_1 \wedge \dots \wedge x_m$$

and the mapping $p\colon X \to X$

$$p(x) = \begin{cases} c_1 & \text{for } x = c_i \ (i = 1,\dots,m), \\ d_1 & \text{for } x \neq c_i \ (i = 1,\dots,m). \end{cases}$$

Hence, $f(c_1,\dots,c_m,d_1,\dots,d_n) = g(c_1,\dots,c_m,d_1,\dots,d_n)$. By the assumption, we get $f\big(p(c_1),\dots,p(c_m),p(d_1),\dots,p(d_n)\big) = g\big(p(c_1),\dots,p(c_n),p(d_1),\dots,p(d_n)\big)$. Then $c_1 \wedge d_1 = c_1$. Now, let $f_1(x_1,x_2) = x_1 \wedge x_2$, $g_1(x_1,x_2) = x_1$ and $p_1(c_1) = d_1$, $p_1(d_1) = c_1$. Then $f_1(c_1,d_1) = g_1(c_1,d_1)$ and $f_1\big(p(c_1),p(d_1)\big) = g_1\big(p(c_1),p(d_1)\big)$, because the family $\mathrm{Ind}(\mathfrak{L}_D, S_0)$ is hereditary. We thus get $d_1 \wedge c_1 = d_1$ and, in consequence, $c_1 = d_1$, which is a contradiction. The result is $\mathrm{Ind}(\mathfrak{L}_D, M) = \mathrm{Ind}(\mathfrak{L}_D, S_0)$ and $\mathrm{Ind}(\mathfrak{L}_D, M) = \mathrm{Ind}(\mathfrak{L}_D, S)$ by $(*)$.

Now, suppose that $X \in \mathrm{Ind}(\mathfrak{L}_D, I)$. Consider the following mapping

$$p_2(c_i) = \begin{cases} c_1 \vee \cdots \vee c_m & \text{for } i = 1\,, \\ c_i & \text{for } i \neq 1\,, \end{cases}$$

$$p_2(d_j) = \begin{cases} d_1 \wedge \cdots \wedge d_n & \text{for } j = 1\,, \\ d_j & \text{for } j \neq 1\,, \end{cases}$$

where $i = 1, \ldots, m$; $j = 1, \ldots, n$. Since X is I-independent, it follows that p_2 is injective. Then $f\big(p_2(c_1), \ldots, p_2(c_m), p_2(d_1), \ldots, p_2(d_n)\big) = g\big(p_2(c_1), \ldots, p_2(c_m), p_2(d_1), \ldots, p_2(d_n)\big)$, which implies $(c_2 \wedge \cdots \wedge c_m) \wedge (d_2 \vee \cdots \vee d_n) = c_2 \wedge \cdots \wedge c_m$. The family $\mathrm{Ind}(\mathfrak{L}_D, I)$ is hereditery, so after k similar steps ($k = \max\{m, n\}$), we get $c_m \wedge d_n = c_n$. Considering injective mapping p_3 defined as follows: $p_3(c_m) = d_n$, $p_3(d_n) = c_m$, we obtain $c_m = d_n$, a contradiction.

Since $\mathfrak{L}_D$ is without constants, $\mathrm{Ind}(\mathfrak{L}_D, M) = \mathrm{Ind}(\mathfrak{L}_D, G)$ (see [5]). □

Moreover (obviously), we have: $\mathrm{Ind}(\mathfrak{L}_D, A_1) = 2^L$.

Since every $\mathfrak{F}_a$ in Stone algebras is a distributive lattice with the greatest element a, the following corollary will be useful:

COROLLARY 2. *Let $\mathfrak{L}_c = (L; \vee, \wedge, c)$ (where $c = \mathbf{1}$ or $c = \mathbf{0}$) be a distributive lattice with the greatest element $\mathbf{1}$ or the least element $\mathbf{0}$, respectively. If $X \subseteq L$, then the following conditions are equivalent:*

- (α) *$X \in \mathrm{Ind}(\mathfrak{L}_c, Q)$, where $Q = M, S, I$ or $X = \{c\}$;*
- (β) *$X \in \mathrm{Ind}(\mathfrak{L}_c, S_0)$;*
- (γ) *$X \setminus \{c\} \in \mathrm{Ind}(\mathfrak{L}_c, G)$;*
- (δ) *$c_1 \wedge \cdots \wedge c_m \not\leq d_1 \vee \cdots \vee d_n$ for each sequence $c_1, \ldots, c_m, d_1, \ldots, d_n$ of different elements of X.*

Like for Boolean algebras (see [13]), for Stone algebras one can define so-called *atom with respect to the elements* $x_1, \ldots, x_n \in L$ indexed by the sequence $(i_1, \ldots, i_n)$, where $i_k \in \{0, 1, 2\}$ ($k = 1, \ldots, n$), as follows:

$$A_{(i_1,\ldots,i_n)}(x_1, \ldots, x_n) = \bigwedge_{k=1}^{n} x_k^{i_k}\,,$$

where $x^0 = x$, $x^1 = x^*$, $x^2 = x^{**}$. Consider the following operations

$$A_J(x_1, \ldots x_n) = \bigvee \{x_1^{i_1} \wedge \cdots \wedge x_n^{i_n} : (i_1, \ldots, i_n) \in J\}$$

for any non-empty set $J \subseteq \{0, 1, 2\}^n$, $A_\emptyset(x_1, \ldots x_n) = \mathbf{0}$.

We recall (see [7]):

THEOREM 3. *For each n-ary term operation f in a Stone algebra $(L;\vee,\wedge,{}^*,\mathbf{0},\mathbf{1})$ there exists $J \subseteq \{0,1,2\}^n$ such that $f(x_1,\dots,x_n) = A_J(x_1,\dots x_n)$.*

In contrast to the analogous representation for the term operations in Boolean algebras, the subset J is not uniquely determined for Stone algebras.

Now, we note some simple properties of term operations of Stone algebras (and their values, especially for elements from congruence classes F_a).

LEMMA 2. *Let $\mathfrak{L} = (L;\vee,\wedge,{}^*,\mathbf{0},\mathbf{1})$ be a Stone algebra, $x \in L$, and $b_1,\dots,b_n \in F_a$ for some $a \in S(L)$. Then:*

(a) $\big(\forall k \in \{1,\dots,n\}\big)\big((i_k = 1) \implies A_{(i_1,\dots,i_n)}(b_1,\dots,b_n) = a^*$
$\qquad\qquad \& \ A_{(i_1,\dots,i_n)}(x,\dots,x) = x^*\big)$,

(b) $\big(\forall k \in \{1,\dots,n\}\big)\big((i_k \neq 1) \implies A_{(i_1,\dots,i_n)}(b_1,\dots,b_n) \in F_a$
$\qquad\qquad \& \ \big(\forall c \in S(L)\big)\big(A_{(i_1,\dots,i_n)}(c,\dots,c) = c\big)\big)$,

(c) $\big(\exists k,l \in \{1,\dots,n\}\big)\big((i_k = 1 \ \& \ i_l \neq 1) \implies A_{(i_1,\dots,i_n)}(b_1,\dots,b_n) = \mathbf{0}$
$\qquad\qquad = A_{(i_1,\dots,i_n)}(x,\dots,x)\big)$.

P r o o f.

(a): Since $(b_k)^1 = (b_k)^* = a^*$ for all $k \in \{1,\dots,n\}$, we get $A_{(i_1,\dots,i_n)}(b_1,\dots,b_n) = b_1^{i_1} \wedge \dots \wedge b_n^{i_n} = a^*$. Similarly, $A_{(1,\dots,1)}(x,\dots,x) = x^*$.

(b): In this case $(b_k)^0 = b_k \in F_a$ or $(b_k)^2 = (b_k)^{**} = a \in F_a$ for all $k \in \{1,\dots,n\}$. So, $b_1^{i_1} \wedge\dots\wedge b_n^{i_n} \in F_a$. As $c^{**} = c$, we obtain $A_{(i_1,\dots,i_n)}(c,\dots,c) = c$.

(c): Let $i_k = 1$ and $i_l = 2$ for some $k,l \in \{1,\dots,n\}$ (say $k < l$). Then $A_{(i_1,\dots,i_n)}(b_1,\dots,b_n) = b_1^{i_1} \wedge \dots \wedge b_k^{i_k} \wedge \dots \wedge b_l^{i_l} \wedge \dots \wedge b_n^{i_n} = b_1^{i_1} \wedge \dots \wedge a^* \wedge \dots \wedge a \wedge \dots \wedge b_n^{i_n} = \mathbf{0}$. The proof for $i_k = 1$ and $i_l = 0$ runs analogously. Since $x^* \wedge x = x^* \wedge x^{**} = \mathbf{0}$, we get $A_{(i_1,\dots,i_n)}(x,\dots,x) = \mathbf{0}$. □

By Theorem 3 and Lemma 2, we obtain the following corollaries:

COROLLARY 3. *Let $\mathfrak{L}$ be a Stone algebra, $b_1,\dots,b_n \in F_a$ for some $a \in S(L)$, and $f \in \mathbb{T}^{(n)}(\mathfrak{L})$. Then $f(b_1,\dots,b_n) \in F_a \cup \varphi_L(a) \cup \{\mathbf{0},a^*\}$. If, additionally, $a \neq \mathbf{0},\mathbf{1}$, then $F_a \cap \varphi_L(a) \cap \{\mathbf{0},a^*\} = \emptyset$.*

For the next theorem we need some "reduced" form of term operations of Stone algebras. Here and subsequently, we write $\bar{J} = J \cap \{0,2\}^n$.

COROLLARY 4. *Let $\mathfrak{L}$ be a Stone algebra, $b_1,\dots,b_n \in F_a$ for some $a \in S(L)$ and $f(b_1,\dots,b_n) = A_J(b_1,\dots,b_n) \in F_a$ for some $J \in \{0,1,2\}^n$. Then $A_J(b_1,\dots,b_n) = A_{\bar{J}}(b_1,\dots,b_n)$.*

Let $f(x_1,\dots,x_n) = A_J(x_1,\dots x_n)$ for some $J \subseteq \{0,1,2\}^n$. We define a mapping $\phi_1 \colon J \to \{0,1\}^n$ by $\phi_1\big((i_1,\dots,i_n)\big) = \big(i_1 \pmod 2,\dots,i_n \pmod 2\big)$

for each $(i_1,\dots,i_n)\in J\subseteq\{0,1,2\}^n$. Write $\bar f(x_1,\dots,x_n)=A_{\phi_1(J)}(x_1,\dots,x_n)$. We observe that $\bar f\in\mathbb{T}^{(n)}(\mathfrak{S}(\mathfrak{L}))$.

Let now $J\subseteq\{0,2\}^n$, and a mapping $\phi_2\colon J\to 2^{\{1,\dots,n\}}$ be given by

$$\phi_2\big((i_1,\dots,i_n)\big)=\{k\in\mathbb{N}:\ i_k=0\}\qquad\text{for each}\quad (i_1,\dots,i_n)\in J.$$

Taking into account $(**)$, we obtain $\tilde f_{\phi_2(\bar J)}\in\mathbb{T}^{(n)}(\mathfrak{F}_a)$.

Assume that $f_A\colon\{1,\dots,n\}\to\{0,1\}$, for $A\subseteq\{1,\dots,n\}$, is the characteristic function of the set A. We define a mapping $\phi_3\colon 2^{\{1,\dots,n\}}\to\{0,2\}^n$ by

$$\phi_3(X)=(i_1,\dots,i_n),\qquad\text{where}\quad i_k=2f_{X'}(k),\ \ k\in\{1,\dots,n\},$$
$$X'=\{1,\dots,n\}\setminus X,\ \ X\in 2^{\{1,\dots,n\}}.$$

It is easy to check that $A_{\bar J}(b_1,\dots,b_n)=A_{\phi_3(\phi_2(\bar J))}(b_1,\dots,b_n)$ for all $b_1,\dots,b_n\in F_a$.

Lemma 3. *Let $\mathfrak{L}$ be a Stone algebra, and $f(x_1,\dots,x_n)=A_J(x_1,\dots x_n)$ for some $J\subseteq\{0,1,2\}^n$. Then*

(a) $(\forall a_1,\dots,a_n\in S(L))\big(f(a_1,\dots,a_n)=A_{\phi_1(J)}(a_1,\dots,a_n)\big)$,

(b) $(\forall a\in S(L))(\forall b_1,\dots,b_n\in F_a)\Big(f(b_1,\dots,b_n)\in F_a \implies f(b_1,\dots,b_n)=\tilde f_{\phi_2(\bar J)}(b_1,\dots,b_n)\Big)$,

(c) $(\forall a\in S(L))(\forall b_1,\dots,b_n\in F_a)(\forall P\subseteq 2^{\{1,\dots,n\}})\big(\tilde f_P(b_1,\dots,b_n)=A_{\phi_3(P)}(b_1,\dots,b_n)\big)$.

P r o o f.

(a): It is obvious, since $a^{**}=a$ for all $a\in S(L)$.

(b): By Corollary 4, we obtain $f(b_1,\dots,b_n)=A_J(b_1,\dots,b_n)=A_{\bar J}(b_1,\dots,b_n)$, where $\bar J=J\cap\{0,2\}^n$. As $b_i^2=b_i^{**}=a$ and $b_i\wedge a=b_i$ for every $b_i\in F_a$, we get $A_{\bar J}(b_1,\dots,b_n)=\tilde f_{\phi_2(\bar J)}(b_1,\dots,b_n)$.

(c): Taking into account that $b_i\wedge a=b_i$ for every $b_i\in F_a$ $(i=1,\dots,n)$, we can easily verify that $\tilde f_P(b_1,\dots,b_n)=A_{\phi_3(P)}(b_1,\dots,b_n)$. □

Theorem 4. *Let $\mathfrak{L}=(L;\vee,\wedge,{}^*,\mathbf{0},\mathbf{1})$ be a Stone algebra and $X\subseteq F_a$ for some $a\in S(L)$. Then $X\in\operatorname{Ind}(\mathfrak{L},S_0)$ iff $X\in\operatorname{Ind}(\mathfrak{F}_a,S_0)$. Moreover, if $a\neq\mathbf{0},\mathbf{1}$, then $X\in\operatorname{Ind}(\mathfrak{L},A_1)$.*

P r o o f. Let $X\subseteq F_a$. First, we verify the implication

$$X\in\operatorname{Ind}(\mathfrak{L},S_0)\implies X\in\operatorname{Ind}(\mathfrak{F}_a,S_0).\tag{$***$}$$

Take two n-ary term operations f, g on $\mathfrak{F}_a$ such that $f(b_1,\dots,b_n)=g(b_1,\dots,b_n)$ for some $b_1,\dots,b_n\in X$. The operations f, g correspond to two terms of type $(2,2,0)$. These terms can also be interpreted in the algebra

$\mathfrak{L} = (L; \vee, \wedge, {}^*, 0, 1)$ as term operations $f_1, g_1 \in \mathbb{T}^{(n)}(\mathfrak{L})$, respectively. Hence, we have $f_1(b_1, \dots, b_n) = f(b_1, \dots, b_n) = g(b_1, \dots, b_n) = g_1(b_1, \dots, b_n)$. Since $b_1, \dots, b_n \in X \in \mathrm{Ind}(\mathfrak{L}, S_0)$, we get

$$\begin{aligned} f_1\big(p(b_1), \dots, p(b_n)\big) &= f\big(p(b_1), \dots, p(b_n)\big) \\ &= g\big(p(b_1), \dots, p(b_n)\big) = g_1\big(p(b_1), \dots, p(b_n)\big) \end{aligned}$$

for all $p \colon X \to X$. Therefore, $X \in \mathrm{Ind}(\mathfrak{F}_a, S_0)$.

Now, we prove the converse implication of $(***)$.

Let $X \in \mathrm{Ind}(\mathfrak{F}_a, S_0)$, $p \in X^X$. Then $p(b_i) \in F_a$ for all $b_i \in X$ $(i = 1, \dots, n)$. Assume that $f(b_1, \dots, b_n) = g(b_1, \dots, b_n)$ for some $f, g \in \mathbb{T}^{(n)}(\mathfrak{L})$. Theorem 3 now gives $f(b_1, \dots, b_n) = A_{J_1}(b_1, \dots, b_n)$, $g(b_1, \dots, b_n) = A_{J_2}(b_1, \dots, b_n)$ for some $J_1, J_2 \in \{0, 1, 2\}^n$.

According to Corollary 3, we need to consider the following four cases:

(1) $f(b_1, \dots, b_n) = \mathbf{0}$,
(2) $f(b_1, \dots, b_n) = a^*$,
(3) $f(b_1, \dots, b_n) \in F_a$,
(4) $f(b_1, \dots, b_n) \in \varphi_L(a)$.

(1) and (2): In these cases all the atoms of f and g are equal to $\mathbf{0}$ or a^*. By Lemma 2(a), (c), we obtain $f\big(p(b_1), \dots, p(b_n)\big) = g\big(p(b_1), \dots, p(b_n)\big)$.

(3): By Corollary 4, we get $f(b_1, \dots, b_n) = A_{\bar{J}_1}(b_1, \dots, b_n) = A_{\bar{J}_2}(b_1, \dots, b_n) = g(b_1, \dots, b_n)$. Then $f(b_1, \dots, b_n) = \tilde{f}_{\phi_2(\bar{J}_1)}(b_1, \dots, b_n) = \tilde{f}_{\phi_2(\bar{J}_2)}(b_1, \dots, b_n) = g(b_1, \dots, b_n)$ and $\tilde{f}_{\phi_2(\bar{J}_i)} \in \mathbb{T}^{(n)}(\mathfrak{F}_a)$ for $i = 1, 2$ by Lemma 3(b). Therefore,

$$\tilde{f}_{\phi_2(\bar{J}_1)}\big(p(b_1), \dots, p(b_n)\big) = \tilde{f}_{\phi_2(\bar{J}_2)}\big(p(b_1), \dots, p(b_n)\big)$$

since $X \in \mathrm{Ind}(\mathfrak{F}_a, S_0)$.

Lemma 3 (c) now implies

$$\tilde{f}_{\phi_2(\bar{J}_i)}\big(p(b_1), \dots, p(b_n)\big) = A_{\phi_3(\phi_2(\bar{J}_i))}\big(p(b_1), \dots, p(b_n)\big) = A_{\bar{J}_i}\big(p(b_1), \dots, p(b_n)\big)$$

for $i = 1, 2$. Consequently, $f\big(p(b_1), \dots, p(b_n)\big) = g\big(p(b_1), \dots, p(b_n)\big)$.

(4): Theorem 3 and Lemma 2 now give

$$f(b_1, \dots, b_n) = A_{\bar{J}_1}(b_1, \dots, b_n) \vee a^* = A_{\bar{J}_2}(b_1, \dots, b_n) \vee a^* = g(b_1, \dots, b_n).$$

Since $\phi \colon F_a \to \varphi_L(a)$ defined by $\phi(x) = x \vee a^*$ is a bijection, we have $A_{\bar{J}_1}(b_1, \dots, b_n) = A_{\bar{J}_2}(b_1, \dots, b_n) \in F_a$. According to the above results, we get $A_{\bar{J}_1}\big(p(b_1), \dots, p(b_n)\big) = A_{\bar{J}_2}\big(p(b_1), \dots, p(b_n)\big)$ for $p \in A_1 \cup S_0$. Therefore, $A_{\bar{J}_1}\big(p(b_1), \dots, p(b_n)\big) \vee a^* = A_{\bar{J}_2}\big(p(b_1), \dots, p(b_n)\big) \vee a^*$ and, in consequence, $f\big(p(b_1), \dots, p(b_n)\big) = g\big(p(b_1), \dots, p(b_n)\big)$.

Now, we prove that $X \in \mathrm{Ind}(\mathfrak{L}, A_1)$ for $X \subseteq F_a$, $a \in S(L)\setminus\{\mathbf{0},\mathbf{1}\}$. Obviously, $a^* \neq \mathbf{0}$, $a^* \notin \varphi_L(a)$ and $F_a \cap \varphi_L(a) = \emptyset$.

Consider all unary term operations as mappings on X. Therefore, we have six mappings $p_j\colon X \to L$, namely: $p_j(x) = x^i$ for $j = 0,1,2$, $p_3(x) = \mathbf{0}$, $p_4(x) = \mathbf{1}$, and $p_5(x) = \phi(x) = x \vee x^*$.

For every $b \in F_a$, we have $p_0(b) = b$, $p_1(b) = a^*$, $p_2(b) = a$, $p_3(b) = \mathbf{0}$, and $p_4(b) = \mathbf{1}$. Moreover, $p_5(b) = b \vee b^* \in D(L) = F_{\mathbf{1}}$.

Let $f(b_1,\dots,b_n) = g(b_1,\dots,b_n)$ for some $b_1,\dots,b_n \in X$ and $f,g \in \mathbb{T}^{(n)}(\mathfrak{L})$.

For the rest of our proof we need to consider cases (1)–(4), mentioned above.

First, suppose that $f(b_1,\dots,b_n) = \mathbf{0}$. Then all the atoms of f and g are equal to $\mathbf{0}$. By Lemma 2(c), we obtain $f\big(p_j(b_1),\dots,p_j(b_n)\big) = \mathbf{0} = g\big(p_j(b_1),\dots,p_j(b_n)\big)$ for $j = 0,\dots,5$.

Let $f(b_1,\dots,b_n) = a^*$. Then all the atoms of f and g are equal to $\mathbf{0}$ or a^*. Therefore, $f\big(p_j(b_1),\dots,p_j(b_n)\big) = [p_j(b_k)]^* = g\big(p_j(b_1),\dots,p_j(b_n)\big)$ for $j = 0,\dots,5$ by Lemma 2(a), (c).

Suppose now $f(b_1,\dots,b_n) \in F_a$. By Corollary 4, we obtain

$$f(b_1,\dots,b_n) = A_{\bar{J}_1}(b_1,\dots,b_n) = A_{\bar{J}_2}(b_1,\dots,b_n) = g(b_1,\dots,b_n)\,.$$

Since $p_j(b_k) \in S(L)$ for $j = 1,\dots,4$, we have $f\big(p_j(b_1),\dots,p_j(b_n)\big) = p_j(b_k) = g\big(p_j(b_1),\dots,p_j(b_n)\big)$ by Lemma 2(b). As $\phi|_{F_a}$ is a lattice-homomorphism, we obtain $f\big(p_5(b_1),\dots,p_5(b_n)\big) = g\big(p_5(b_1),\dots,p_5(b_n)\big)$.

The consideration of case (4) is similar to the same case for S_0-independence.

By Proposition 1(ii), it is evident that $X \notin \mathrm{Ind}(\mathfrak{L}, A_1)$ for every $X \subseteq D(L)$. □

Now, we have the following Proposition.

PROPOSITION 3. *Let $\mathfrak{L}$ be a Stone algebra and $X \subseteq D(L)$. Then*

$$X \in \mathrm{Ind}(\mathfrak{L}, G) \iff X \setminus \{\mathbf{1}\} \in \mathrm{Ind}\big(\mathfrak{D}(\mathfrak{L}), G\big)\,.$$

Proof. First, we observe that the implication

$$X \in \mathrm{Ind}(\mathfrak{L}, G) \implies X \setminus \{\mathbf{1}\} \in \mathrm{Ind}\big(\mathfrak{D}(\mathfrak{L}), G\big)$$

can be verified similarly to $(***)$ of the proof of Theorem 4.

To prove the converse implication, we can assume that $\mathbf{1} \notin X \in \mathrm{Ind}\big(\mathfrak{D}(\mathfrak{L}), G\big)$ since $X \in \mathrm{Ind}(\mathfrak{L}, G)$ iff $X \setminus \{\mathbf{1}\} \in \mathrm{Ind}(\mathfrak{L}, G)$ (see [5]).

Let $f(b_1,\dots,b_n) = g(b_1,\dots,b_n)$ for some $b_1,\dots,b_n \in X$, $f,g \in \mathbb{T}^{(n)}(\mathfrak{L})$. By Theorem 3, we have $f(x_1,\dots,x_n) = A_{J_1}(x_1,\dots,x_n)$, $g(x_1,\dots,x_n) = A_{J_2}(x_1,\dots,x_n)$ for some $J_1, J_2 \in \{0,1,2\}^n$. Therefore,

$$f(b_1,\dots,b_n) = A_{J_1}(b_1,\dots,b_n) = A_{J_2}(b_1,\dots,b_n) = g(b_1,\dots,b_n)\,.$$

Lemma 2 now yields $A_{(i_1,\dots,i_n)}(b_1,\dots,b_n) = \mathbf{0}$ if $i_k = 1$ for some $k = 1,\dots,n$ and, in consequence, $f(b_1,\dots,b_n) = A_{\bar{J}_1}(b_1,\dots,b_n) = A_{\bar{J}_2}(b_1,\dots,b_n) = g(b_1,\dots,b_n)$. Then $\tilde{f}_{\phi_2(J_1)}(b_1,\dots,b_n) = \tilde{f}_{\phi_2(J_2)}(b_1,\dots,b_n)$ and $\tilde{f}_{\phi_2(J_i)} \in \mathbb{T}^{(n)}(\mathfrak{D}(\mathfrak{L}))$ for $i = 1,2$ by Lemma 3(b). An arbitrary mapping p: $\{b_1,\dots,b_n\} \to D(L)$ is diminishing, by Corollary 1. Since $X \in \operatorname{Ind}(\mathfrak{D}(\mathfrak{L}),G)$, it follows that

$$\tilde{f}_{\phi_2(J_1)}(p(b_1),\dots,p(b_n)) = \tilde{f}_{\phi_2(J_2)}(p(b_1),\dots,p(b_n)) .$$

By Lemma 3(c), we obtain $\tilde{f}_{\phi_2(J_i)}(p(b_1),\dots,p(b_n)) = A_{\phi_3(\phi_2(J_i))}(p(b_1),\dots,p(b_n)) = A_{J_i}(p(b_1),\dots,p(b_n))$ for $i = 1,2$. Hence, $f(p(b_1),\dots,p(b_n)) = g(p(b_1),\dots,p(b_n))$. □

5. Connections between Q-independence in $\mathfrak{L}$ and $\mathfrak{S}(\mathfrak{L})$

In the Boolean algebra $\mathfrak{B}$ we have (see [5])

$$\operatorname{Ind}(\mathfrak{B},M) = \operatorname{Ind}(\mathfrak{B},S) = \operatorname{Ind}(\mathfrak{B},S_0) \setminus \{\{\mathbf{0}\},\{\mathbf{1}\}\}$$

and

$$I \in \operatorname{Ind}(\mathfrak{B},G) \iff I \setminus \{\mathbf{0},\mathbf{1}\} \in \operatorname{Ind}(\mathfrak{B},M) .$$

We also recall (see [13]):

THEOREM 5. *Let $\mathfrak{B} = (B;\vee,\wedge,^*,\mathbf{0},\mathbf{1})$ be a Boolean algebra and $X \subseteq B$. Then $X \in \operatorname{Ind}(\mathfrak{B},M)$ iff $A_{(i_1,\dots,i_n)}(a_1,\dots,a_n) \neq \mathbf{0}$ for each sequence $(i_1,\dots,i_n) \in \{0,1\}^n$ and for every different $a_1,\dots,a_n \in X$.*

Now, we characterize the family of t-independent subsets of Stone algebra $\mathfrak{L}$ using corresponding elements from its skeleton (i.e. the Boolean algebra $\mathfrak{S}(\mathfrak{L}) = (S(L);\vee,\wedge,^*,\mathbf{0},\mathbf{1})$).

THEOREM 6. *Let $\mathfrak{L} = (L;\vee,\wedge,^*,0,1)$ be a Stone algebra, $X \subseteq L$. Then $X \in t\text{-}\operatorname{Ind}_{\mathfrak{L}}$ iff for any finite system of different elements $b_1,\dots,b_n \in X$ there exist $a_1,\dots,a_n \in S(L)$ such that $a_i \neq b_i \in F_{a_i}$ $(i = 1,\dots,n)$ and $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{S}(\mathfrak{L}),M)$.*

P r o o f. Let $X \in t\text{-}\operatorname{Ind}_{\mathfrak{L}}$ and $b_1,\dots,b_n$ be different elements of X. By Propositions 1(iii) and 2, we get $b_i \in F_{a_i}$ and $b_i \neq a_i$ $(i = 1,\dots,n)$ for some different $a_1,\dots,a_n \in S(L) \setminus \{\mathbf{0},\mathbf{1}\}$.

Suppose, contrary to our claim, that $\{a_1,\dots,a_n\} \notin \operatorname{Ind}(\mathfrak{S}(\mathfrak{L}),M)$. By Theorem 5, we have $a_1^{i_1} \wedge \dots \wedge a_n^{i_n} = \mathbf{0}$ for some $i_k \in \{0,1\}$ $(k = 1,\dots,n)$. Then $b_1^{i_1} \wedge \dots \wedge b_n^{i_n} \in F_0 = \{\mathbf{0}\}$. So, we get $b_1^{i_1} \wedge \dots \wedge b_n^{i_n} = \mathbf{0}$.

First, suppose that $i_1 = \cdots = i_n = 0$. Then $b_1^* \vee \cdots \vee b_n^* = \mathbf{1}$, which gives $b_1 = b_1 \wedge (b_1^* \vee \cdots \vee b_n^*) = b_1 \wedge (b_2^* \vee \cdots \vee b_n^*)$, by distributivity. Consider $f(x_1, \dots, x_n) = x_1 \wedge (x_2^* \vee \cdots \vee x_n^*)$. Then we have $f(b_1, \dots, b_n) = b_1$, $f(\mathbf{1}, \dots, \mathbf{1}) = \mathbf{0}$ and $f \in \mathbb{T}^{(n)}(\mathfrak{L})$.

Now, let $i_k = 1$ for some $k \in \{1, \dots, n\}$. Without any loss of generality, we can assume that $k = 1$. So $b_1^* \wedge b_2^{i_2} \wedge \cdots \wedge b_n^{i_n} = \mathbf{0}$. Then $b_1 \vee (b_1^* \wedge b_2^{i_2} \wedge \cdots \wedge b_n^{i_n}) = b_1$. Put $g(x_1, \dots, x_n) = x_1 \vee (x_1^* \wedge x_2^{i_2} \wedge \cdots \wedge x_n^{i_n})$. Thus, we get $g(b_1, \dots, b_n) = b_1$, $g \in \mathbb{T}^{(n)}(\mathfrak{L})$ and g is not a projection. Hence, we have a contradiction in both cases.

Finally, suppose that for any finite system of different elements $b_1, \dots, b_n \in X$ there exist different $a_1, \dots, a_n \in S(L)$ such that $a_i \neq b_i \in F_{a_i}$ $(i = 1, \dots, n)$ and $\{a_1, \dots, a_n\} \in \mathrm{Ind}(\mathfrak{S}(\mathfrak{L}), M)$.

Assume $X \notin t\text{-}\mathrm{Ind}_{\mathfrak{L}}$. Therefore, $b_k = f(b_1, \dots, b_n)$ for some $f \in \mathbb{T}^{(n)}(\mathfrak{L})$. Then $f(b_1, \dots, b_n) \in F_{f(a_1, \dots, a_n)}$ and $b_k \in F_{a_k}$. We thus get $a_k = f(a_1, \dots, a_n) = \bar{f}(a_1, \dots, a_n)$ and $\bar{f} \in \mathbb{T}^{(n)}(\mathfrak{S}(\mathfrak{L}))$ by the definition of ϕ_1 and Lemma 3(a). This gives $\{a_1, \dots, a_n\} \notin t\text{-}\mathrm{Ind}_{\mathfrak{S}(\mathfrak{L})}$. But in every Boolean algebra $\mathfrak{B}$ we have $\mathrm{Ind}(\mathfrak{B}, M) = t\text{-}\mathrm{Ind}_{\mathfrak{B}}$ (see [8]). Hence, we have a contradiction. $\square$

Proposition 1 deals with one-element subsets. Now, we will consider subsets with at least two elements.

Lemma 4. *Let $\mathfrak{L}$ be a Stone algebra, $a \in D(L)$ and $b \notin D(L)$ (or $a = \mathbf{0}$ and $b \neq \mathbf{0}$). Then $\{a, b\} \notin \mathrm{Ind}(\mathfrak{L}, S_0)$.*

Proof. Let $a \in D(L)$ and $b \notin D(L)$. Consider two binary term operations $f(x, y) = x^* \wedge y$ and $g(x, y) = x^* \wedge y^*$ and a mapping $p\colon \{a, b\} \to \{a, b\}$, $p(x) = b$. Then $f(a, b) = g(a, b)$, but $f\big(p(a), p(b)\big) = f(b, b) = \mathbf{0} \neq b^* = g\big(p(a), p(b)\big)$.

Now, suppose that $a = \mathbf{0}$ and $b \neq \mathbf{0}$. Consider the following term operations $f(x, y) = x \wedge y$ and $g(x, y) = \mathbf{0}$. Therefore, $f(a, b) = g(a, b)$ and $f\big(p(a), p(b)\big) = f(b, b) = b \neq \mathbf{0} = g\big(p(a), p(b)\big)$. So $\{a, b\} \notin \mathrm{Ind}(\mathfrak{L}, S_0)$. $\square$

Proposition 4. *Let $\mathfrak{L}$ be a Stone algebra, $b_i \in F_{a_i}$ $(i = 1, \dots, n,\ n > 1)$ for some different $a_1, \dots, a_n \in S(L)$. If $\{a_1, \dots, a_n\} \notin \mathrm{Ind}\big(\mathfrak{S}(\mathfrak{L}), M\big)$, then $\{b_1, \dots, b_n\} \notin \mathrm{Ind}(\mathfrak{L}, Q)$, where $Q = M, S, S_0$ or I.*

Proof. Suppose that $b_i \in F_{a_i}$ $(i = 1, \dots, n,\ n > 1)$ for some different $a_1, \dots, a_n \in S(L)$ and $\{a_1, \dots, a_n\} \notin \mathrm{Ind}\big(\mathfrak{S}(\mathfrak{L}), M\big)$. According to Theorem 5, we have $a_1^{i_1} \wedge \cdots \wedge a_n^{i_n} = \mathbf{0}$ for some $i_k \in \{0, 1\}$ $(k = 1, \dots, n)$. Then $b_1^{i_1} \wedge \cdots \wedge b_n^{i_n} = \mathbf{0}$. Consider the following n-ary term operations $f(x_1, \dots, x_n) = x^{i_1} \wedge \cdots \wedge x^{i_n}$, $g(x_1, \dots, x_n) = \mathbf{0}$. Of course, $f(b_1, \dots, b_n) = g(b_1, \dots, b_n)$ and $f \neq g$.

First, suppose that $\{b_1,\dots,b_n\} \in \operatorname{Ind}(\mathfrak{L},S_0)$. Consider a mapping $p\colon \{b_1,\dots,b_n\} \to \{b_1,\dots,b_n\}$, given by

$$p(b_k)=\begin{cases} b_1 & \text{for } i_k=0\,,\\ b_2 & \text{for } i_k=1\end{cases} \qquad (k=1,\dots,n).$$

Then $f\big(p(b_1),\dots,p(b_n)\big) = g\big(p(b_1),\dots,p(b_n)\big)$. If $i_k = 0$ (or $i_k = 1$) for all $k=1,\dots,n$, then we obtain $b_1=\mathbf{0}$ (respectively $b_2^*=\mathbf{0}$, i.e. $b_2\in D(L)$) and we get a contradiction by Lemma 4. In the case $i_k=0$, $i_l=1$ for some $k,l\in\{1,\dots,n\}$, we get $b_1\wedge b_2^*=\mathbf{0}$, so $a_1\wedge a_2^*=\mathbf{0}$ and $a_1\le a_2^{**}=a_2$. Since the family $\operatorname{Ind}(\mathfrak{L},S_0)$ is hereditary, we obtain $\{b_1,b_2\}\in \operatorname{Ind}(\mathfrak{L},S_0)$. We can consider two binary term operations $f_1(x,y)=x\wedge y^*$, $g_1(x,y)=\mathbf{0}$ and a mapping $p_1\colon \{b_1,b_2\}\to\{b_1,b_2\}$, defined by $p_1(b_1)=b_2$, $p_1(b_2)=b_1$. Then $f_1\big(p_1(b_1),p_1(b_2)\big)=f_1(b_2,b_1)=b_2\wedge b_1^*=\mathbf{0}=g_1(b_2,b_1)$. So, $a_2\wedge a_1^*=\mathbf{0}$ and $a_2\le a_1$. Therefore, $a_1=a_2$, which is a contradiction.

By $(*)$ the same conclusion can be drawn for M-independence and S-independence.

Now, let $\{b_1,\dots,b_n\}\in\operatorname{Ind}(\mathfrak{L},I)$. Proposition 1 gives now $\{b_1,\dots,b_n\}\cap S(L)=\emptyset$. Consider the injective mapping

$$p_2(b_k)=\begin{cases} \mathbf{1} & \text{for } k=1 \text{ and } i_1=0\,,\\ \mathbf{0} & \text{for } k=1 \text{ and } i_1=1\,,\\ b_k & \text{for } k>1\,.\end{cases}$$

Therefore, $f\big(p_2(b_1),\dots,p_2(b_n)\big) = g\big(p_2(b_1),\dots,p_2(b_n)\big)$, which yields $b_2^{i_2}\wedge\dots\wedge b_n^{i_n}=\mathbf{0}$. Since the family $\operatorname{Ind}(\mathfrak{L},I)$ is hereditary, after $n-1$ similar steps, we get $b_n=\mathbf{0}$ or $b_n^*=\mathbf{0}$, contrary to Lemma 4. □

PROPOSITION 5. *Let $\mathfrak{L}$ be a Stone algebra, $X=\{b_k:\ b_k\in F_{a_k}\ a_k\in S(L),\ a_k$ different, $k\in K\}$ and $Y=\{a_k:\ (\exists k\in K)(b_k\in X\cap F_{a_k})\}$. If $Y\setminus\{\mathbf{0},\mathbf{1}\}\notin \operatorname{Ind}\big(\mathfrak{S}(\mathfrak{L}),M\big)$, then $X\notin\operatorname{Ind}(\mathfrak{L},G)$.*

Proof. Let $Y\setminus\{\mathbf{0},\mathbf{1}\}\notin\operatorname{Ind}\big(\mathfrak{S}(\mathfrak{L}),M\big)$. Contrary to our claim, suppose that $X\in\operatorname{Ind}(\mathfrak{L},G)$. By Theorem 5, there exist $a_1,\dots,a_n\in Y$ $(n\le\operatorname{card}(Y))$ such that $a_1^{i_1}\wedge\dots\wedge a_n^{i_n}=\mathbf{0}$ for some $i_k\in\{0,1\}$ $(k=1,\dots,n)$. Then $b_1^{i_1}\wedge\dots\wedge b_n^{i_n}=\mathbf{0}$. Consider the following n-ary term operations $f(x_1,\dots,x_n)=x_1^{i_1}\wedge\dots\wedge x_n^{i_n}$, $g(x_1,\dots,x_n)=\mathbf{0}$. Of course, $f(b_1,\dots,b_n)=g(b_1,\dots,b_n)$ and $f\ne g$. According to Corollary 1, the following mapping

$$p(b_k)=\begin{cases} \mathbf{1} & \text{for } i_k=0\,,\\ \mathbf{0} & \text{for } i_k=1\,.\end{cases}$$

is diminishing. Since the family $\operatorname{Ind}(\mathfrak{L},G)$ is hereditary, we get $f\big(p(b_1),\dots,p(b_n)\big) = g\big(p(b_1),\dots,p(b_n)\big)$, which yields $\mathbf{1}=\mathbf{0}$, a contradiction. □

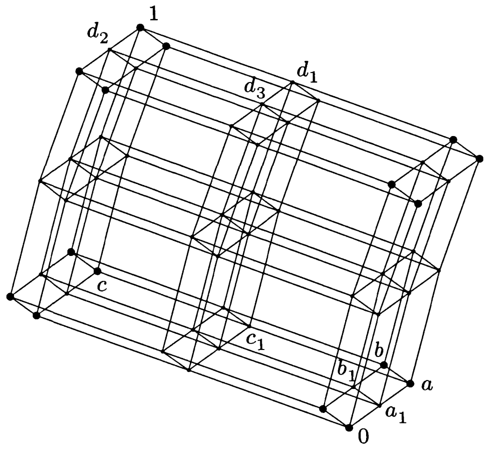

FIGURE 1.

The converse statements of Propositions 4 and 5 do not hold.

EXAMPLE. Consider the Stone algebra $\mathfrak{L}$ presented in Figure 1 (it is a direct product of the 2-element Stone algebra by three copies of the 3-element Stone algebra). It is easy to check that $\{b,c\} \in \operatorname{Ind}(\mathfrak{S}(\mathfrak{L}),M)$. But $\{b_1,c_1\} \notin \operatorname{Ind}(\mathfrak{L},Q)$, where $Q = M,S,S_0,I,G$. Indeed, consider two binary term operations $f(x,y) = x \wedge y$ and $g(x,y) = x \wedge y^{**}$. Then $f(b_1,c_1) = a_1 = b_1 \wedge c_1^{**} = g(b_1,c_1)$. If $p(b_1) = c_1$ and $p(c_1) = b_1$, then $p \in M \cup S \cup S_0 \cup G \cup I$. We have $f(p(b_1),p(c_1)) = f(c_1,b_1) = a_1$ and $g(p(b_1),p(c_1)) = c_1 \wedge b_1^{**} = c_1 \wedge b = a$.

PROPOSITION 6. *Let $\mathfrak{L}$ be a Stone algebra and $a_1,\dots,a_n$ be different elements of $S(L)$. If $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{S}(\mathfrak{L}),M)$, then $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{L},Q)$, where $Q = S, S_0$, and G.*

Proof. Let $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{S}(\mathfrak{L}),M)$, $f(a_1,\dots,a_n) = g(a_1,\dots,a_n)$ for some $f,g \in \mathbb{T}^{(n)}(\mathfrak{L})$ and $p\colon \{a_1,\dots,a_n\} \to \langle a_1,\dots,a_n\rangle_{\mathfrak{L}}$. By M-independence, we get $f = g$ in $\mathfrak{S}(\mathfrak{L})$. As $\mathfrak{S}(\mathfrak{L})$ is a subalgebra of $\mathfrak{L}$, we have $p(a_i) \in S(L)$. So $f(p(a_1),\dots,p(a_n)) = g(p(a_1),\dots,p(a_n))$. This gives $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{L},S) \subseteq \operatorname{Ind}(\mathfrak{L},S_0)$.

Since all diminishing mappings in $\mathfrak{L}$ preserves $S(L)$, we can see that $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{L},G)$ by the same method. □

PROPOSITION 7. *Let $\mathfrak{L}$ be a Stone algebra and $b_i \in F_{a_i}$, $i = 1,\dots,n$, for some different $a_1,\dots,a_n \in S(L)$. If $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{S}(\mathfrak{L}),M)$ and $\phi(b_1) = \dots = \phi(b_n)$, then $\{b_1,\dots,b_n\} \in \operatorname{Ind}(\mathfrak{L},A_1)$.*

Proof. Suppose that $b_i \in F_{a_i}$, $i = 1,\dots,n$, for some different $a_1,\dots,a_n \in S(L)$, $\{a_1,\dots,a_n\} \in \operatorname{Ind}(\mathfrak{S}(\mathfrak{L}),M)$, $\phi(b_1) = \dots = \phi(b_n)$ and $f(b_1,\dots,b_n) =$

$g(b_1,\dots,b_n)$ for some $f,g \in \mathbb{T}^{(n)}(\mathfrak{L})$. Since $f(b_1,\dots,b_n) \in F_{f(a_1,\dots,a_n)}$ and $g(b_1,\dots,b_n) \in F_{g(a_1,\dots,a_n)}$, it follows that $f(a_1,\dots,a_n) = g(a_1,\dots,a_n)$. Then $f = g$ in $\mathfrak{S}(\mathfrak{L})$ by M-independence.

The set of unary term operations in Stone algebras consists of six elements: $p_1(x) = x$, $p_2(x) = x^*$, $p_3(x) = x^{**}$, $p_4(x) = \mathbf{0}$, $p_5(x) = \mathbf{1}$ and $\phi(x) = x \vee x^*$. The equality $f\big(p_i(b_1),\dots,p_i(b_n)\big) = g\big(p_i(b_1),\dots,p_i(b_n)\big)$ is obvious for $i = 1$. For $i = 2,\dots,5$ we see that $p_i(b_k) \in S(L)$ for $k = 1,\dots,n$, so we have the same equality.

It remains to be proved that $f\big(\phi(b_1),\dots,\phi(b_n)\big) = g\big(\phi(b_1),\dots,\phi(b_n)\big)$. By Theorem 1, we obtain $f(x_1,\dots,x_n) = A_{J_1}(x_1,\dots,x_n)$ and $g(x_1,\dots,x_n) = A_{J_2}(x_1,\dots,x_n)$ for some $J_1, J_2 \in \{0,1,2\}^n$.

As $\phi(x) \in D(L)$ for all $x \in L$, we get $\left[\phi(b_i)\right]^* = \mathbf{0}$ and $\left[\phi(b_i)\right]^{**} = \mathbf{1}$. Therefore, $f\big(\phi(b_1),\dots,\phi(b_n)\big) = A_{\bar{J}_1}\big(\phi(b_1),\dots,\phi(b_n)\big) = \tilde{f}_{\phi_2(\bar{J}_1)}\big(\phi(b_1),\dots,\phi(b_n)\big)$, where $\tilde{f}_{\phi_2(\bar{J}_1)}$ is defined in Section 4. In the same manner, we can see that $g\big(\phi(b_1),\dots,\phi(b_n)\big) = \tilde{f}_{\phi_2(\bar{J}_2)}\big(\phi(b_1),\dots,\phi(b_n)\big)$, and $\tilde{f}_{\phi_2(\bar{J}_2)} \in \mathbb{T}^{(n)}\big(\mathfrak{D}(\mathfrak{L})\big)$. Since $\mathfrak{D}(\mathfrak{L})$ is a distributive lattice and the term operations $\vee$, $\wedge$ are idempotent, we conclude that $f\big(\phi(b_1),\dots,\phi(b_n)\big) = \tilde{f}_{\phi_2(\bar{J}_1)}\big(\phi(b_1),\dots,\phi(b_1)\big) = \tilde{f}_{\phi_2(\bar{J}_2)}\big(\phi(b_1),\dots,\phi(b_1)\big) = g\big(\phi(b_1),\dots,\phi(b_n)\big)$. □

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