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ON THE MEASURABILITY OF REAL FUNCTIONS DEFINED ON PRODUCT-SPACES

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Let $(X_1, \varrho_1, \mathfrak{M}_1, \mu_1)$ and $(X_2, \varrho_2, \mathfrak{M}_2, \mu_2)$ be complete metric spaces with metrics ϱ_1 and ϱ_2, respectively, and with σ-finite regular complete measures μ_1 and μ_2, defined over a σ-field $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of subsets of X_1 and X_2, respectively.

(1) Let $\mathcal{F}_1 \subset \mathfrak{M}_1$ and $\mathcal{F}_2 \subset \mathfrak{M}_2$ be families of closed sets with nonempty interiors and positive and finite measures μ_1, μ_2.

Definition 1. *The sequence $\{I_n\}_{n=1}^{\infty}$ of sets from $\mathcal{F}_1$ ($\mathcal{F}_2$) is said to converge to the point $x_0 \in X_1$ ($y_0 \in X_2$) iff $x_0 \in \operatorname{Int}(I_n)$ ($y_0 \in \operatorname{Int}(I_n)$) for $n = 1, 2, \ldots$ and the sequences of diameters $\mathrm{d}(I_n)$ converge to zero while $n \to \infty$.*

The convergence of $\{I_n\}_{n=1}^{\infty}$ to x_0 will be denoted by $I_n \to x_0$.

(2) Assume that the family $\mathcal{F}_1$ ($\mathcal{F}_2$) is countable and that for every $x_0 \in X_1$ ($y_0 \in X_2$) there is a sequence of sets $\{I_n\}_{n=1}^{\infty}$ from F_1 (F_2) converging to x_0 (y_0).

Let $A_1 \subset X_1$, $A_2 \subset X_2$, $x_1 \in X_1$, $x_2 \in X_2$. Denote by μ_1^* and μ_2^* (μ_{1*} and μ_{2*}) the outer (inner) measures corresponding to μ_1 and μ_2, respectively.

Definition 2. *The upper (lower) vound of the set of numbers $\lim_{n\to\infty} \frac{\mu_1^*(A_1 \cap I_n)}{\mu_1(I_n)}$ taken from all the sequences $\{I_n\}_{n=1}^{\infty}$ converging to x_1 is called the upper (lower) external density of A_1 in x_1 with respect to $\mathcal{F}_1$ and is denoted by $D_u^*(x_1, A_1)$ ($D_l^*(x_1, A_1)$).*

If $D_u^*(x_1, A_1) = D_l^*(x_1, A_1)$, then their common value is called the external density of the set A_1 in x_1 with respect to $\mathcal{F}_1$ and is denoted by $D^*(x_1, A_1)$.

If we replace in the above definitions set A_1 by A_2 point x_1 by x_2, we get the upper external density $D_u^*(x_2, A_2)$, the lower external density $D_l^*(x_2, A_2)$ and the external density $D^*(x_2, A_2)$ of the set A_2 in the point x_2 with respect to $\mathcal{F}_2$.

If $A_1 \in \mathfrak{M}_1$ ($A_2 \in \mathfrak{M}_2$), then the respective external densities are called densities with respect to $\mathcal{F}_1$ ($\mathcal{F}_2$) and denoted by $D_u(x_1, A_1)$, $D_l(x_1, A_1)$, and $D(x_1, A_1)$ ($D_u(x_2, A_2)$, $D_l(x_2, A_2)$ and $D(x_2, A_2)$), respectively.

Point x_1 is called a density point of the set A with respect to $\mathcal{F}_1$ if there exists a set $B \in \mathfrak{M}_1$ such that $B \subset A$ and $D(x_1, B) = 1$.

Moreover assume that

(3) for every set $A_1 \in \mathfrak{M}_1$ ($A_2 \in \mathfrak{M}_2$) the μ_1 (μ_2)-measure of the set $\{x: x \in A_1, D_1(x, A_1) < 1\}$ ($\{y: y \in A_2, D_1(y, A_2) < 1\}$) is equal to zero.

Examples of the spaces that satisfy (1), (2) and (3):

1. If $X_1 = R^2$, ϱ_1 is the Euclidean metric R^2, $\mathfrak{M}_1$ a family of Lebesque measurable sets, μ_1 the Lebesque measure and $\mathcal{F}_1$ consists of circle centres with rational centre coordinates and radius, then the conditions (1), (2), (3) are met.

2. Let $X_1 = R^1$, $\varrho_1(x, y) = |x - y|$ for $x, y \in R^1$, let $\mathfrak{M}_1$ be a σ-field of Borel sets of the straight line, μ_1-a regular σ-finite measure, such that all intervals are μ_1-measurable and their μ_1-measure is non-zero and let $\mathcal{F}_1$ be a family of closed intervals with rational ends. Then the conditions (1, (2), (3) are satisfied.

In order to show this it suffices to prove that for every set $A \in \mathfrak{M}_1$ condition (3) holds.

Let $E \in \mathfrak{M}_1$. Since μ_1 is σ-finite, we may assume that $\mu_1(E) < \infty$. Since $\frac{\mu_1(I \cap E)}{\mu_1(I)} \leq 1$ for $I \in \mathcal{F}_1$, the set Z of those points $x \in E$, for which $D_l(x, E) < 1$ is equal to $\bigcup_{k=1}^{\infty} H\left(1 - \frac{1}{k}\right)$, where $H = H(\alpha) = E \cap \{x: D_1(x, E) < \alpha\}$. And so it is sufficient to prove that $0 < \alpha < 1$ implies $\mu_1(H) = 0$.

Let ε be any positive number and G be an open set containing H and such that

$$\mu_1(G) \leq \mu_1(H) + \varepsilon.$$

It follows from the definition of the set H that the family of closed intervals I contained in G and such that $\frac{\mu_1(I \cap E)}{\mu_1(I)} < \alpha$ is a Vitaly cover of H, and so (see [4]) there exists a sequence $\{I_\nu\}$ of mutually disjoint intervals of this family, for which $\mu_1(H - \bigcup_{\nu=1}^{\infty} I_\nu) = 0$. Since $H = \bigcup_{\nu=1}^{\infty} (H \cap I_\nu) \cup \left(H - \bigcup_{\nu=1}^{\infty} I_\nu\right) \subset \bigcup_{\nu=1}^{\infty} (E \cap I_\nu) \cup \left(H - \bigcup_{\nu=1}^{\infty} I_\nu\right)$,

then

$$\mu_1(H) \leq \sum_{\nu=1}^{\infty} \mu_1(E \cap I_\nu) < \alpha \cdot \sum_{\nu=1}^{\infty} \mu_1(I_\nu) \leq \alpha \cdot \mu_1(G) \leq \alpha \cdot (\mu_1(H) + \varepsilon).$$

Hence, while $\varepsilon \to 0$, we get $\mu_1(H) \leq \alpha \cdot \mu_1(H)$ and since $0 < \alpha < 1$, we get $\mu_1(H) = 0$.

Lemma 1. ([2], Lemma 4.1) *Let* $(X_3, \varrho_3, \mathfrak{M}_3, \mu_3) = (X_1 \times X_2, \varrho_1 \times \varrho_2, \overline{\mathfrak{M}_1 \times \mathfrak{M}_2}, \overline{\mu_1 \times \mu_2})$ ($\overline{\mu_1 \times \mu_2}$ *stands for the completion of the measure* $\mu_1 \times \mu_2$).

Let $A \in \mathfrak{M}_3$ and $\mu_3(A) < \infty$. Then the set B of all points $(x, y) \in A$, for which the section $A_x = \{y: y \in X_2, (x, y) \in A\}$ is μ_2-measurable of positive measure μ_2 and $D(y, A_x) = 1$ is μ_3-measurable and $\mu_3(A - B) = 0$.

Definition 3. *Let $A \in \mathfrak{M}_3$ and $B \in \mathfrak{M}_3$. By $A \Subset B$ we denote the statement that*

(i) $A \subset B$

(ii) *any point y_0 belonging to $A_{x_0} = \{y: y \in X_2, (x_0, y) \in A\}$ is a density point of the set $B_{x_0} = \{y: y \in X_2, (x_0, y) \in B\}$ with respect to $\mathscr{F}_2$ and*

(iii) *any point x_0 belonging to $A = \{x: x \in X_1, (x, y_0) \in A\}$ is a density point of the set $B^{y_0} = \{x: x \in X_1, (x, y_0) \in B\}$ with respect to $\mathscr{F}_1$.*

Lemma 2. *If $A \in \mathfrak{M}_3$, then there exists an F_σ set $B \subset A$ such that $\mu_3(A - B) = 0$ and $B \Subset B$.*

Proof. If $\mu_3(A) = 0$, then we may take the empty set for B. Otherwise let A' be such a F_σ that $\mu_3(A - A') = 0$. Let B_1 be the set of all points $(x, y) \in A'$ such that the section A'^y belongs to $\mathfrak{M}_1$, the measure $\mu_1(A'^y)$ is positive and x is a density point of A'^y with respect to $\mathscr{F}_1$. In accordance with Lemma 1 $B_1 \in \mathfrak{M}_3$ and $\mu_3(A' - B_1) = 0$. Let B_2 be a G_δ which contains $A' - B_1$ with the μ_3-measure equal to zero.

Let $A_1 = \{y: y \in X_2, \mu_2(B_2^y) > 0\}$. Evidently $\mu_2(A_1) = 0$.

Let $A_2 \subset X_2$ be a G_δ which contains the set A_1 with the μ_2-measure equal to zero. We put $B_3 = A - ((X_1 \times A_2) \cup B_2)$. The set $B_3 \subset X_3$ is an F_σ and $\mu_3(B_1 - B_3) = 0$ and any point $x \in (B_3)^y$ is a density point of the section $(B_3)^y$ with respect to $\mathscr{F}_1$. Let B_4 be a set of all points $(x, y) \in B_3$ for which the section $(B_3)_x$ is μ_2-measurable, $\mu_2((B_3)_x) > 0$ and y is a density point of the section $(B_3)_x$ with respect to $\mathscr{F}_2$. Once more $\mu_3(B_3 - B_4) = 0$. Denote by B_5 a G_δ of the μ_3-measure equal to zero containing $B_3 - B_4$. Let $A_3 = \{x: x \in X_1, \mu_2((B_5)_x) > 0\}$. It is clear that $\mu_1(A_3) = 0$.

Let $A_4 = \{y: y \in X_2, \mu_1(B_5^y) > 0\}$. Let $A_5 \subset X_1$ be a G_δ with the μ_1-measure equal to zero containing A_3 and let A_6 be a G_δ of μ_2-measure equal to zero containing A_4. For B take $B = B_3 - [(A_5 \times X_2) \cup (X_1 \times A_6) \cup B_5]$. By this definition B meets the conditions of the Lemma and this completes the proof.

Definition 4. *The function $f: X_1 \to R$ is called approximately upper (lower) semicontinous in the point $x_1 \in X_1$ with respect to $\mathscr{F}_1$ iff for every $a \in R$ if $f(x_1) < a$ $f(x_1) > a$), then there exists the set $F \in \mathfrak{M}_1$ such that $F \subset \{x: x \in X_1, f(x) < a\}$, $(F \subset \{x: x \in X_1, f(x) > a\})$ and $D(x_1, F) = 1$.*

A function that is simultaneously approximately lower and upper semicontinuous in $x_1 \in X_1$ with respect to $\mathscr{F}_1$, is called approximately continuous in x_1 with respect to $\mathscr{F}_1$.

A function that is approximately continuous (approximately lower semicontinuous) ((approximately upper semicontinuous)) in any $x \in X_1$ with respect to $\mathscr{F}_1$ is called approximately continuous (approximately lower semicontinuous) (approximately upper semicontinuous)) with respect to $\mathscr{F}_1$.

Lemma 3. *A function f: $X_1 \to R$ that is almost everywhere approximately lower semicontinuous with respect to $\mathcal{F}_1$, is μ_1-measurable.*

In order to prove Lemma 3 we first show

Lemma 3'. *The set $M \subset X_1$, whose almost every point is its density point with respect to $\mathcal{F}_1$ is μ_1-measurable.*

Proof. Decompose the set M into two disjoint sets M_1 and M_2 such that $M = M_1 \cup M_2$, $M_1 \in \mathfrak{M}_1$ and $\mu_{1*}(M_2) = 0$. The set M_1 belongs to $\mathfrak{M}_1$, so that by property (3) of the family $\mathcal{F}_1$ its density with respect to $\mathcal{F}_1$ equals 1 in almost every of its points and equals 0 in almost every point of the set M_2. Since the inner density of M with respect to $\mathcal{F}_1$ is positive in almost every of its points, the μ_1-measure of the M_2 equals zero, i. e. $M = M_1 \cup M_2 \in \mathfrak{M}_1$.

Now we shall prove Lemma 3. Let us fix an $a \in R$. It remains to be shown that the set $M = \{x \colon x \in X_1,\ f(x) > a\}$ belongs to $\mathfrak{M}_1$.

Let $x_1 \in M$ and let the function f be approximately lower semicontinuous in x_1 with respect to $\mathcal{F}_1$. Hence $f(x_1) > a$ and there exists a set $F \in \mathfrak{M}_1$ such that $F \subset M$ and $D(x_1, F) = 1$. Thus the set M has the density 1 with respect to $\mathcal{F}_1$ in almost each of its points and thus, by Lemma 3', $M \in \mathfrak{M}_1$. With that proof Lemma 3 is completed.

Definition 5. ([2], def. 4.2) *The function f: $X_2 \to R$ has the property* (K) *with respect to F_2 iff it is pointwise noncontinuous over any closed set, whose set of density points is dense in it with respect to F_2.*

It follows from the above definition that

(4) every function belonging to the Baire class I has the property (K).

Lemma 4. ([2], Lemma 4.2) *If the function g: $X_2 \to R$ has the property* (K) *with respect to $\mathcal{F}_2$, then for every set $F \in \mathfrak{M}_2$ of positive μ_2-measure and for every positive ε there exists a set $J \in \mathcal{F}_2$ such that $\mu_2(J \cap F) > 0$ and $\operatorname{osc}_{U \cap J} g \leq \varepsilon$, where U is the set of density points of F with respect to $\mathcal{F}_2$.*

Denote by $\Phi(f)$ the set of all points $(x_0, y_0) \in X_3$ such that either the function $f^{y_0}(x) = f(x, y_0)$ (called section) is not approximately continuous in x_0 with respect to $\mathcal{F}_1$ or the section $f_{x_0}(y) = f(x_0, y)$ is not approximately continuous in y_0 with respect to $\mathcal{F}_2$.

Lemma 5. *Let f: $X_3 \to R$ be a μ_2-measurable function. Then $\mu_3(\Phi(f)) = 0$.*

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be the sequence of all open intervals with rational endpoints such that $U_i \neq U_j$ for $i \neq j$. We put $A_n = f^{-1}(U_n)$. Lemma 2 implies that every set A_n contains a subset B_n such that $\mu_3(A_n - B_n) = 0$ and $B_n \Subset B_n$. Let $C_n = A_n - B_n$ and

$$(*) \qquad D = X_3 - \bigcup_{n=1}^{\infty} C_n .$$

Let $(x_0, y_0) \in D$ and $\varepsilon > 0$. Assume that $f(x_0, y_0) \in U_{n_0} \subset (f(x_0, y_0) - \varepsilon, f(x_0, y_0) + \varepsilon)$. The point (x_0, y_0) belongs to B_{n_0} and x_0 is a density point of the counterimage $(f^{y_0})^{-1} \in (f(x_0, y_0) - \varepsilon, f(x_0, y_0) + \varepsilon)$. Therefore the function f^{y_0} is approximately continuous in x_0 with respect to $\mathscr{F}_1$. The proof that the section f_{x_0} is approximately continuous in y_0 with respect to $\mathscr{F}_2$ is similar. Hence $D \cap \Phi(f) = \emptyset$. By (*) $\Phi(f) \subset \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (A_n - B_n)$. The latter set has the measure 0, and so $\mu_3(\Phi(f)) = 0$.

Lemma 6. ([1], lemma 2) *Let $(X, \mathfrak{M}, \mu)$ be a measurable space with the σ-finite measure μ. Let $g\colon X \to R$ be such that for any $\varepsilon > 0$ for class of sets*

$D_\varepsilon\{D\colon D \in \mathfrak{M}, \operatorname{osc}_D g \leq \varepsilon\}$ satisfies the following condition:

(d) *for any set $B \subset X$ with a positive measure μ there exists a set $D \in D_\varepsilon$ such that $D \subset B$ and $\mu(D) > 0$.*

Then the function g is $\bar{\mu}$-measurable, where $\bar{\mu}$ stands for the completion of μ.

(Davies has proved the Lemma under the assumption that μ is finite, whereas σ-finiteness is sufficient).

Definition 6. *The function $g\colon X_1 \to R$ is said to be degenerated in the point $x_1 \in X_1$ when there exists a closed interval I such that $g(x_1) \in \operatorname{Int}(I)$ and the external density with respect to $\mathscr{F}_1$ of the counterimage $g^{-1}(I)$ is in x_1 equal to zero.*

For the function $f\colon X_3 \to R$ we define $A(f)$ as the set of all points $(x, y) \in X_3$ such that the section f^y is degenerated in x.

Let $B(f)$ denote the set of all points $(x, y) \in X_3$ such that the section f_x is not approximately continuous with respect to $\mathscr{F}_2$ in y.

Theorem 1. *Let $f\colon X_3 \to R$ be a function such that all its sections f^y are μ_1-measurable. The function f is measurable if and only if $\mu_3(A(f) \cup B(f)) = 0$.*

Proof. The necessity of the condition follows from Lemma 5 as $A(f) \cup B(f) \subset \Phi(f)$. We shall therefore now show the sufficiency of the condition. Let $A = X_3 - [A(f) \cup B(f)]$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of closed sets with a μ_3 positive measure such that $A_i \subset A_{i+1}$ for $i = 1, 2, \ldots$ and $\mu_3\left(A - \bigcup_{i=1}^{\infty} A_i\right) = 0$.

Put

$$f_n(x, y) = \begin{cases} f(x, y) & \text{for } (x, y) \in A_n \\ 0 & \text{for } (x, y) \notin A_n . \end{cases}$$

As almost everywhere $\lim_{n\to\infty} f_n(x, y) = f(x, y)$ with respect to the measure μ_3, it is sufficient to show that the functions f_n satisfy the assumptions concerning the function g of Lemma 6 in the case when $X = X_3$ and $\mu = \mu_3$. Let $E \in \mathfrak{M}_3$, $0 < \mu_3(E) < \infty$.

Let $\varepsilon > 0$. Denote by Q the set of all points $x \in X_1$ such that the sections $E_x \in \mathfrak{M}_2$ and $\mu_2(E_x) > 0$ and $\mu_2([A(f) \cup B(f)]_x) = 0$. It follows from Fubini's Theorem that $Q \subset X_1$ is a μ_1-measurable set with a positive measure μ_1. For $x \in Q$ the sections $f_x(y)$ are almost everywhere approximately continuous and therefore by Lemma 3:

(1.1) For any $x \in Q$ the sections f_x are μ_2-measurable.

Let $\{J_k\}_{k=1}^{\infty}$ be the sequence of all sets belonging to $\mathscr{F}_2$ and let $\{K_k\}_{k=1}^{\infty}$ be the sequence of all closed intervals with rational ends and lengths smaller then ε. Denote by $Q_{r,s}$ the set of all points $x \in Q$ such that

(i) $\mu_2(J_r \cap E_x) > 0$
(ii) if $D(y, E_x) = 1$ and $y \in J_r$, then $f_n(x, y) \in K_s$.

Notice that

(1.2) for any $x \in Q$, any $n \in N$ and any set $Z \in \mathfrak{M}_2$ with a positive measure μ_{22} and for any $\delta > 0$ there exists a set $J \in \mathscr{F}_2$ such that $\mu_2(J \cap Z) > 0$ and $\operatorname{osc}_{U \cap J} (f_n)_x \leqslant \delta$, where U is the set of density points of Z with respect to $\mathscr{F}_2$.

Indeed. Let $Z \in \mathfrak{M}_2$ be a set with a positive measure μ_2 and $\delta > 0$. We discuss two cases.

1. If $\mu_2(Z - (A_n)_x) > 0$, then there exists a point $y' \in X_2$ such that $y' \in Z - (A_n)_x$ and $D(y', Z - (A_n)_x) = 1$. As the set $(A_n)_x$ is closed, it follows from property (2) of the family $\mathscr{F}_2$ that there exists $J \in \mathscr{F}_2$ such that $y' \in \operatorname{Int}(J)$ and $J \cap (A_n)_x = \emptyset$. Therefore for $y \in J$ we have $f_n(x, y) = (f_n)_x(y) = 0$. Hence $\operatorname{osc}_{U \cap J}(f_n)_x = 0 < \delta$.

2. If $\mu_2(Z - (A_n)_x) = 0$, then we notice that in this case all density points of Z belong to $(A_n)_x$. In order to show that

(1.2) holds in this case it is sufficient to show that

(1.3) there exists a set $I \in \mathscr{F}_2$ such that $\mu_2(I \cap Z \cap (A_n)_x) > 0$ and $\operatorname{osc}_{U_1 \cap I} (f_n)_x \leqslant \delta$, where U_1 is the set of density points of $Z \cap (A_n)_x$ with respect to $\mathscr{F}_2$.

Assume that (1.3) does not hold. Then we have

(1.4) if for the set $J \in \mathscr{F}_2$ the inequality $\mu_2(J \cap Z \cap (A_n)_x) > 0$ holds, then $\operatorname{osc}_{U_1 \cap J} (f_n)_x > \delta$.

Let $y_1 \in Z \cap (A_n)_x$ and

(1.5) $D(y_1, Z\cap(A_n)_x)=1.$

Such a point y_1 exists according to property (3) of $\mathscr{F}_2$.
Let $I_1 \in \mathscr{F}_2$ be a set such that

(1.6) $y_1 \in \operatorname{Int}(I_1)$

(1.7) $\dfrac{\mu_2(I_1\cap Z\cap(A_n)_x)}{\mu_2(I_1)}>\dfrac{3}{4}$

(1.8) $\dfrac{\mu_2\left(I_1\cap\left\{y\colon y\in X_2,\ |(f_n)_x(y)-(f_n)_x(y_1)|<\frac{\delta}{8}\right\}\right)}{\mu_2(I_1)}>\dfrac{3}{4}.$

The existence of I follows from (1.5) and from the fact that y_1 is a point of approximative continuity of the section $(f_n)_x$ with respect to $\mathscr{F}_2$.

Let $G_1 = \left\{y\colon y\in I_1\cap Z\cap(A_n)_x,\ |(f_n)_x(y)-(f_n)_x(y_1)|>\frac{\delta}{2}\right\}.$

(1.9) $\mu_2(G_1)>0.$

Indeed. Assume that $\mu_2(G_1)=0$. Then for points $y\in[I_1\cap Z\cap(A_n)_x]-G_1$ the inequality $|(f_n)_x(y)-(f_n)_x(y_1)|\leq\frac{\delta}{2}$ holds and therefore

(1.10) $\operatorname*{osc}_{[I_1\cap Z\cap(A_n)_x]-G_1}(f_n)\leq\delta$ and

(1.11) $\{y\colon y\in\operatorname{Int}(I_1),\ D(y_1, Z\cap(A_n)_x)=1\}\subset[I_1\cap Z\cap(A_n)_x]-G_1.$

Indeed. Assume that (1.11) does not hold. Then there exists a point $y_1'\in \operatorname{Int}(I_1)\cap Z\cap(A_n)_x$ such that $D(y_1', Z\cap(A_n)_x)=$ and $y_1'\in G_1$.

Then $|(f_n)_x(y_1')-(f_n)_x(y_1)|>\frac{\delta}{2}$. The point y_1' is a density point of the set $(A_n)_x$ with respect to $\mathscr{F}_2$ and therefore it is a point of approximative continuity of the section $(f_n)_x$ with respect to $\mathscr{F}_2$.

Assume that $(f_n)_x(y_1')>(f_n)_x(y_1)$. Denote $\eta = (f_n)_x(y_1') - (f_n)_x(y_1) - \frac{\delta}{2}$. The number η chosen in this way is positive. Consider the set

$$H_1=\{y\colon y\in I_1\cap Z\cap(A_n)_x,\ |(f_n)_x(y)-(f_n)_x(y_1')|<\eta\}$$

being a subset of G_1. As y_1' is a point of approximative continuity of the function $(f_n)_x$, it is also a density point of the set $H_1\subset G_1$ which contradicts the condition $\mu_2(G_1)=0$. In the case when $(f_n)_x(y_1')<(f_n)_x(y_1)$, the reasoning is analogous. Thus we have shown that the nodensity point of $Z \cap (A_n)_x$ belonging to Int (I_1) can belong to G_1 too. We have proved (1.11) in spite of the assumption that (1.11) does not hold. Therefore (1.11) must be true. Therefrom and from (1.10) we infer that

(1.12) $$\operatorname*{osc}_{U_1 \cap I_1} (f_n)_x \leq \delta.$$

As $I_1 \in \mathcal{F}_2$ and because of (1.7) $\mu_2(I_1 \cap Z \cap (A_n)_x) > 0$, therefore by (1.4) we obtain $\operatorname*{osc}_{U_1 \cap I_1} (f_n)_x > \delta$, which contradicts (1.12). Thus negation of (1.9) leads to a contradiction and therefore (1.9) must be true.

(1.13) There exists a point $y_2 \in G_1 \cap \text{Int}(I_1)$ such that $D(y_2, G_1) = 1$.

Indeed. Assume that (1.13) does not hold. Then

$\{y: y \in G_1 \cap \text{Int}(I_1), D(y, G_1) = 1\} = \emptyset$ and therefore

(1.14) $$\mu_2(G_1 \cap \text{Imt}(I_1)) = 0.$$

The inequality $|(f_n)_x(y) - (f_n)_x(y_1)| \leq \frac{\delta}{2}$ must hold for all $y \in [\text{Int}(I_1) \cap Z \cap (A_n)_x] - G_1$ and therefore

(1.15) $$\operatorname*{osc}_{[\text{Int}(I_1) \cap Z \cap (A_n)_x] - G_1} (f_n)_x \leq \delta.$$

On the other hand (1.6) and (1.5) hold true. Therefore there exist $I_1' \in \mathcal{F}_2$ such that $I_1' \subset \text{Int}(I_1)$ and

(1.16) $$\mu_2(I_1' \cap Z \cap (A_n)_x) > 0.$$

We shall prove that

(1.17) $$G_1 \cap (U_1 \cap I_1') = \emptyset.$$

Assume that there exists a point $y_1'' \in G_1 \cap I_1' \cap U_1$. Then the inequality $|(f_n)_x(y_1'') - (f_n)_x(y_1)| > \frac{\delta}{2}$ holds. Point y_1'' is a density point of $(A_n)_x$ and is therefore a point of approximative continuity of the section $(f_n)_x$. Assume that $(f_n)_x(y_1'') > (f_n)_x(y_1)$. Denote $\eta = (f_n)_x(y_1'') - (f_n)_x(y_1) - \frac{\delta}{2}$. The number chosen in this way is positive. Define

$$H_2 = \{y: y \in \text{Int}(I_1) \cap Z \cap (A_n)_x, |(f_n)_x(y_1'') - (f_n)x(y)| < \eta\}.$$

Obviously $H_2 \subset G_1 \cap \text{Int}(I_1)$. As y_1'' is a point of approximative continuity of the section $(f_n)_x$ it is also a density point of $H_2 \subset G_1 \cap \text{Int}(I_1)$, which contradicts (1.14).

In the case when $(f_n)_x(y_1'') < (f_n)_x(y_1)$ the reasoning is analogous. The assumption that (1.17) does not hold leads to a contradiction. Thus (1.17) holds. From this and from (1.15) we get

$$\operatorname*{osc}_{U_1 \cap I_1'} (f_n)_x \leq \delta. \tag{1.18}$$

As $I_1' \in \mathcal{F}_2$ and (1.16), we obtain by (1.4) $\operatorname*{osc}_{U_1 \cap I_1'} (f_n)_x > \delta$, which contradicts (1.18). We have shown that the negation of (1.13) leads to a contradiction. Therefore (1.13) leads to a contradiction. Therefore (1.13) houlds true. Thus $|(f_n)_x(y_2) - (f_n)_x(y_1)| > \frac{\delta}{2}$.

Let $I_{22} \subset \operatorname{Int}(I_1)$ be such a set belonging to the family $\mathcal{F}_2$ that $y_2 \in \operatorname{Int}(I_2)$ and $d(I_2) < \frac{1}{2}$ and

$$\frac{\mu_2(I_2 \cap Z \cap (A_n)_x)}{\mu_2(I_2)} > \frac{3}{4} \quad \text{and} \tag{1.19}$$

$$\frac{\mu_2\left(I_2 \cap \left\{y \colon y \in X_2,\ |(f_n)_x(y_2) - (f_n)_x(y)| < \frac{\delta}{8}\right\}\right)}{\mu_2(I_2)} > \frac{3}{4}. \tag{1.20}$$

The existence of I_2 follows from the fact that y_2 is a point of approximative continuity of the section $(f_n)_x$. Similarly as before the set

$$G_2 = \left\{y \colon y \in I_2 \cap Z \cap (A_n)_x,\ |(f_n)_x(y_2) - (f_n)_x(y)| > \frac{\delta}{2}\right\},$$

being a subset of $I_2 \cap Z \cap (A_n)_x$, is μ_2-measurable and has a positive measure μ_2.

Let $y_3 \in G_2 \cap \operatorname{Int}(I_2)$ be a density point of G_2 with respect to $\mathcal{F}_2$. Evidently $|(f_n)_x(y_3) - (f_n)_x(y_2)| > \frac{\delta}{2}$. Proceeding analogously we define the sequence $\{I_k\}_{k=1}^{\infty}$ of the sets from $\mathcal{F}_2$ such that $I_{i+1} \subset \operatorname{Int}(I_i)$, $d(I_i) < \frac{1}{2^{i-1}}$ for $i = 1, 2, \ldots$ and the sequence points $\{y_k\}_{k=1}^{\infty}$ such that $y_k \in \operatorname{Int}(I_k)$ $(k = 1, 2, \ldots)$ and

$$|(f_n)_x(y_{i+1}) - (f_n)_x(y_i)| > \frac{\delta}{2} \quad \text{for} \quad i = 1, 2, \ldots. \tag{1.21}$$

The set $\bigcap_{i=1}^{\infty} I_i$ consists of one point y_0. As the section f_x is approximately continuous in y_0 $\left(\text{as } y_0 \in \bigcap_{i=1}^{\infty} (I_i \cap (A_n)_x)\right)$, we have $D(y_0, K) = 1$, where $K = \left\{y \colon y \in X_2,\ |f_x(y_0) - f_x(y)| < \frac{\delta}{8}\right.$.

Moreover $\frac{\mu_2(I_k \cap (A_n)_x)}{\mu_2(I_k)} > \frac{3}{4}$, hence there exists N such that for $k > N$ $\frac{\mu_2(I_k \cap (A_n)_x \cap K)}{\mu_2(I_k)} \geq \frac{3}{4}$ and consequently also

$$\frac{\mu_2\left(I_k \cap \left\{y: y \in X_2, |(f_n)_x(y_0) - (f_n)_x(y)| < \frac{\delta}{8}\right\}\right)}{\mu_2(I_k)} > \frac{1}{2}.$$

On the other hand

$$\frac{\mu_2\left(I_k \cap \left\{y: y \in X_2\ |(f_n)_x(y) - af_n)_x(y_k)| < \frac{\delta}{8}\right\}\right)}{\mu_2(I_k)} > \frac{3}{4}$$

(cf. (1 8) and (1.20)).

Therefore for

$$k > N \left\{y: y \in X_2, |(f_n)_x(y) - (f_n)_x(y_0)| < \frac{\delta}{8}\right\} \cap$$

$$\cap \left\{y: y \in X_2, |(f_n)_x(y) - (f_n)_x(y_k)| < \frac{\delta}{8}\right\} \neq 0.$$

Thence for $k > N$ the following inequality holds $|(f_n)_x(y_k) - (f_n)_x(y_0)| < \frac{\delta}{4}$, which contradicts (1.21). Thus the negation of (1.3) leads to a contradiction. Therefore (1 3) holds true and also (1.2) holds true, because both possible cases have been proved. Therefore $Q = \bigcup_{r,s} Q_{r,s}$. Thus there exists a couple of positive integers (r_0, s_0) such that $\mu_1(Q_{r_0, s_0}) > 0$. Put $P = \{x: x \in X_1, D^*(x, Q_{r_0, s_0}) = 1\}$. Set $P \in \mathfrak{M}_1$ and $\mu_1(P) > 0$. Let $F = E \cap (P \times I_{r_0})$. Set $F \in \mathfrak{M}_3$ and $\mu_3(F) > 0$, because for any $x \in Q_{r_0, s_0}$ $\mu_2(F_x) > 0$. By Lemma 2 there exists the sets $G \subset A_n$, $H \subset (X_3 - A_n)$ and $L \subset F$ of the F_σ type such that $\mu_3(A_n - G) = 0$, $\mu_3((X_3 - A_n) - H) = 0$, $\mu_3(F - L) = 0$ and $G \subset G$, $H \subset H$ and $L \subset L$. Let $M = L \cap (G \cup H)$. Notice that $M \in \mathfrak{M}_3$ has a positive measure ($\mu_3(X_3 - (G \cup H)) = 0$ and $\mu_3(L) > 0$).

To prove the theorem it is sufficient to show that $f_n(x, y) \in K_{s_0}$ for any point $(x, y) \in M$. Let $(\xi, \eta) \in M$ and let $\delta > 0$. Denote by α the upper density of $(f_n^\eta)^{-1}(f_n(\xi, \eta) - \delta, f_n(\xi, \eta) + \delta)$ in ξ with respect to $\mathcal{F}_1$. Evidently $\alpha > 0$. Discuss the case when $(\xi, \eta) \in G$. Then ξ is a density point of G^η and therefore also of $(A_n)^\eta$. As the upper density of the counterimage $(f^\eta)^{-1}(f(\xi, \eta) - \delta, f(\xi, \eta) + \delta)$ is positive in ξ with respect to $\mathcal{F}_1$, the same must hold for the density of the counterimage $(f_n^\eta)^{-1}(f_n(\xi, \eta) - \delta, f_n(\xi, \eta) + \delta)$ in ξ with respect to $\mathcal{F}_1$. In the case when $(\xi, \eta) \in H$, then ξ is a density point of H^η with respect to $\mathcal{F}_1$ and therefore also of $(X_3 - A_n)^\eta$.

On this set f_n is a constant equal to zero. The density of the counterimage $(f_n^\eta)^{-1}(f_n(\xi, \eta)-\delta,\ f_n(\xi, \eta)+\delta)$ in ξ with respect to $\mathscr{F}_1$ is in this case equal to 1.

For $\delta > 0$ there exists a set $I \in \mathscr{F}_1$ containing ξ such that $\frac{\mu_1(I \cap M^\eta)}{\mu_1(I)} > 1 - \frac{\alpha}{4}$,

$$\frac{\mu_1(I \cap \{x\colon x \in X_1,\ |f_n(x, \eta) - f_n(\xi, \eta)| < \delta\})}{\mu_1(I)} > 1 - \frac{\alpha}{4}.$$

Hence all these three sets have a common point $x_0 \in I$. As $(x_0, \eta) \in M$, the section F_{x_0} is μ_2-measurable and has a positive measure μ_2 and η is a density point of F_{x_0} with respect to $\mathscr{F}_2$. Moreover $\eta \in I_{r_0}$ and $x_{00} \in Q_{r_0, s_0}$. As $x_0 \in \{x\colon |f_n(x, \eta) - f_n(\xi, \eta) < \delta\}$, we have $f_n(x_0, \eta) \in K_{s_0}$. From this we infer that the distance from $f_n(\xi, \eta)$ to K_{s_0} is smaller than δ. As δ is an arbitrary number and K_{s_0} is closed, there is $f_n(\xi, \eta) \in K_{s_0}$. The proof of the theorem is completed. Theorem 1 is a generalization of Theorem 6 of [3].

Let $f\colon X_1 \times X_2 \to R$ be founded, where $X_2 = R^n$. Let ϱ_2 be a Euclidean metric in R^n and μ_2 an arbitrary regular complete measure σ-finite and defined on some σ-field $\mathfrak{M}_2$ enclosing Borel sets.

Denote by $M_k^y(x_0, y_0)$ the upper bound of the functions $\varphi(y) = f(x_0, y)$ in the open sphere $K\left(y_0, \frac{1}{k}\right) \subset R^n$.

Let $A \subset X_1 \times X_2$ and $f\colon X_1 \times X_2 \to R$. The definition of density of the set $A \subset X_1 \times X_2$ in (x, y) with respect to $\mathscr{F}_3$ is analogous to Definition 2.

Also similar to Definition 4 is the definition of the approximative lower semicontinuity of the function $f\colon X_1 \times X_2 \to R$ with respect to $\mathscr{F}_3$.

Lemma 7. *If all sections f^y are approximately lower semicontinuous with respect to $\mathscr{F}_1$, then M_k^y considered as a function of two variables x and y is approximately lower semicontinuous with respect to $\mathscr{F}_3$, where*

$$\mathscr{F}_3 = \mathscr{F}_1 \times \mathscr{F}_2 = \{F\colon F = F_1 \times F_2,\ F_1 \in \mathscr{F}_1,\ F_2 \in \mathscr{F}_2\}.$$

Proof. Fix the point $(x_0, y_0) \in X_1 \times R^n$ and a number $a \in R$. Assume that $M_k^y(x_0, y_0) > a$. Let $0 < \varepsilon < M_k^y(x_0, y_0) - a$. It follows from the definition of $M_k^y(x_0, y_0)$ that in the sphere $K\left(y_0, \frac{1}{k}\right)$ there exists a point y_1 for which the inequality

$$(1) \qquad f(x_0, y_1) > M_k^y(x_0, y_0) - \frac{\varepsilon}{2} \text{ holds.}$$

According to the approximative lower semicontinuity of the function f^{y_1} with respect to $\mathscr{F}_1$ in x_0 for the set $E = \{x\colon x \in X_1,\ f^{y_1}(x) > b\}$, where $b = f^{y_1}(x_0) - \frac{\varepsilon}{2}$, there exists the set $F \in \mathfrak{M}_1$ such that $F \subset E$ and $D(x_0, F) = 1$. Therefore for any $x \in F$ the inequality

(2) $$f(x, y_1) > f(x_0, y_1) - \frac{\varepsilon}{2} \text{ holds.}$$

Let $\beta = \varrho(y_1, \operatorname{Fr} K\left(y_0, \frac{1}{k}\right))$, where $\operatorname{Fr} K\left(y_0, \frac{1}{k}\right)$ denotes the border of $K\left(y_0, \frac{1}{k}\right)$. Then $\varrho(y, y_0) < \beta$ implies $y_1 \in K\left(y, \frac{1}{k}\right)$. As $y_1 \in K\left(y, \frac{1}{k}\right)$ for $y \in K(y_0, \beta)$, for these points y the inequality

(3) $$M_k^y(x, y) \geqslant f(x, y)$$

holds (according to the definition of M_k^y in (x, y)).

From (1), (2), (3) we obtain $M_k^y(x, y) > M_k^y(x_0, y_0) - \varepsilon$ for all points $(x\ y)$ belonging to the set

$$A = (F \times X_2) \cap (X_1 \times K(y_0, \beta))$$

As $D((x_0, y_0), A) = 1$ and

$$A \subset \{(x, y) : (x, y) \in X_1 \times R^n, M_k^y(x, y) > a\},$$

because $0 < \varepsilon < M_k^y(x_0, y_0) - a$ we come to the conclusion that M_k^y is approximately lower semicontinuous in (x_0, y_0) with respect to $\mathcal{F}_3$. The proof of the Lemma is completed.

Theorem 2. *If all sections f^y of the function $f: X_1 \times R^n \to R$ are approximately lower semicontinuous with respect to $\mathcal{F}_1$ and all sections f_x of this function are upper semicontinuous, then f is a point limit of a non-increasing sequence of functions approximately lower semicontinuous with respect to $\mathcal{F}_3$.*

If we assume that the family $\mathcal{F}_3$ satisfied (3) *for the families $\mathcal{F}_1$ and $\mathcal{F}_2$, then f would be μ_3-measurable.*

Proof. Denote $M^y(x_0, y_0) = \lim_{k\to\infty} M_k^y(x_0, y_0)$. As the sections f_x are upper semicontinuous, $M^y(x_0, y_0) = f(x_0, y_0)$. The function $M^y: (x, y) \to M_k^y(x, y)$ is the point limit of a non-increasing sequence of functions M_k^y, which are according to Lemma 7 approximately lower semicontinuous with respect to $\mathcal{F}_3$. This ends the proof of the Theorem.

Theorem 2 is a generalization of Theorem 2.1 of [2].

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ИЗМЕРИМОСТЬ ДЕИСТВИТЕЛЬНЫХ ФУНКЦИЙ, ЗАДАНЫХ НА ДЕКАРТОВОМ ПРОИЗВЕДЕНИИ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

Гражина Квециньска

Резюме

В настоящей работе находится необходимое и достаточное условие измеримости функций, заданных на декартовом произведении двух метрических пространств с мерами, которы удовлетворяют некоторым дополнительным условиям.