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# ALMOST EVERY BIPARTITE GRAPH HAS NOT TWO VERTICES OF MINIMUM DEGREE

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ABSTRACT. We rectify the assertion of Z. Palka [5] concerning the limit distribution of minimum (maximum) degrees in a random bipartite graph $K_{m,n,p}$, where $m$ and $n$ are growing nearly equally.

By Palka, in this case, $K_{m,n,p}$ has, almost surely, two vertices of the minimum (maximum) degree, one in each part of $K_{m,n,p}$. Contrary to this, we prove that the difference between minimum (maximum) degrees in the parts of $K_{m,n,p}$ is almost surely not bounded above by an arbitrary slowly growing function $\omega(n)$, $\omega(n) \to \infty$ as $n \to \infty$.

## Introduction

Let $K_{m,n}$ be a complete bipartite graph with $m$ labelled vertices in one part and $n$ labelled vertices in the other part. Let each edge of $K_{m,n}$ be independently removed with the same probability $q = 1 - p$. Denote the resulting random graph by $K_{m,n,p}$. Obviously, then each of $n \cdot m$ possible edges remains in $K_{m,n,p}$ with the same probability $p$, independently of the other.

We distinguish vertices of different parts from $K_{m,n,p}$ by colouring all $m$ labelled vertices of one part red and all $n$ labelled vertices of the other part with blue.

A random graph $K_{m,n,p}$ has a property $Q$ almost surely (a.s.) if probability $\mathbf{P}(K_{m,n,p} \text{ has } Q) \to 1$ as $n \to \infty$.

The symbol $\mathbf{E}(X)$ denotes the mean of a random variable $X$, $\mathbf{E}_k(X)$ denotes the $k$-th factorial moment of $X$.

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As we shall frequently encounter the binomial distribution we write

$$b(k;n,p) = \binom{n}{k} p^k q^{n-k},$$

$$B(t;n,p) = \sum_{k=0}^{t} b(k;n,p).$$

For simplicity $p$ shall be omitted from $b(k;n,p)$ and $B(t;n,p)$ in cases when misunderstanding is excluded.

We use the following well-known approximation of the binomial distribution (see, e.g. [4, Chap. 7]):

If $t = np - x \cdot (npq)^{1/2}$ where $p = p(n)$, $0 < p < 1$ and $x = x(n)$ is a function such that $x(n) \to \infty$ as $n \to \infty$ but $x^3 \cdot (npq)^{-1/2} \to 0$, then

$$B(t;n,p) \backsim (2\pi)^{-1/2} \frac{1}{x} \mathrm{e}^{-x^2/2}, \tag{1}$$

$$b(t;n,p) \backsim (2\pi npq)^{-1/2} \mathrm{e}^{-x^2/2}. \tag{2}$$

Denote by $d(u)$ the degree of a vertex $u$ and $\delta_b$ ($\delta_r$) the minimum degree of blue (red) vertices in a random bipartite graph $K_{m,n,p}$.

We estimate the difference between minimum degrees of both parts to show that in $K_{m,n,p}$ with $m$ and $n$ growing nearly equally, a vertex of minimum degree is a.s. unique.

We note that Erdős and Wilson [3] showed that almost every graph has a unique vertex of minimum degree and a unique vertex of maximum degree. The random graph $K_{n,p}$ has this property in case $pn/\log n \to \infty$ and $p \leq 0.5$ as was shown by Bollobás [2].

## Results

Using the following lemmas we can bound the $\delta_r$, the minimum degree of the red vertices in $K_{m,n,p}$. The next result is a direct consequence of the more general Lemma 1 in [5].

**LEMMA 1.** *Let the random variable $Y_t$ denote the number of red vertices of degree at most $t$ in $K_{m,n,p}$.*

*Let $t = t(n)$ and $T = T(n)$ be natural numbers such that $\mathbf{E}(Y_t) = o(1)$, $\mathbf{E}(Y_T) = o(1)^{-1}$ and $\mathbf{E}_2(Y_T) \leq \mathbf{E}(Y_T)^2\big(1 + o(1)\big)$.*

*Then*

$$t < \delta_r < T \qquad a.s.$$

**LEMMA 2.** *Let* $m = c \cdot n$ *with* $c = c(n)$ *such that* $\liminf_{n\to\infty} c(n) > 0$ *and*

$$t = np - \sqrt{npq} \cdot \left[\sqrt{2 \cdot logn} - \frac{\log\log n}{\sqrt{8 \cdot \log n}} + \frac{\varphi(n)}{\sqrt{2 \cdot \log n}}\right],$$

$$T = np - \sqrt{npq} \cdot \left[\sqrt{2 \cdot \log n} - \frac{\log\log n}{\sqrt{8 \cdot \log n}} - \frac{\varphi(n)}{\sqrt{2 \cdot \log n}}\right],$$

*where* $\varphi(n) \to \infty$ *as* $n \to \infty$.

*If* $p = p(n)$, $0 < p < 1$ *such that* $\frac{(\log n)^3}{npq} \to 0$ *as* $n \to \infty$, *then* $t < d_r < T$ *a.s.*

Proof. The proof is similar to the proof of Theorem 1 in [5]. Obviously $\mathbf{E}(Y_t) = n \cdot B(t; m, p)$. It is therefore enough to show that this choice of $t$ and $T$ satisfies the conditions of Lemma 1. It can be done easily by estimate (1) for the tail of the binomial distribution. □

Having obtained upper and lower bounds for the minimum degree of the red vertices in $K_{m,n,p}$, we turn to the study of the difference between the minimum degree of the blue vertices and the minimum degree of the red ones.

**THEOREM 1.** *Let* $\alpha(n) \to 0$, $\frac{m}{n} \to 1$ *as* $n \to \infty$.

*If the edge probability* $p$ *satisfies* $\frac{(\log n)^3}{npq} \to 0$ *as* $n \to \infty$, *then for the minimum degree of the red and blue vertices in a random bipartite graph* $K_{m,n,p}$ *it is valid that*

$$|\delta_r - \delta_b| > \frac{\alpha(n) \cdot \sqrt{pqn}}{\log n} \qquad a.s.$$

Proof. Put $\varphi(n) = \frac{1}{4}\log\log n$ in Lemma 2.

According to Lemma 2 it is sufficient to consider only a random bipartite graph with property $t < \delta_r < T$.

Put $s = \frac{\alpha(n) \cdot \sqrt{pqn}}{\log n}$.

We define an indicator random variable $Z$ to be:

1 if there exist two vertices $u$, $v$ ($u$ of red colour and $v$ of blue), $t < d(u) < T$ and $|d(u) - d(v)| \le s$ and

0 otherwise.

Clearly from $|\delta_b - \delta_r| \le s$ follows $Z = 1$.

Since $\mathbf{P}(Z = 1) \le \mathbf{E}(Z)$, the theorem will be proved if we show that $\mathbf{E}(Z) \to 0$ as $n \to \infty$.

For the expectation of $Z$ is valid

$$\begin{aligned}\mathbf{E}(Z) = n{\cdot}m \cdot \Big\{ &\mathbf{P}\big(uv \text{ is an edge in } K_{m,n,p}\,,\ t < d(u) < T,\ |d(u) - d(v)| \le s\big)\\ &+ \mathbf{P}\big(uv \text{ is not an edge in } K_{m,n,p}\,,\ t < d(u) < T,\\ &\qquad |d(u) - d(v)| \le s\big)\Big\}\\ \le n{\cdot}m \cdot \Big\{ &\sum_{k=t}^{T} \sum_{l=k-s}^{k+s} p \cdot b(k-1, n-1) \cdot b(l-1, m-1)\\ &+ \sum_{k=t}^{T} \sum_{l=k-s}^{k+s} q \cdot b(k, n-1) \cdot b(l, m-1)\Big\}.\end{aligned}$$

To see this, we note that

$$n \cdot m \cdot \big[p \cdot b(k-1, n-1) \cdot b(l-1, m-1) + q \cdot b(k, n-1) \cdot b(l, m-1)\big] \tag{3}$$

is the expected number of such ordered pairs of vertices, which consist of a red and blue vertex of degree $k$ and $l$ respectively. The first term in (3) is the contribution of adjacent pairs.

Since the terms $b(l, n, p)$ (according to $l$) strictly increase up while $l < p \cdot n$

$$\begin{aligned}\mathbf{E}(Z) &\le 2snm \cdot \Big\{ p \cdot \sum_{k=t}^{T} b(k-1, n-1) \cdot b(k+s, m-1)\\ &\qquad + q \cdot \sum_{k=t}^{T} b(k, n-1) \cdot b(k+s, m-1)\Big\}\\ &\le 2snm \cdot b(T+s, m-1) \cdot \sum_{k=t}^{T} b(k, n-1)\\ &\le 2snm \cdot b(T+s, m-1) \cdot B(T, n-1)\,.\end{aligned}$$

Using (2) to estimate $b(T+s, m-1)$ and (1) to estimate $B(T, n-1)$ we find that if $n$ is sufficiently large, then

$$\mathbf{E}(Z) \le O(1) \cdot \alpha(n)\,.$$

Since $\alpha(n) \to 0$, the expected number of the ordered pairs tends to $0$. Hence the proof is completed. □

A similar argument gives the following assertion for the maximum degrees.

**THEOREM 2.** *Denote by $\Delta_b$ ($\Delta_r$) the maximum degree of the blue (red) vertices in a random bipartite graph $K_{m,n,p}$.*

*Let $\alpha(n) \to 0$, $\frac{m}{n} \to 1$ as $n \to \infty$.*

*If the edge probability $p$ satisfies $\frac{(\log n)^3}{npq} \to 0$ as $n \to \infty$, then for the maximum degrees of the red and blue vertices it is valid that*

$$|\Delta_b - \Delta_r| > \frac{\alpha(n) \cdot \sqrt{pqn}}{\log n} \qquad a.s.$$

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