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ON A TENSOR PRODUCT OF WEAKLY COMPACT MAPPINGS

MILOSLAV DUCHOŇ

Let S and T be Hausdorff locally compact spaces, $C_0(S)$ and $C_0(T)$ the spaces of all continuous complex-valued functions on S and T, respectively, vanishing at infinity, with the sup norm; further let X and Y be Hausdorff locally convex spaces. If $u\colon C_0(S)\to X$ and $v\colon C_0(T)\to Y$ are continuous linear mappings, then their tensor product, i. e. the linear mapping $u\otimes v\colon C_0(S)\otimes C_0(T)\to X\otimes Y$ such that $u\otimes v(f\otimes g)=u(f)\otimes v(g)$, $f\in C_0(S)$, $g\in C_0(T)$, is continuous, if both tensor products $C_0(S)\otimes C_0(T)$ and $X\otimes Y$ are endowed with the ε-topology; thus the mapping $u\otimes v\colon C_0(S)\otimes_\varepsilon C_0(T)\to X\otimes_\varepsilon Y$ is continuous, moreover $u\otimes v$ has the extension onto $C_0(S\times T)$ [11, p. 275 and p. 288]. If u and v are compact, then their tensor product is also compact as it follows from [11, p. 285]. It is the main purpose of this paper to prove that if u and v are weakly compact mappings, then their tensor product is a weakly compact mapping from $C_0(S\times T)$ into $X\otimes_\varepsilon Y$. Moreover the approach used makes it possible to show that the ε-topology on $X\otimes Y$ may be replaced by the s-topology or summing topology [15, p. 313], [10, p. 22] generally finer than the ε-topology and weaker than the projective topology or π-topology. That the mapping $u\otimes v$ is weakly compact under the ε-topology if u and v are weakly compact can be also deduced from our results in [4] or from the results of the [14] obtained by different methods in the case when X and Y are Banach spaces; compare also the paper [12]. In the present paper we obtain this result as a corollary of a more general result.

1. The tensor product of regular Borel vector measures

Denote by $B_a(S)$, $B(S)$ and $B_w(S)$ the sigma ring of Baire, Borel and weakly Borel sets in S, i. e. the sigma ring generated by compact G_δ, compact and closed sets, respectively; $B_a(S)\times B_a(T)$, $B(S)\times B(T)$ and $B_w(S)\times B_w(T)$ denote the sigma rings in $S\times T$ generated by Baire, Borel and weakly Borel rectangles, respectively. The following relations hold

(1) $B_a(S)\times B_a(T)=B_a(S\times T)$,
(2) $B(S)\times B(T)\subset B(S\times T)$,
(3) $B_w(S)\times B_w(T)\subset B_w(S\times T)$,

in other words, Baire sets "multiply"; in general, Borel sets and weakly Borel sets do not, for there exist compact topological groups G for which the inclusion

$$B(G) \times B(G) \subset B(G \times G)$$

is proper [1], [8], [9].

Let X and Y be Hausdorff locally convex spaces with the corresponding systems $P=(p)$ and $Q=(q)$ of continuous seminorms determining their locally convex topologies $\tilde{X}$ denoting the quasicompletion of X .

A sigma additive set function m_a: $B_a(S) \to X$, m: $B(S) \to X$ and m_w: $B_w(S) \to X$ is called a Baire, Borel and weakly Borel vector measure, respectively, on S with values in X.

Let m: $B(S) \to X$ and n: $B(T) \to Y$ be Borel vector measures. From [7] it follows that there exists their tensor product $l = m \times n$ as a vector measure on $B(S) \times B(T)$ with values in the tensor product $X \otimes Y$, such that $m \times n(A \times B) = m(A) \otimes n(B)$, $A \in B(S)$, $B \in B(T)$ if l is allowed to take its values in $X \tilde{\otimes} \varepsilon Y$, the completion of the algebraic tensor product $X \otimes Y$ in the ε-topology or the topology of bi-equicontinuous convergence [11]. Similarly for Baire and weakly Borel measures. In [10] this result was improved allowing a topology on the algebraic tensor product $X \otimes Y$ in which l is sigma additive and such that all values of l belong to the completion of $X \otimes Y$ under the topology called the τ-topology or the s-topology or the summing topology [15, p. 313] finer than the ε-topology and coarser than the projective tensor topology or the π-topology. It follows from [13] that when $X = Y = l^2$ the ε, τ or s and π-topologies are all distinct. Recall that the s-topology (or summing topology) on $X \otimes Y$ is defined by the system of seminorms

$$p \otimes_{sl} q(z) = \inf \sup p \left(\sum_{i=1}^{k} c_i q(y_i)\, x_i \right), \quad p \in P, \; q \in Q,$$

[the sl-topology]

$$p \otimes_{sr} q(z) = \inf \sup q \left(\sum_{i=1}^{k} c_i p(x_i)\, y_i \right), \quad p \in P, \; q \in Q,$$

[the sr-topology]

where the supremum is taken over all choices of complex numbers c_i, $|c_i| \leq 1$, $i = 1, \ldots, k$, and the infimum is taken over all expressions of z in the form

$$z = \sum_{i=1}^{k} x_i \otimes y_i$$

with $x_i \in X$, $y_i \in Y$, $i = 1, \ldots, k$. We have

$$p \otimes_\varepsilon q \leq p \otimes_{sl} q \leq p \otimes_\pi q$$

$$p\otimes_\varepsilon q \leqslant p\otimes_{sr} q \leqslant p\otimes_\pi q,$$

$$p\otimes_{sl} q(x\otimes y) = p\otimes_{sr} q(x\otimes y) = p(x)\,q(y) \quad \text{for all } x\in X,\ y\in Y.$$

It follows that the identity mapping on $X\otimes Y$ extends uniquely to continuous inclusions $X\otimes_\pi Y \to X\otimes_{si} Y \to X\otimes_\varepsilon Y$, $i=l$ or r, where $X\otimes_\pi Y$, $X\otimes_s Y$ and $X\otimes_\varepsilon Y$ denotes the quasicompletion of $X\otimes Y$ equipped with the π-topology, the s-topology and the ε-topology, respectively. In [10] there is considered still another topology on $X\otimes Y$, namely for $p\in P$ and $q\in Q$ one defines a seminorm by the formula

$$p\tau q = \frac{1}{2}\,(p\otimes_{sl} q + p\otimes_{sr} q).$$

Clearly we have

$$p\tau q(x\otimes y) = p(x)\,q(y), \quad x\in X, \quad y\in Y,$$

$$p\otimes_\varepsilon q \leqq p\tau q \leqq p\otimes_\pi q.$$

$X\otimes_\tau Y$ denotes the quasicompletion of $X\otimes Y$ endowed with τ-topology. Again the corresponding continuous inclusions could be strict.

According to Theorem proved in [10] we obtain using (1) the following.

Theorem 1. *Let m_a: $B_a(S)\to X$ and n_a: $B_a(T)\to Y$ be Baire vector measures. Then there exists a unique Baire vector measure $l_a = m_a\times n_a$: $B_a(S\times T)\to X\otimes_\tau Y$ such that*

$$l_a(A\times B) = m_a(A)\otimes n_a(B), \qquad A\in B_a(S), \quad B\in B_a(T).$$

A similar theorem can be stated if the Borel sets or the weakly Borel sets multiply. If the Borel sets do not multiply we must proceed in the same way as that used for the ε-topology in [4]. However the satisfactory result can be obtained only in the case that the vector measures are regular.

Let $R(S)$ be a ring of subsets of S and m_0: $R(S)\to X$ an additive set function. We say that m_0 is regular if for each E in $R(S)$ and every $d>0$, for each p in P there exist a compact set C in $R(S)$ and an open set V in $R(S)$, $C\subset E\subset V$, such that we have $p(m_0(H))<d$ for every H in $R(S)$ with $H\subset V-C$. Recall that if $m_0: R(S)\to X$ is additive and regular, then m_0 is countably additive [3, p. 510, Theorem 3]. Recall that every Baire vector measure m_a: $B_a(S)\to X$ is regular [3, p. 511] or [6, Th. 2]. Moreover every Baire vector measure m_a: $B_a(S)\to X$ can be extended uniquely to a regular Borel vector measure m: $B(S)\to \tilde{X}$ and every regular Borel vector measure m: $B(S)\to X$ can be extended uniquely to a regular weakly Borel vector measure m_w: $B_w(S)\to \tilde{X}$ [6, Theorem 5 and Theorem 6].

Now if m: $B(S)\to X$ and n: $B(T)\to Y$ are regular Borel vector measures then there exists their τ-tensor product as a vector measure $l = m\times_\tau n$: $B(S)\times B(T)\to X\otimes_\tau Y$ such that $l(A\times B) = m(A)\otimes n(B)$, $A\in B(S)$, $B\in B(T)$, but this may fail to be a Borel vector measure since in general its domain of definition is not satisfactory if Borel sets do not multiply. The same situation appears in the case of weakly Borel vector measures. If we take in such a case m_a and n_a as the Baire restrictions of m and n, then the τ-tensor product of m_a and n_a, namely $m_a \times_\tau n_a$, is according to Theorem 1 a Baire vector measure on $S \times T$ with values in $X \bigotimes_\tau Y$. Now we obtain a regular Borel vector measure r on $S \times T$ with values in $X \bigotimes_\tau Y$ as the unique regular extension of $m_a \times_\tau n_a$ which always exists. We must prove that r is an extension of $m \times_\tau n$.

Theorem 2. *If m: $B(S) \to X$ and n: $B(T) \to Y$ are regular Borel vector measures, then there exists one and only one regular Borel vector measure on $S \times T$ with values in $X \bigotimes_\tau Y$ which extends $m \times_\tau n$. This measure is simply the measure r described above.*

This theorem is a corollary of the following more general result proved in [4, p. 326].

Theorem 3. *Suppose that m_0: $B(S) \times B(T) \to X$ is a vector measure such that*

(i) *for each compact set C in S, the correspondence*

$$B \to m_0(C \times B), \quad B \in B(T),$$

is a regular Borel vector measure on T with values in X, and

(ii) *for each compact set D in T, the correspondence*

$$A \to m_0(A \times D), \quad A \in B(S),$$

is a regular Borel vector measure on S with values in X. Then m_0 may be extended to one and only one regular Borel vector measure r on $S \times T$ with values in X, r: $B(S \times T) \to \tilde{X}$.

Proof of Theorem 2. We apply Theorem 3 to the τ-tensor product $m_0 = m \times_\tau n$; conditions (i) and (ii) are verified using the fact that

$$m_0(A \times B) = m(A) \otimes n(B), \quad p\tau q(m_0(A \times B)) = p(m(A))\, q(n(B))$$

for all rectangles with Borel sides.

The next theorem shows that if $m \times_\tau n$ is non-zero, then no regular Borel extension of $m \times_\tau n$ is possible, unless m and n are both regular.

Theorem 4. *If there exists a non-zero regular Borel vector measure l: $B(S \times T) \to X \bigotimes_\tau Y$ which extends $m \times_\tau n$, then both m and n are regular.*

Proof. If l is a regular Borel extension of $m \times_\tau n$, it is a regular extension of the Baire measure $m_a \times_\tau n_a$, m_a and n_a being the restrictions of m and n to the Baire sets, respectively. It follows from Theorem 2 that l extends $m' \times_\tau n'$, where m' and n' are regular Borel extensions of m_a and n_a, respectively. Hence $m' \times_\tau n' = m \times_\tau n$, and thus

$$m'(A) \otimes n'(B) = m(A) \otimes n(B)$$

for all Borel sets A in S and B in T. Since l, and hence $m' \times_\tau n'$, are non-zero, it follows that $m = m'$ and $n = n'$. For if we choose a Baire set B in $B_a(T)$ such that $n'(B) = n(B) = n_a(B) \neq 0$, we have $m'(A) \otimes n(B) = m(A) \otimes n(B)$ for all Borel sets A in $B(S)$; there is q in Q such that $q(n(B)) \neq 0$; for all p in P we have $p(m'(A))\, q(n(B)) = p(m(A))\, q(n(B))$, which implies $p(m'(A)) = p(m(A))$ for all p in P and thus $m'(A) = m(A)$; since this holds for all A in $B(S)$ we have $m' = m$ and similarly $n' = n$.

A bilinear mapping $b\colon X \times Y \to Z$, where Z is also a Hausdorff locally convex space, is said to be τ-hypercontinuous or s-hypercontinuous if the linear mapping induced by it on the tensor product $X \otimes Y$ is continuous under the τ-topology or the s-topology. We have the following

Theorem 5. *Let $b\colon X \times Y \to Z$ be a τ-hypercontinuous bilinear mapping and Z a sequentially complete space. If $m\colon B(S) \to X$ and $n\colon B(T) \to Y$ are regular Borel vector measures, then there exists one and only one regular Borel vector measure $r\colon B(S \times T) \to Z$ for which*

$$r(A \times B) = b(m(A), n(B)), \quad A \in B(S), \quad B \in B(T).$$

Proof. If r_0 is the unique regular Borel vector measure on $S \times T$ with values in $X \otimes_\tau Y$ which extends $m \times_\tau n$ as in Theorem 2 and b' is a linear mapping induced by b, we define a set function $r\colon B(S \times T) \to Z$ as follows:

$$r(G) = b'(r_0(G)), \quad G \in B(S \times T);$$

since r_0 is regular and b' is continuous, r is a regular Borel vector measure on $S \times T$ with values in Z.

A generalization of Theorem 2 for weakly Borel sets is the following.

Theorem 6. *Let $m_w\colon B_w(S) \to X$ and $n_w\colon B_w(T) \to Y$ be regular weakly Borel vector measures. Then there exists one and only one regular weakly Borel measure $l_w\colon B_w(S \times T) \to X \otimes_\tau Y$, extending the τ-tensor product $r_w = m_w \times_\tau n_w\colon B_w(S) \times B_w(T) \to X \otimes_\tau Y$.*

The validity of the last theorem follows from an adaptation of Theorem 3 for weakly Borel measures.

Theorem 7. *Let r_w be a vector measure on the sigma algebra $B_w(S) \times B_w(T)$ with values in X such that*

(i) *for each closed set A in S, the correspondence*

$$F \to r_w(A \times F), \quad F \in B_w(T),$$

is a regular weakly Borel vector measure on T, and

(ii) *for each closed B in T, the correspondence*

$$E \to r_w(E \times B), \quad E \in B_w(S),$$

is a regular weakly Borel vector measure on S. Then r_w may be extended to one and only one regular weakly Borel vector measure m_w: $B_w(S\times T)\to \tilde{X}$.

The proof of this theorem is similar to that of theorem 3 in [4] and is based on a theorem in [5, Theorem 5] asserting that Theorem 7 is valid for complex-valued measures.

2. Tensor product of weakly compact mappings

We shall see that theorems 2 and 6 are in fact the results which imply that τ-tensor product or s-tensor product of weakly compact linear mappings u: $C_0(S)\to X$ and v: $C_0(T)\to Y$ is a weakly compact linear mapping from $C_0(S\times T)$ into $X\otimes_\tau Y$.

If u: $C_0(S)\to X$ is a weakly compact mapping, there exists a unique regular Borel vector measure m: $B(S)\to X$ such that

$$u(f)=\int_S f\,\mathrm{d}m$$

for all $f\in C_0(S)$ as it follows from [2, Théorème 12]. Similarly if V: $C_0(T)\to Y$ is a weakly compact mapping, there exists a unique regular Borel vector measure n: $B(T)\to Y$ such that

$$v(g)=\int_T g\,\mathrm{d}n$$

for all $g\in C_0(T)$. Take now the extended τ-tensor product $l=m\otimes_\tau n$: $B(S\times T)\to X\otimes_\tau Y$ existing according to Theorem 2. Since it is a regular Borel vector measure the relation

$$w(h)=\int_{S\times T} h\,\mathrm{d}m\otimes_\tau n,\quad h\in C_0(S\times T),$$

defines a weakly compact linear mapping from $C_0(S\times T)$ into $X\otimes_\tau Y$. This follows from [2, Théorème 13].

We must prove that for $h=f\otimes g$, $f\in C_0(S)$, $g\in C_0(T)$ we have

$$w(h)=w(f\otimes g)=u(f)\otimes v(g),$$

i. e.

$$\int_{S\times T} h\,\mathrm{d}m\otimes_\tau n=\int_S f\,\mathrm{d}m\otimes\int_T g\,\mathrm{d}n.$$

But this holds for all bounded Borel functions f and g with compact support on S and T, respectively. In fact, let $f=c_A$ and $g=c_B$ be the characteristic functions of bounded Borel sets A and B, respectively, in S and T. Then we have

$$\int_{S\times T} c_A \otimes c_B \, dm \otimes_s n = m \otimes_s n(A \times B) = m(A) \otimes n(B) = \int_S c_A \, dm \otimes \int_T c_B \, dn.$$

If

$$f = \sum_{i=1}^{k} a_i c_{A_i}, \quad g = \sum_{j=1}^{l} b_j c_{B_j}$$

are simple Borel functions with compact supports, we have

$$\int_{S\times T} f \otimes g \, dm \otimes_s n = \int_S f \, dm \otimes \int_T g \, dn.$$

If f and g are bounded Borel functions with compact supports on S and T, respectively, then they are uniform limits of a sequence of Borel simple functions with compact supports,

$$f = \lim_{k\to\infty} f_k, \quad g = \lim_{l\to\infty} g_l.$$

We have

$$\int_{S\times T} f \otimes g \, dm \otimes_s n = \lim_{k\to\infty} \lim_{l\to\infty} \int_{S\times T} f_k \otimes g_l \, d_m \otimes_s n =$$

$$= \lim_{k\to\infty} \lim_{l\to\infty} \left(\int_S f_k \, dm \right) \otimes_s \left(\int_T g_l \, dn \right) = \int_S f \, dm \otimes_s \int g \, dn.$$

We may summarize.

Theorem 8. *Let X and Y be quasi-complete Hausdorff locally convex spaces. If u: $C_0(S) \to X$ and v: $C_0(T) \to Y$ are weakly compact linear mappings, then the τ-tensor or the s-tensor product of u and v is the weakly compact linear mapping on $C_0(S\times T)$ into $X \otimes_\tau Y$ or $X \otimes_s Y$,* i. e. *the linear mapping $w = u \otimes v$: $C_0(S\times T) \to X \otimes_\tau Y$ or $X \otimes_s Y$ such that $u \otimes v(f \otimes g) = u(f) \otimes v(g)$, $f \in C_0(S)$, $g \in C_0(T)$, is weakly compact.*

For $w = u \otimes v$ is weakly compact as a mapping from $C_0(S) \otimes_\varepsilon C_0(T)$ into $X \otimes_\tau Y$ or $X \otimes_s Y$ and $C_0(S) \otimes_\varepsilon C_0(T)$ is isomorphic with $C_0(S\times T)$ [11, p. 288].

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О ТЕНЗОРНОМ ПРОИЗВЕДЕНИИ СЛАБО КОМПАКТНЫХ ОТОБРАЖЕНИЙ

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Резюме

В работе применяются результаты о тензорном произведении векторных мер к доказательству следующего утверждения: Пусть u: $C_0(S)\to X$ и v: $C_0(T)\to Y$ слабо компактные линейные отображения пространства непрерывнаых функций, нулевых в бесконечности, в локально выпуклое пространство X и Y, соответственно. Тогда s-тензорное произведение отображений u и v слабо компактно из $C_0(S\times T)$ в $X\otimes_s Y$, квазипополнение тензорного произведения при s-топологии, вообще сильнее ε-топологии и слабее π-топологии.