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ON THE HSU CONDITION IN A REPLICATED REGRESSION MODEL

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Introduction

In a regression model $\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{Y}$ is an n-dimensional random vector, $\mathbf{X}$ is a given $n \times k$ matrix with the rank $R(\mathbf{X}) = k < n$, $\boldsymbol{\beta} \in \mathcal{R}^k$ (k-dimensional Euclidean space) is an unknown parameter, the error vector $\boldsymbol{\varepsilon}$ is supposed to have the mean value $E(\boldsymbol{\varepsilon}) = \boldsymbol{0}$ and the covariance matrix $\operatorname{Var}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identical matrix and $\sigma^2 \in (0, \infty)$.

If $\boldsymbol{\varepsilon}$ is normally distributed, then $\boldsymbol{Y}'\mathbf{M}\boldsymbol{Y}/(n-k)$, where $\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, is an estimator of the parameter σ^2 with the following properties. Its realization does not depend on the parameter $\boldsymbol{\beta} \in \mathcal{R}^k$ (i.e. it is invariant with respect to $\boldsymbol{\beta}$) and its variance, depending on σ^2, is minimal in the class of all the quadratic unbiased and invariant estimators at each $\sigma^2 \in (0, \infty)$ (i.e. it is the uniformly — with respect to σ^2 — minimum variance quadratic unbiased and invariant — with respect to $\boldsymbol{\beta}$ — estimator of σ^2). The notation UMVQUIE is used for an estimator with such properties.

If $\boldsymbol{\varepsilon}$ is not normally distributed, then two cases are investigated: a) the components $\varepsilon_1, \ldots, \varepsilon_n$ are stochastically independent, b) they are i.i.d. (independent identically distributed). Neither in the case a), nor in the case b) the mentioned estimator is in general the UMVQUIE. It is caused by the fact that the variance of an quadratic estimator depends not only on σ^2, but also on the parameters $\gamma_i = [E(\varepsilon_i^4)/\sigma^4] - 3$, $i = 1, \ldots, n$. Thus only the γ_0-locally best quadratic invariant unbiased estimator can be constructed (i.e. its variance is minimal in the class of all the quadratic invariant unbiased estimators under the condition that the vector $\boldsymbol{\gamma}$ belonging to the error vector $\boldsymbol{\varepsilon}$ is equal to $\boldsymbol{\gamma}_0 = (\gamma_{01}, \ldots, \gamma_{0n})'$; for other values of the vector $(\gamma_1, \ldots, \gamma_n)'$ the considered estimator need not have the minimal variance).

P. L. Hsu (1938) [2] gave the necessary and sufficient condition for the estimator $\boldsymbol{Y}'\mathbf{M}\boldsymbol{Y}/(n-k)$ to be the γ_0-locally best quadratic invariant unbiased estimator (case a) and the UMVQUIE (case b) in the class of all probability distributions with finite values $\boldsymbol{\gamma}$ (i.e. uniformly with respect to σ^2 and $\boldsymbol{\gamma}$).

H. Drygas and G. Hupet (1977) [1] gave a new proof of the Hsu statement and J. Kleffe (1979) [3] analysed this condition within the multivariate regression.

The aim of the paper is to comment the Hsu condition within a replicated regression model $\boldsymbol{Y} = (\boldsymbol{i} \otimes \mathbf{X})\boldsymbol{\beta} + \underline{\varepsilon}$, $\underline{\varepsilon}' = (\varepsilon_1', \ldots, \varepsilon_m')$, $\mathrm{Var}(\underline{\varepsilon}) = \mathbf{I}_m \otimes \sigma^2 \mathbf{I}_n$, $\boldsymbol{i} = (1_1, \ldots, 1_m)'$, $Y_1, \ldots, Y_m$ are i.i.d. vectors. Components $\varepsilon_{i,j}$, $j = 1, \ldots, n$, of the vector ε_i, $i = 1, \ldots, m$, are independent (case a)) or they are i.i.d. (case b)).

1. Definitions and auxiliary statements

The following notations are used:

$\boldsymbol{\Psi} = E[(\varepsilon_i \varepsilon_i') \otimes (\varepsilon_i \varepsilon_i')]$, $i = 1, \ldots, m$ (the symbol $\otimes$ denotes the tensor product),

$\boldsymbol{\Psi}_{kl} = E(\varepsilon_{ik}\varepsilon_{il}\varepsilon_i\varepsilon_i')$, $i = 1, \ldots, m$; $k, l = 1, \ldots, n$,

$\underline{\boldsymbol{\Psi}} = E[(\underline{\varepsilon}\underline{\varepsilon}') \otimes (\underline{\varepsilon}\underline{\varepsilon}')]$,

$\underline{\boldsymbol{\Psi}}_{ij} = E[(\varepsilon_i \varepsilon_j') \otimes (\underline{\varepsilon}\underline{\varepsilon}')]$, $i, j = 1, \ldots, m$,

$\boldsymbol{\Psi}_{ik,jl} = E(\varepsilon_{ik}\varepsilon_{jl}\underline{\varepsilon}\underline{\varepsilon}')$, $i, j = 1, \ldots, m$; $k, l = 1, \ldots, n$,

$\mathrm{diag}(\mathbf{A}) = (a_{11}, a_{22}, \ldots, a_{nn})'$, $\mathbf{A}$ is an $n \times n$ matrix with entries $\{\mathbf{A}\}_{i,j} = a_{ij}$, $i, j = 1, \ldots, n$,

$$\mathrm{Diag}(\mathbf{A}) = \begin{bmatrix} a_{11}, & 0, & \ldots, & 0 \\ 0, & a_{22}, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & a_{nn} \end{bmatrix}.$$

Lemma 1.1. *Let $\varepsilon_1, \ldots, \varepsilon_m$ be n-dimensional i.i.d. random vectors with the zero mean value and the covariance matrix* $\mathrm{Var}(\varepsilon_i) = \boldsymbol{\Sigma}$, $i = 1, \ldots, m$. *Then*

$$\underline{\boldsymbol{\Psi}} = \begin{bmatrix} \underline{\boldsymbol{\Psi}}_{11}, & \ldots, & \underline{\boldsymbol{\Psi}}_{1m} \\ \ldots & \ldots & \ldots \\ \underline{\boldsymbol{\Psi}}_{m1}, & \ldots, & \underline{\boldsymbol{\Psi}}_{mm} \end{bmatrix}, \quad \underline{\boldsymbol{\Psi}}_{ij} = \begin{bmatrix} \boldsymbol{\Psi}_{i1,j1}, & \boldsymbol{\Psi}_{i1,j2}, & \ldots, & \boldsymbol{\Psi}_{i1,jn} \\ \ldots & \ldots & \ldots & \ldots \\ \boldsymbol{\Psi}_{in,j1}, & \boldsymbol{\Psi}_{in,j2}, & \ldots, & \boldsymbol{\Psi}_{in,jn} \end{bmatrix},$$

$$\boldsymbol{\Psi}_{ik,jl} = \begin{cases} \sigma_{kl}(\mathbf{I} \otimes \boldsymbol{\Sigma}) + \boldsymbol{e}_i^{(m)}\boldsymbol{e}_i^{(m)\prime} \otimes (\boldsymbol{\Psi}_{kl} - \sigma_{kl}\boldsymbol{\Sigma}), & i = j, \\ \boldsymbol{e}_i^{(m)}\boldsymbol{e}_j^{(m)\prime} \otimes \boldsymbol{\sigma}_{.k}\boldsymbol{\sigma}_{l.} + \boldsymbol{e}_j^{(m)}\boldsymbol{e}_i^{(m)\prime} \otimes \boldsymbol{\sigma}_{.l}\boldsymbol{\sigma}_{k.}, & i \neq j, \end{cases}$$

$i, j = 1, \ldots, m$; $k, l = 1, \ldots, n$. Here $\sigma_{kl} = \{\boldsymbol{\Sigma}\}_{kl}$, $\boldsymbol{\sigma}_{.k}$ is the kth column of the matrix $\boldsymbol{\Sigma}$, $\boldsymbol{\sigma}_{l.}$ is the lth row of the matrix $\boldsymbol{\Sigma}$ and $\boldsymbol{e}_i^{(m)} = (0_1, \ldots, 0_{i-1}, 1_i, 0_{i+1}, \ldots, 0_m)'$, $i = 1, \ldots, m$. If $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}$, then

$$\boldsymbol{\Psi}_{kl} = \begin{cases} \sigma^4\mathbf{I} + (2\sigma^4 + \gamma_k\sigma^4)\,\boldsymbol{e}_k^{(n)}\boldsymbol{e}_k^{(n)\prime}, & k = l, \\ \sigma^4(\boldsymbol{e}_k^{(n)}\boldsymbol{e}_l^{(n)\prime} + \boldsymbol{e}_l^{(n)}\boldsymbol{e}_k^{(n)\prime}), & k \neq l, \end{cases}$$

$k, l = 1, \ldots, n$. Here $\gamma_k = [E(\varepsilon_{ik}^4)/\sigma^4] - 3$, $k = 1, \ldots, n$; $i = 1, \ldots, m$.

Proof. It follows from the assumption on the vectors $\varepsilon_1, \ldots, \varepsilon_m$ and from the definition of the matrices $\underline{\Psi}$, $\underline{\Psi}_{ij}$, $\Psi_{ik,jl}$, and Ψ_{kl}.

Lemma 1.2 (Kleffe–Volaufová). *Let* $E(\underline{Y}) = (\boldsymbol{i} \otimes \mathbf{X})\boldsymbol{\beta}$, $\operatorname{Var}(\underline{Y}) = \mathbf{I} \otimes \boldsymbol{\Sigma}$ *be the replication of the model* $E(\boldsymbol{Y}) = \mathbf{X}\boldsymbol{\beta}$, $\operatorname{Var}(\boldsymbol{Y}) = \boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$; *the symmetric matrices* $\mathbf{V}_1, \ldots, \mathbf{V}_p$ *are given, the vector* $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)'$ *is unknown. If a function* $\gamma(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$ $(\boldsymbol{f} \in \mathscr{R}^p$ *is a given vector),* $\boldsymbol{\vartheta} \in \underline{\vartheta}$ *(a parametric space with an unempty interior)* $\subset \mathscr{R}^p$, *is unbiasedly estimated, then for every unbiased estimator* $\underline{Y}'\underline{\mathbf{T}}\underline{Y}$ $(\underline{\mathbf{T}}$ *is a symmetric* $(mn) \times (mn)$ *matrix) of the function* $\gamma(\cdot)$ *there exists an estimator* $\underline{Y}'(\mathbf{M}_m \otimes \mathbf{T}_1 + \mathbf{P}_m \otimes \mathbf{T}_2)\underline{Y}$ $(= (m-1)\operatorname{Tr}(\mathbf{S}\mathbf{T}_1) + m\bar{\boldsymbol{Y}}'\mathbf{T}_2\bar{\boldsymbol{Y}})$ *with the property* $\forall\{\boldsymbol{\beta} \in \mathscr{R}^k\}\,\forall\{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\ \operatorname{Var}(\underline{Y}'\underline{\mathbf{T}}\underline{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta}) \geqslant \operatorname{Var}\{[(m-1)\operatorname{Tr}(\mathbf{S}\mathbf{T}_1) + m\bar{\boldsymbol{Y}}'\mathbf{T}_2\bar{\boldsymbol{Y}}] \mid \boldsymbol{\beta}, \boldsymbol{\vartheta}\}$. *Here* $\mathbf{P}_m = (1/m)\boldsymbol{i}\boldsymbol{i}'$, $\boldsymbol{i} = (1_1, \ldots, 1_m)'$, $\mathbf{M}_m = \mathbf{I} - \mathbf{P}_m$, $\bar{\boldsymbol{Y}} = (1/m)\sum_{i=1}^{m} \boldsymbol{Y}_i$, $\mathbf{S} = [1/(m-1)]\sum_{i=1}^{m}(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})'$.

Proof. See [4] or Theorem 5.6.11 in [5].

Lemma 1.3 (Hsu). *Let* $E(\boldsymbol{Y}) = \mathbf{X}\boldsymbol{\beta}$, $\operatorname{Var}(\boldsymbol{Y}) = \sigma^2\mathbf{I}$, $Y_1, \ldots, Y_n$ *be independent components of the vector* $\boldsymbol{Y}$ *and let the rank of the* $n \times k$ *matrix* $\mathbf{X}$ *be* $R(\mathbf{X}) = k < n$. *Let* $\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, $\{\mathbf{M}\}_{ij} = m_{ij}$, $i, j = 1, \ldots, n$ *and* $\boldsymbol{\gamma}$ *the diagonal matrix with the diagonal* $(\gamma_1, \ldots, \gamma_n)$.

a) *The estimator* $\boldsymbol{Y}'\mathbf{M}\boldsymbol{Y}/\operatorname{Tr}(\mathbf{M})$ *is the* $\boldsymbol{\gamma}$-LMVQUIE *(locally minimum variance quadratic invariant estimator) of the parameter* σ^2 *in the class of all probability distributions if and only if*

$$(\mathbf{M} * \mathbf{M})\operatorname{diag}(\mathbf{M}) = \{[\operatorname{diag}(\mathbf{M})]'\boldsymbol{\gamma}\operatorname{diag}(\mathbf{M})/\operatorname{Tr}(\mathbf{M})\}\operatorname{diag}(\mathbf{M})$$

(here $*$ *denotes the Hadamard multiplication of matrices;* $\{\mathbf{A} * \mathbf{B}\}_{ij} = \{\mathbf{A}\}_{ij}\{\mathbf{B}\}_{ij}$, $i, j = 1, \ldots, n$*).*

b) *If* $\varepsilon_1, \ldots, \varepsilon_n$ *are i.i.d. random variables, then* $\boldsymbol{Y}'\mathbf{M}\boldsymbol{Y}/\operatorname{Tr}(\mathbf{M})$ *is the* UMVQUIE *of the parameter* σ^2 *in the class of all probability distributions iff*

$$(\mathbf{M} * \mathbf{M})\operatorname{diag}(\mathbf{M}) = \{[\operatorname{diag}(\mathbf{M})]'\operatorname{diag}(\mathbf{M})/\operatorname{Tr}(\mathbf{M})\}\operatorname{diag}(\mathbf{M}).$$

Proof. See [2].

2. The Hsu condition and estimators in a replicated regression model

In this section the symbol $\boldsymbol{\gamma}$ denotes the diagonal matrix with the diagonal $(\gamma_1, \ldots, \gamma_n)$, $\gamma_k = [E(\varepsilon_{ik}^4)/\sigma^4] - 3$, $k = 1, \ldots, n$; $i = 1, \ldots, m$, $\underline{\boldsymbol{\varepsilon}}' = (\boldsymbol{\varepsilon}_1', \ldots, \boldsymbol{\varepsilon}_m')$, $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_m$ are i.i.d. vectors, $\varepsilon_{i1}, \ldots, \varepsilon_{in}$ are stochastically independent random variables, $i = 1, \dots, m$, $\mathrm{Var}(\varepsilon_i) = \sigma^2 \mathbf{I}_n$, $i = 1, \dots, m$, and $\mathbf{T}_1$, $\mathbf{T}_2$ are symmetric $n \times n$ matrices.

Theorem 2.1. *In a replicated model* $\underline{Y} = (\boldsymbol{i} \otimes \mathbf{X})\boldsymbol{\beta} + \underline{\varepsilon}$, $\mathrm{Var}(\underline{\varepsilon}) = \mathbf{I}_m \otimes \sigma^2 \mathbf{I}_n$:

a) *If* $\gamma \neq \gamma \mathbf{I}$, *then* $\underline{Y}'\underline{\mathbf{M}}\underline{Y}/\mathrm{Tr}(\underline{\mathbf{M}}) = [(m-1)\,\mathrm{Tr}(\mathbf{S}) + m\bar{Y}'\mathbf{M}\bar{Y}]/[n - k + n(m-1)]$ *is the* γ-LMVQUIE *of the parameter* σ^2 *in the class of all probability distributions iff*

$$\gamma[(m-1)\mathbf{I} + (\mathbf{M} * \mathbf{M})]\,\boldsymbol{i} = d\boldsymbol{i},$$

where $d = [(m-1)\boldsymbol{i} + \mathrm{diag}(\mathbf{M})]'\gamma[(m-1)\boldsymbol{i} + \mathrm{diag}(\mathbf{M})]/[n - k + n(m-1)]$.

b) *If* $\gamma = \gamma\mathbf{I}$, *then* $\underline{Y}'\underline{\mathbf{M}}\underline{Y}/\mathrm{Tr}(\underline{\mathbf{M}})$ *is the* UMVQUIE *of the parameter* σ^2 *in the class of all probability distributions iff*

$$(\mathbf{M} * \mathbf{M})\,\boldsymbol{i} = [(n-k)/n]\,\boldsymbol{i}.$$

Here $\underline{\mathbf{M}} = \mathbf{I}_m \otimes \mathbf{I}_n - (\boldsymbol{i} \otimes \mathbf{X})[(\boldsymbol{i}' \otimes \mathbf{X}')(\boldsymbol{i} \otimes \mathbf{X})]^{-1}(\boldsymbol{i}' \otimes \mathbf{X}')$.

Proof. With respect to Lemma 1.3 the assertion a) is valid if and only if

$$(*) \qquad (\underline{\mathbf{M}} * \underline{\mathbf{M}})(\mathbf{I} \otimes \gamma)\,\mathrm{diag}(\underline{\mathbf{M}}) = \underline{d}\,\mathrm{diag}(\underline{\mathbf{M}}),$$

where $\underline{d} = [\mathrm{diag}(\underline{\mathbf{M}})]'(\mathbf{I} \otimes \gamma)\,\mathrm{diag}(\underline{\mathbf{M}})/\mathrm{Tr}(\underline{\mathbf{M}})$.

As $\underline{\mathbf{M}} = \mathbf{M}_m \otimes \mathbf{I} + \mathbf{P}_m \otimes \mathbf{M}$,

$$\mathrm{diag}(\underline{\mathbf{M}}) = \boldsymbol{i} \otimes (1/m)[(m-1)\boldsymbol{i} + \mathrm{diag}(\mathbf{M})],$$

$$(\underline{\mathbf{M}} * \underline{\mathbf{M}}) = \mathbf{M}_m \otimes (1/m)[2\mathrm{Diag}(\mathbf{M}) + (m-2)\mathbf{I}] + \mathbf{P}_m \otimes (1/m)[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}],$$

$$\mathbf{M}_m \boldsymbol{i} = 0, \quad \mathbf{P}_m \boldsymbol{i} = \boldsymbol{i}, \quad \mathrm{Tr}(\underline{\mathbf{M}}) = n - k + n(m-1)$$

and

$$(m-1)\boldsymbol{i} + \mathrm{diag}(\mathbf{M}) = [(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\,\boldsymbol{i} \quad (\Leftarrow \mathbf{M}^2 = \mathbf{M}),$$

the l.h.s. of $(*)$ can be rewritten as

$$\boldsymbol{i} \otimes (1/m^2)\{[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\gamma[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\,\boldsymbol{i}$$

and the r.h.s. of $(*)$ as

$$\boldsymbol{i} \otimes (1/m^2)\{[(m-1)\boldsymbol{i} + \mathrm{diag}(\mathbf{M})]'\gamma[(m-1)\boldsymbol{i} + \mathrm{diag}(\mathbf{M})]/[n - k + n(m-1)]\}[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\,\boldsymbol{i}.$$

Thus $(*)$ is equivalent to

$$[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\gamma[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\,\boldsymbol{i} = d[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\,\boldsymbol{i} \Leftrightarrow \gamma[[m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\,\boldsymbol{i} = d\boldsymbol{i}.$$

b) If in the last relationship $\gamma = \gamma\mathbf{I}$, then

$$(m-1)\boldsymbol{i} + (\mathbf{M} * \mathbf{M})\,\boldsymbol{i} = \Big\{\Big[(m-1)^2 n + 2(m-1)(n-k) +$$

$$+ \sum_{i=1}^{n} m_{ii}^2 \Big] \Big/ [n-k+n(m-1)] \Big\} \boldsymbol{i} \Rightarrow \operatorname{diag}(\mathbf{M}) = (\mathbf{M} * \mathbf{M}) \boldsymbol{i} =$$

$$= \Big\{ \Big[(m-1)(n-k) + \sum_{i=1}^{n} m_{ii}^2 \Big] \Big/ [n-k+n(m-1)] \Big\} \boldsymbol{i}.$$

As

$$\operatorname{Tr}(\mathbf{M}) = \boldsymbol{i}' \operatorname{diag}(\mathbf{M}) = n-k = n \Big[(m-1)(n-k) + \sum_{i=1}^{n} m_{ii}^2 \Big] \Big/ [n-k+n(m-1)]$$

we obtain the condition (*) in the following equivalent form

$$(\mathbf{M} * \mathbf{M}) \boldsymbol{i} = [(n-k)/n] \boldsymbol{i}.$$

Remark. In the case a) the condition (*) changes with respect to m: in the case b) the condition (*) is invariant with respect to m.

Lemma 2.1. *The variance of the random variable* $\underline{\boldsymbol{\varepsilon}}'(\mathbf{M}_m \otimes \mathbf{T}_1 + \mathbf{P}_m \otimes \mathbf{T}_1)\underline{\boldsymbol{\varepsilon}}$ *is*

$$\operatorname{Var}[\underline{\boldsymbol{\varepsilon}}'(\mathbf{M}_m \otimes \mathbf{T}_1 + \mathbf{P}_m \otimes \mathbf{T}_2)\underline{\boldsymbol{\varepsilon}}] = 2\sigma^4[(m-1)\operatorname{Tr}(\mathbf{T}_1^2) + \operatorname{Tr}(\mathbf{T}_2^2)] + \sigma^4\{(m-1)^2[\operatorname{diag}(\mathbf{T}_1)]'(\gamma/m)\operatorname{diag}(\mathbf{T}_1) + 2(m-1)[\operatorname{diag}(\mathbf{T}_1)]' \cdot (\gamma/m)\operatorname{diag}(\mathbf{T}_2) + [\operatorname{diag}(\mathbf{T}_2)]'(\gamma/m)\operatorname{diag}(\mathbf{T}_2).$$

Proof. In Lemma 1.1 let $\mathbf{\Sigma} = \sigma^2\mathbf{I}$. Then

$$\mathbf{\Psi}_{jl,ik} = \begin{cases} \mathbf{I}_m \otimes \sigma^4 \mathbf{I}_n + \boldsymbol{e}_i^{(m)}\boldsymbol{e}_i^{(m)\prime} \otimes (2\sigma^4 + \sigma^4\gamma_k)\boldsymbol{e}_k^{(n)}\boldsymbol{e}_k^{(n)\prime}, & i=j,\ k=l, \\ \boldsymbol{e}_i^{(m)}\boldsymbol{e}_i^{(m)\prime} \otimes \sigma^4(\boldsymbol{e}_k^{(n)}\boldsymbol{e}_l^{(n)\prime} + \boldsymbol{e}_l^{(n)}\boldsymbol{e}_k^{(n)\prime}), & i=j,\ k \neq l, \\ \boldsymbol{e}_i^{(m)}\boldsymbol{e}_j^{(m)\prime} \otimes \sigma^4\boldsymbol{e}_k^{(n)}\boldsymbol{e}_l^{(n)\prime} + \boldsymbol{e}_j^{(m)}\boldsymbol{e}_i^{(m)\prime} \otimes \sigma^4\boldsymbol{e}_l^{(n)}\boldsymbol{e}_k^{(n)\prime}, & i \neq j, \end{cases}$$

$k, l = 1, \dots, n$; $i, j = 1, \dots, m$. For an arbitrary symmetric $(mn) \times (mn)$ matrix $\underline{\mathbf{A}}$ there holds

$$\operatorname{Var}(\underline{\boldsymbol{\varepsilon}}'\underline{\mathbf{A}}\underline{\boldsymbol{\varepsilon}}) = \operatorname{Tr}[(\underline{\mathbf{A}} \otimes \underline{\mathbf{A}})\underline{\mathbf{\Psi}}] - \sigma^4[\operatorname{Tr}(\underline{\mathbf{A}})]^2.$$

If $\underline{\mathbf{A}} = \mathbf{M}_m \otimes \mathbf{T}_1 + \mathbf{P}_m \otimes \mathbf{T}_2$, then with respect to the assumption $\mathbf{\Sigma} = \sigma^2\mathbf{I}$ and with respect to Lemma 1.1 the assertion follows from the expression for $\operatorname{Var}(\underline{\boldsymbol{\varepsilon}}'\underline{\mathbf{A}}\underline{\boldsymbol{\varepsilon}})$.

Theorem 2.2. *In a replicated regression model* $\underline{\boldsymbol{Y}} = (\boldsymbol{i} \otimes \mathbf{X})\boldsymbol{\beta} + \underline{\boldsymbol{\varepsilon}}$, $\operatorname{Var}(\underline{\boldsymbol{Y}}) = \mathbf{I} \otimes \sigma^2\mathbf{I}$ *with the matrix* γ *the* γ-LMVQUIE *of the parameter* σ^2 *is*

$$\hat{\sigma}^2 = \underline{\boldsymbol{Y}}'(\mathbf{M}_m \otimes \mathbf{T}_1 + \mathbf{P}_m \otimes \mathbf{T}_2)\underline{\boldsymbol{Y}} = (m-1)\operatorname{Tr}(\mathbf{S}\mathbf{T}_1) + m\bar{\boldsymbol{Y}}'\mathbf{T}_2\bar{\boldsymbol{Y}},$$

where

$$\operatorname{diag}(\mathbf{T}_1) = (1/c)\{\mathbf{I} + (1/2)(\gamma/m)[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\}^{-1}\boldsymbol{i},$$

$$c = [(m-1)\boldsymbol{i} + \operatorname{diag}(\mathbf{M})]'\{\mathbf{I} + (1/2)(\gamma/m)[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\}^{-1}\boldsymbol{i},$$

$$\mathbf{T}_1 = \operatorname{diag}(\mathbf{T}_1), \quad \mathbf{T}_2 = \mathbf{M}\mathbf{T}_1\mathbf{M}.$$

Proof. With respect to Lemma 1.2 it is sufficient to consider an estimator of the form $Y'(\mathbf{M}_m \otimes \mathbf{T}_1 + \mathbf{P}_m \otimes \mathbf{T}_2) Y$, where $\mathbf{T}_2\mathbf{X} = \mathbf{0}$ (invariance) and

$$\mathrm{Tr}(\mathbf{M}_m \otimes \mathbf{T}_1 + \mathbf{P}_m \otimes \mathbf{T}_2) = (m-1)\,\mathrm{Tr}(\mathbf{T}_1) + \mathrm{Tr}(\mathbf{T}_2) = 1$$

(unbiasedness). In what follows the Lagrange procedure with indefinite multipliers is used; with respect to Lemma 2.1 the auxiliary Lagrange function is

$$\Phi(\mathbf{T}_1, \mathbf{T}_2) = 2(m-1)\,\mathrm{Tr}(\mathbf{T}_1^2) + \mathrm{Tr}(\mathbf{T}_2^2) + (m-1)^2[\mathrm{diag}(\mathbf{T}_1)]'(\gamma/m)\,\mathrm{diag}(\mathbf{T}_1) +$$
$$+ 2(m-1)[\mathrm{diag}(\mathbf{T}_1)]'(\gamma/m)\,\mathrm{diag}(\mathbf{T}_2) + [\mathrm{diag}(\mathbf{T}_2)]'(\gamma/m)\,\mathrm{diag}(\mathbf{T}_2) -$$
$$- \lambda[(m-1)\,\mathrm{Tr}(\mathbf{T}_1) + \mathrm{Tr}(\mathbf{T}_2) - 1] - \mathrm{Tr}(\boldsymbol{\varkappa}\mathbf{T}_2\mathbf{X}),$$

where λ is a Lagrange multiplier $\boldsymbol{\varkappa}$ is a matrix of Lagrange multipliers.

$\partial\Phi(\mathbf{T}_1, \mathbf{T}_2)/\partial\mathbf{T}_1 = \mathbf{0} \Rightarrow$

$$\begin{aligned}&\mathbf{T}_1 - (1/2)\,\mathrm{Diag}(\mathbf{T}_1) + (1/4)(m-1)(\gamma/m)\,\mathrm{Diag}(\mathbf{T}_1) + \\ &+ (1/4)(\gamma/m)\,\mathrm{Diag}(\mathbf{T}_2) = (\lambda/8)\,\mathbf{I},\end{aligned} \tag{1}$$

$\partial\Phi(\mathbf{T}_1, \mathbf{T}_2)/\partial\mathbf{T}_2 = \mathbf{0} \Rightarrow$

$$\begin{aligned}&\mathbf{T}_2 - (1/2)\,\mathrm{Diag}(\mathbf{T}_2) + (1/4)(m-1)(\gamma/m)\,\mathrm{Diag}(\mathbf{T}_1) + \\ &+ (1/4)(\gamma/m)\,\mathrm{Diag}(\mathbf{T}_2) - (1/8)[\mathbf{X}\boldsymbol{\varkappa} + \boldsymbol{\varkappa}'\mathbf{X}' - \mathrm{Diag}(\mathbf{X}\boldsymbol{\varkappa})] = \mathbf{0}.\end{aligned} \tag{2}$$

Diag (2) + (2) (the operation Diag is applied to the equation (2) and added to (2)) $\Rightarrow$

$$\begin{aligned}&\mathbf{T}_2 + (1/2)(m-1)(\gamma/m)\,\mathrm{Diag}(\mathbf{T}_1) + (1/2)(\gamma/m)\,\mathrm{Diag}(\mathbf{T}_2) - \\ &- (1/8)(\mathbf{X}\boldsymbol{\varkappa} + \boldsymbol{\varkappa}'\mathbf{X}') = (\lambda/4)\,\mathbf{I}.\end{aligned} \tag{2'}$$

(1) implies $\mathbf{T}_1 = \mathrm{Diag}(\mathbf{T}_1)$, thus

$$\mathbf{T}_1 + (1/2)(m-1)(\gamma/m)\mathbf{T}_1 + (1/2)(\gamma/m)\,\mathrm{Diag}(\mathbf{T}_2) = (\lambda/4)\,\mathbf{I}. \tag{1'}$$

Comparing $\mathbf{M}(1')\mathbf{M}$ and $\mathbf{M}(2')\mathbf{M}$ and taking into account the relations $\mathbf{M}\mathbf{T}_2\mathbf{M} = \mathbf{T}_2$ ($\Leftarrow \mathbf{T}_2\mathbf{X} = \mathbf{0}$) and $\mathbf{M}\mathbf{X} = \mathbf{0}$ we get $\mathbf{T}_2 = \mathbf{M}\mathbf{T}_1\mathbf{M}$, i.e. $\mathrm{diag}(\mathbf{T}_2) = (\mathbf{M} * \mathbf{M})\,\mathrm{diag}(\mathbf{T}_1)$.

$$\begin{aligned}&\mathrm{diag}(1') \Rightarrow \\ &\{\mathbf{I} + (1/2)(\gamma/m)[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\}\,\mathrm{diag}(\mathbf{T}_1) = (\lambda/4)\,\boldsymbol{i}.\end{aligned}$$

As

$$[(m-1)\,\mathrm{Tr}(\mathbf{T}_1) + \mathrm{Tr}(\mathbf{T}_2) =]\ 1 = \boldsymbol{i}'[(m-1)\,\mathrm{diag}(\mathbf{T}_1) + \mathrm{diag}(\mathbf{T}_2)]$$

and $(\mathbf{M} * \mathbf{M})\,\boldsymbol{i} = \mathrm{diag}(\mathbf{M})$, we have $\lambda/4 = 1/c$.

Corollary. *As* $\mathbf{M}\bar{Y} = \bar{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} = \bar{\boldsymbol{v}}$, *where* $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\bar{Y}$, *the estimator* $\hat{\sigma}^2$ *from Theorem 2.2 can be expressed in the form*

$$\hat{\sigma}^2 = (1/c)\,\boldsymbol{i}'\{\mathbf{I} + (1/2)[(m-1)\mathbf{I} + \mathbf{M} * \mathbf{M}]\gamma/m\}^{-1}\,\mathrm{diag}[(m-1)\mathbf{S} + m\bar{\boldsymbol{v}}\bar{\boldsymbol{v}}'].$$

If $\gamma = \mathbf{0}$, *then* $c = n - k + n(m-1)$ *and*

$$\hat{\sigma}^2 = \left[(m-1)\sum_{i=1}^{n} S_{ii} + m\sum_{i=1}^{n} \bar{v}_i^2\right]\Big/[(m-1)n + n - k] = \underline{Y}'\underline{\mathbf{M}}\,\underline{Y}/\mathrm{Tr}(\underline{\mathbf{M}}).$$

Lemma 2.2. *Let* $\mathbf{V}$, $\mathbf{U}$ *be* $m \times m$ *and* $n \times n$ *matrices, respectively, satisfying the condition* $(\mathbf{V} \otimes \mathbf{U})(\boldsymbol{i} \otimes \mathbf{X}) = \mathbf{0}$. *Then*

$$\mathrm{Var}(\underline{Y}'(\mathbf{V} \otimes \mathbf{U})\,\underline{Y}) = 2\sigma^4\,\mathrm{Tr}(\mathbf{V}^2)\,\mathrm{Tr}(\mathbf{U}^2) + \sigma^4 \sum_{i=1}^{m} V_{ii}^2 \sum_{j=1}^{n} \gamma_j U_{jj}^2 .$$

Proof. It is an analogy of the proof of Lemma 2.1.

Theorem 2.3. *Consider a replicated regression model* $\underline{Y} = (\boldsymbol{i} \otimes \mathbf{X})\boldsymbol{\beta} + \underline{\boldsymbol{\varepsilon}}$, $\mathrm{Var}(\underline{Y}) = \mathbf{I} \otimes \sigma^2\mathbf{I}$. *The following assertions are valid*:

1. *If*

$$\mathbf{U}_1 = (1/\{(m-1)\,\mathrm{Tr}[(\mathbf{I} + (1/2)(m-1)\gamma/m)^{-1}]\})[\mathbf{I} + (1/2)(m-1)\gamma/m]^{-1},$$

then

$$\hat{\sigma}_1^2 = \underline{Y}'(\mathbf{M}_m \otimes \mathbf{U}_1)\,\underline{Y} = (m-1)\,\mathrm{Tr}(\mathbf{S}\mathbf{U}_1) =$$
$$= \left\{1\Big/\sum_{i=1}^{n} [2 + (\gamma_i/m)(m-1)]^{-1}\right\} \sum_{j=1}^{n} S_{jj}\Big/[2 + (\gamma_j/m)(m-1)]$$

is the γ-LMVQUIE *in the class of estimators* $\{\underline{Y}'(\mathbf{M}_m \otimes \mathbf{T}_1)\,\underline{Y}: \mathbf{T}_1 = \mathbf{T}_1', (m-1)\cdot \mathrm{Tr}(\mathbf{T}_1) = 1\}$ and for $m \to \infty$

$$\mathrm{Var}(\hat{\sigma}_1^2 \mid \gamma) = [1/(m-1)]\,\sigma^4\Big/\sum_{i=1}^{n} [2 + (\gamma_i/m)(m-1)]^{-1} \to 0.$$

2. *If*

$$\mathrm{diag}(\mathbf{U}_2) = (1/\{\boldsymbol{i}'[\mathbf{I} + (1/2)(\mathbf{M} * \mathbf{M})\gamma/m]^{-1}\,\mathrm{diag}(\mathbf{M})\})\cdot$$
$$\cdot[\mathbf{I} + (1/2)(\gamma/m)(\mathbf{M} * \mathbf{M})]^{-1}\,\mathrm{diag}(\mathbf{M})$$

and

$$\mathbf{U}_2 = (1/\{\boldsymbol{i}'[\mathbf{I} + (1/2)(\mathbf{M} * \mathbf{M})\gamma/m]^{-1}\,\mathrm{diag}(\mathbf{M})\})\cdot$$
$$\cdot\mathbf{M} - (1/2)\mathbf{M}(\gamma/m)\,\mathrm{Diag}(\mathbf{U}_2)\,\mathbf{M},$$

then $\hat{\sigma}_2^2 = \underline{Y}'(\mathbf{P}_m \otimes \mathbf{U}_2)\,\underline{Y} = m\bar{\mathbf{Y}}'\mathbf{U}_2\bar{\mathbf{Y}}$ *is the* γ-LMVQUIE *in the class of estimators* $\{\underline{Y}'(\mathbf{P}_m \otimes \mathbf{T}_2)\underline{Y}: \mathbf{T}_2 = \mathbf{T}_2', \mathrm{Tr}(\mathbf{T}_2) = 1, \mathbf{T}_2\mathbf{X} = \mathbf{0}\}$ *and* $\mathbf{U}_2 \to [1/(n-k)]\,\mathbf{M}$,

$$\mathrm{Var}(\hat{\sigma}_2^2 \mid \gamma) = 2\sigma^4\,\mathrm{Tr}(\mathbf{U}_2^2) + \sigma^4 \sum_{j=1}^{n} (\gamma_j/m)\,U_{2,jj}^2 \to 2\sigma^4/(n-k),$$

for $m \to \infty$.

3. $$\hat{\sigma}_3^2 = \underline{Y}'\underline{\mathbf{M}}\,\underline{Y}/\mathrm{Tr}(\underline{\mathbf{M}}) = \left[(m-1)\sum_{i=1}^{n} S_{ii} + m\bar{\boldsymbol{v}}'\bar{\boldsymbol{v}}\right]\Big/[(m-1)n + n - k] \Rightarrow$$
$$\Rightarrow \mathrm{Var}(\hat{\sigma}_3^2 \mid \gamma) = \{2\sigma^4/[(m-1)n + n - k]\} + \{\sigma^4/[(m-1)n + n - k]^2\}\cdot$$

$$\cdot\left\{(m-1)^2\sum_{i=1}^{n}(\gamma_i/m)+2(m-1)\sum_{i=1}^{n}(\gamma_i/m)m_{ii}+\sum_{i=1}^{n}(\gamma_i/m)m_{ii}^2\right\}\to 0,$$

for $m\to\infty$.

4. $$\hat{\sigma}_4^2=\underline{\boldsymbol{Y}}'(\mathbf{M}_m\otimes\{1/[n(m-1)]\}\mathbf{I})\underline{\boldsymbol{Y}}=(1/n)\sum_{i=1}^{n}S_{ii}\Rightarrow$$

$$\Rightarrow \operatorname{Var}(\hat{\sigma}_4^2\,|\,\boldsymbol{\gamma})=\{2\sigma^4/[n(m-1)]\}+(\sigma^4/n^2)\sum_{i=1}^{m}\gamma_i/m\to 0,$$

for $m\to\infty$.

5. $$\hat{\sigma}_5^2=\underline{\boldsymbol{Y}}'\{\mathbf{P}_m\otimes[1/(n-k)]\,\mathbf{M}\}\,\underline{\boldsymbol{Y}}=m\bar{\boldsymbol{Y}}'\mathbf{M}\,\bar{\boldsymbol{Y}}/(n-k)\Rightarrow$$

$$\Rightarrow \operatorname{Var}(\hat{\sigma}_5^2\,|\,\boldsymbol{\gamma})=[2\sigma^4/(n-k)]+[\sigma^4/(n-k)^2]\sum_{i=1}^{n}m_{ii}^2\gamma_i/m\to 2\sigma^4/(n-k),$$

for $m\to\infty$.

Proof. With respect to Lemma 2.2

1. $$\operatorname{Var}[\underline{\boldsymbol{Y}}'(\mathbf{M}_m\otimes\mathbf{U}_1)\,\underline{\boldsymbol{Y}}\,|\,\boldsymbol{\gamma}]=2\sigma^4(m-1)\operatorname{Tr}(\mathbf{U}_1^2)+\sigma^4(m-1)^2\sum_{j=1}^{n}U_{1.\,jj}^2\gamma_j/m$$

and

2. $$\operatorname{Var}[\underline{\boldsymbol{Y}}'(\mathbf{P}_m\otimes\mathbf{U}_2)\,\underline{\boldsymbol{Y}}\,|\,\boldsymbol{\gamma}]=2\sigma^4\operatorname{Tr}(\mathbf{U}_2^2)+\sigma^4\sum_{j=1}^{n}U_{2.\,jj}^2\gamma_j/m.$$

In the first case the quantity $\operatorname{Var}[\underline{\boldsymbol{Y}}'(\mathbf{M}_m\otimes\mathbf{U}_1)\,\underline{\boldsymbol{Y}}]$ has to be minimized by a proper choice of the symmetric matrix $\mathbf{U}_1$ satisfying the condition $(m-1)\cdot\operatorname{Tr}(\mathbf{U}_1)=1$ (unbiasedness); the invariance is guaranteed by the form of the estimator considered because of $(\mathbf{M}_m\otimes\mathbf{U}_1)(\boldsymbol{i}\otimes\mathbf{X})=\mathbf{0}$. In the second case the matrix $\mathbf{U}_2$ has to satisfy the conditions $\operatorname{Tr}(\mathbf{U}_2)=1$ (unbiasedness) and $\mathbf{U}_2\mathbf{X}=\mathbf{0}$ (invariance). Further we continue similarly as in the proof of Theorem 2.2. The assertions 3, 4 and 5 follows from Lemma 2.2.

Corollary. *If* $Y_{1i},\ldots,Y_{mi}$ *are i.i.d. random variables with* $\gamma_i=E\{[Y_{ji}-E(Y_{ji})]^4\}/(E\{[Y_{ji}-E(Y_{ji})]^2\})^2-3$, $j=1,\ldots,m$, *then* $E\{[\bar{Y}_i-E(\bar{Y}_i)]^4\}/(E\{[\bar{Y}_i-E(\bar{Y}_i)]^2\})^2-3=\gamma_i/m$, *where* $\bar{Y}_i=(1/m)\sum_{j=1}^{m}Y_{ji}$. *Thus it can be expected that the estimators* $\hat{\sigma}_3^2$, $\hat{\sigma}_4^2$ *and* $\hat{\sigma}_5^2$ *used in the case of normality of the vector* $\underline{\boldsymbol{Y}}$ *deviate unsubstantially for sufficiently large m from the estimators* $\hat{\sigma}^2$ *(Theorem* 2.2*),* $\hat{\sigma}_1^2$ *and* $\hat{\sigma}_2^2$. *Therefore the estimator* $\hat{\sigma}^2$ *can be replaced by the estimator* $\hat{\sigma}_3^2$.

In consequence of the statements 4 *and* 5 *of Theorem* 2.3 *the contribution of the term* $m\bar{\mathbf{v}}'\bar{\mathbf{v}}/[(n-k)+n(m-1)]$ *to the quality of the estimator* $\hat{\sigma}_3^2$ *is negligible, thus for a sufficiently large m the estimator* $\hat{\sigma}_3^2$ *can be replaced by the estimator* $\hat{\sigma}_4^2$. *It is a good approximation of the* UMVQUIE *regardless of the Hsu condition being fulfilled or not.*

If $\boldsymbol{\gamma}=\gamma\mathbf{I}$, *then obviously* $\hat{\sigma}_1^2=\hat{\sigma}_4^2$ *and*

$$\mathrm{Var}(\hat{\sigma}_1^2) = \{2\hat{\sigma}^4/[n(m-1)]\} + \sigma^4\gamma/(nm).$$

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ОБ УСЛОВИИ ХСУ В ПОВТОРЕННОЙ РЕГРЕССИОННОЙ МОДЕЛИ

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Резюме

Условие Хсу, рассматриванное в m-раза повторенной основной регрессионной модели, выражено эквивалентно в терминах основной модели. В общем случае это условие зависит от числа повторений m. Далее показано, что если условие Хсу не выполнено, то для достаточно большего m обыкновенная оценка вариации незначительно отклоняется от наилучшей оценки.