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MORPHISMS AND PASTING OF ORTHOALGEBRAS

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ABSTRACT. We study orthoalgebra morphisms, we show how orthoalgebra monomorphisms and epimorphisms ought to be defined, and we show how orthoalgebras may be "pasted" together to yield new orthoalgebras.

1. Introduction

Historically, Greechie [3] – [5] was the first one who presented a method, which he called the "*paste job*", for constructing "new" orthomodular lattices from "old" by "pasting" together isomorphic parts of the "old" ones. Following Greechie's method of pasting, other authors [1], [8], [9] gave similar methods of pasting together families of orthomodular lattices or posets or Boolean algebras.

It is the purpose of this paper to show that Greechie's constructions for pasting orthomodular lattices can be extended to a suitable class of partially ordered sets; namely, *orthoalgebras*. In Section 2, we study orthoalgebra *morphisms*, and we show how orthoalgebra *monomorphisms* and *epimorphisms* ought to be defined. In Section 3, we study the pasting of two orthoalgebras. Following Greechie [3], we show (see Theorem 3.6) that the pasting of any two disjoint orthoalgebras along "corresponding sections" is an orthoalgebra. We show, further (see Theorem 3.8), that this orthoalgebra is an orthomodular poset in case it is the pasting of orthomodular posets. These results can be used to construct important examples of orthoalgebras or orthomodular posets (see Example 3.9). We conclude this section by showing that the pasting of two orthoalgebras can be viewed as a *pushout* for certain morphisms, or as a *coproduct* of these orthoalgebras. The following is a list of basic definitions and facts from the elementary theory of orthoalgebras ([2], [6], [7], [10]).

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An *orthoalgebra* (OA) is a quadruple $(L, \oplus, 0, 1)$, where L is a set containing two special elements 0, 1, and $\oplus$ is a partially defined binary operation on L that satisfies the following conditions $\forall p, q, r \in L$:

(OA1) (*Commutativity*) If $p\oplus q$ is defined, then $q\oplus p$ is defined and $p\oplus q = q\oplus p$.

(OA2) (*Associativity*) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(OA3) (*Orthocomplementation*) For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(OA4) (*Consistency*) If $p \oplus p$ is defined, then $p = 0$.

In what follows, we shall write L for the orthoalgebra $(L, \oplus, 0, 1)$. Let L be an OA and let $p, q \in L$. We say that p is *orthogonal* to q in L and write $p \perp q$ if and only if $p \oplus q$ is defined in L. We define $p \leq q$ to mean that there exists $r \in L$ such that $p \perp r$ and $q = p \oplus r$; in this case, we define $q - p = r$. The unique element q corresponding to p in condition (OA3) above is called the *orthocomplement* of p and is written as p'. It can be easily proved (see [2]) that $p \perp q$ if and only if $p \leq q'$, that $0 \leq p \leq 1$ holds for all $p \in L$, that "$\leq$" as defined above is a partial ordering on L, and that the unary operation $p \mapsto p' \colon L \to L$ as defined above is an orthocomplementation on L; hence $(L, \leq, ', 0, 1)$ is an orthoposet. Also, it can be proved (see [2], [7], [10]) that if $p \leq q$, then $q = p \oplus (p \oplus q')'$. This is called the *orthomodular identity* (OMI).

Let L be an OA. A subset $A \subseteq L$ is called a *suborthoalgebra* (sub-OA) if $0, 1 \in A$, $p' \in A$ whenever $p \in A$ and $p \oplus q \in A$ whenever $p, q \in A$ and $p \perp q$. A sub-OA of an OA is, of course, an OA in its own right. An *orthomodular poset* (OMP) is an orthoalgebra P that satisfies the following condition:

$$p, q \in P\,,\ \ p \perp q \implies p \vee q \text{ exists and } p \vee q = p \oplus q\,,$$

where the notation $p \vee q$ stands for the supremum (i.e., the least upper bound of $\{p, q\}$ in P). It can be shown (see [2], [10]) that this condition is equivalent to the condition that for $p, q, r \in P$, $p \perp q \perp r \perp p \implies (p \oplus q) \perp r$. A σ-*orthocomplete* OMP is an OMP in which every countable pairwise orthogonal subset has a least upper bound. An *orthomodular lattice* (OML) is an OMP which is also a lattice. A *Boolean algebra* is a *distributive* OML.

2. Morphisms

In this section, unless otherwise stated, L and Q will always denote orthoalgebras.

2.1. Definition. A mapping $\phi\colon L \to Q$ is called a *morphism* if and only if $\phi(1) = 1$ and for $p, q \in L$, $p \perp q \implies \phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$.

Note that the class of all orthoalgebras together with the class of all orthoalgebra morphisms form a *category*.

2.2. Notation. Hereafter we shall denote the composite of any two morphisms ϕ and ψ of OAs by $\psi\phi$, where ϕ is applied first, and ψ is applied second, and the identity map id: $L \to L$ of any OA L by 1_L.

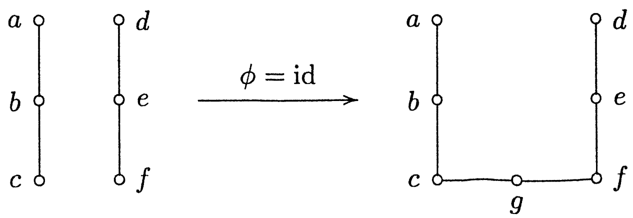
2.3. Definition. Let $\mathcal{B}$ be a nonempty set of Boolean algebras such that

$$A, B \in \mathcal{B} \text{ and } A \neq B \implies A \cap B = \{0, 1\}.$$

Set $L = \bigcup \mathcal{B}$. Define $\leq$ on L by $\leq := \bigcup_{B \in \mathcal{B}} \leq_B$ and define a unary operation $'\colon L \to L$ by $x' := x'^B$ if $x \in B$, where $'^B$ is the unary operation $x \mapsto x'^B\colon B \to B$ on B.

It has been shown in [8; p. 59] that $(L, \leq, ')$ carries in a natural way the structure of an orthomodular lattice. Such an orthomodular lattice is called the *horizontal sum* of the members of $\mathcal{B}$ and is denoted by $\circ\mathcal{B}$.

Let $\phi\colon L \to Q$ be a morphism. Evidently, for every $p \in L$, $\phi(p') = \phi(p)'$, and for $p, q \in L$, $p \leq q \implies \phi(p) \leq \phi(q)$ and $\phi(q - p) = \phi(q) - \phi(p)$. It should be noted that $\phi(L) := \{\phi(p) : p \in L\}$ need not be a sub-OA of Q. To see this, let L be the horizontal sum of two copies of $\mathbf{2}^3$, the eight-element Boolean algebra, and let $Q = D_{16}$ be Dilworth's orthomodular lattice [8]. Figure 1 below shows the Greechie diagrams of L and Q. For reading Greechie diagrams, we refer the reader to [8]. Define $\phi\colon L \to Q$ to be the identity map. Evidently, ϕ is a morphism. However, $\phi(L)$ is not a sub-OA of Q since $c, f \in \phi(L)$ and $c \perp f$, but $c \oplus f = g' \notin \phi(L)$.



$L = \mathbf{2}^3 \circ \mathbf{2}^3$ $\qquad$ $Q = D_{16}$

Figure 1.

2.4. DEFINITION. Let $\phi\colon L \to Q$ be a morphism. We say that ϕ is *special* if it satisfies the following condition:

$$u, v \in \phi(L)\,,\quad u \perp v \implies \exists\, x, y \in L \text{ with } x \perp y\,,\ \ \phi(x) = u\,,\ \text{ and } \phi(y) = v\,.$$

It is easy to check that if $\phi\colon L \to Q$ is a special morphism, then $\phi(L)$ is a sub-OA of Q. The converse of this statement need not be true as the following example shows. Let L be the horizontal sum of three copies of $\mathbf{2}^3$, where the Greechie diagram of L is given in Figure 2 below. Note that L is an OA (in fact, an OML). Let Q be the orthocomplemented poset whose Greechie diagram is given in Figure 2 below. Let $\phi\colon L \to Q$ be the identity map. Evidently, ϕ is bijective and is a morphism; hence $\phi(L) = Q$ is a sub-OA of Q. However, ϕ is not special (because $\phi(c) = c \perp_Q f = \phi(f)$ in Q, but $c \not\perp_L f$ in L).

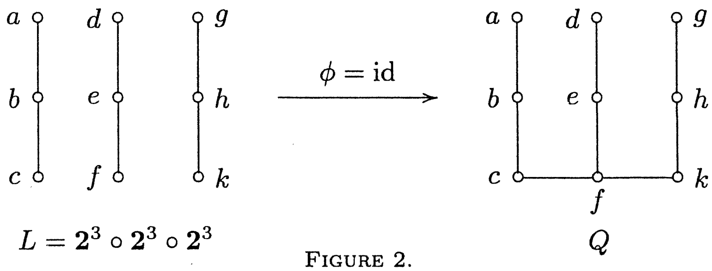


FIGURE 2.

2.5. DEFINITION. Let $\phi\colon L \to Q$ be a morphism. We say that ϕ is

(i) a *monomorphism* if and only if ϕ is special, and there is a morphism $\psi\colon \phi(L) \to L$ such that $\psi\phi = 1_L$;

(ii) an *epimorphism* if and only if there is a morphism $\psi\colon Q \to L$ such that $\phi\psi = 1_Q$;

(iii) an *isomorphism* if and only if there is a morhpism $\psi\colon Q \to L$ such that $\psi\phi = 1_L$, and $\phi\psi = 1_Q$.

Evidently, every monomorphism of OAs is an injection, every epimorphism of OAs is a surjection, and every isomorphism of OAs is a bijection, and its inverse is also an isomorphism. It is also evident that for a morphism $\phi\colon L \to Q$ of OAs, the statement that $\forall\, p_1, p_2 \in L$, $p_1 \perp p_2 \iff \phi(p_1) \perp \phi(p_2)$ is equivalent to the statement that $\forall\, p_1, p_2 \in L$, $p_1 \le p_2 \iff \phi(p_1) \le \phi(p_2)$.

The following theorem characterizes monomorphisms and isomorphisms of orthoalgebras.

2.6. THEOREM. *Let $\phi\colon L \to Q$ be a morphism. Then*

(i) *ϕ is a monomorphism if and only if for all $p_1, p_2 \in L$,*

$$\phi(p_1) \le \phi(p_2) \implies p_1 \le p_2\,;$$

(ii) *ϕ is an isomorphism if and only if it is a monomorphism and $\phi(L) = Q$.*

Proof.

(i) ($\Longrightarrow$): This part is obvious from the definition of a monomorphism.

($\Longleftarrow$): Assume that ϕ satisfies the stated condition. Then this and the hypothesis that ϕ is a morphism imply that $\forall p_1, p_2 \in L$, $p_1 \le p_2 \iff \phi(p_1) \le \phi(p_2)$. It follows that ϕ is one-to-one and, by the remark following Definition 2.5, that ϕ is special; therefore, $\phi(L)$ is a sub-OA of Q. Define a mapping $\psi\colon \phi(L) \to L$ by $\psi(\phi(x)) := x \quad \forall x \in L$. Since ϕ is one-to-one, ψ is well-defined. Clearly, $\psi\phi = 1_L$, and the stated condition shows that ψ is a morphism.

(ii): This follows from (i) and the definition of an isomorphism. □

2.7. Remarks.

(i) *If $\phi\colon L \to Q$ is a bijective morphism, then $\phi^{-1}\colon Q \to L$ need not be a morphism.* Indeed, the example following Definition 2.4 is an example of a morphism ϕ that is bijective, but ϕ^{-1} does not preserve $\oplus$.

(ii) *Let $\phi\colon L \to Q$ be a bijective morphism, and let L be a Boolean algebra. Then $\phi^{-1}\colon Q \to L$ is a morphism, and hence an isomorphism.*

To prove this statement, note first that $\forall u \in Q$, $\phi^{-1}(u') = (\phi^{-1}(u))'$. Next, let $u, v \in Q$, $u \perp v$. Then there exist $x_1, y_1 \in L$ such that $\phi(x_1) = u$ and $\phi(y_1) = v$. The use of the OMI and the hypothesis that L is Boolean yield that $\phi(x_1) = \phi(x_1 \wedge y_1')$, which implies that $x_1 = x_1 \wedge y_1'$, since ϕ is one-to-one. It follows that $x_1 \le y_1'$, i.e., $x_1 \perp y_1$. Thus $\phi^{-1}(u) \perp \phi^{-1}(v)$, and $\phi^{-1}(u \oplus v) = \phi^{-1}(\phi(x_1) \oplus \phi(y_1)) = x_1 \oplus y_1 = \phi^{-1}(u) \oplus \phi^{-1}(v)$.

(iii) Note that the example following Definition 2.4 also shows that *the hypothesis of* (ii), *that L is a Boolean algebra, cannot be weakened to assuming that L is an orthomodular lattice.*

2.8. Definition. Let $\phi\colon L \to Q$ be a special morphism and define an equivalence relation $\sim_\phi$ on L as follows: For each $p_1, p_2 \in L$, $p_1 \sim_\phi p_2$ if and only if $\phi(p_1) = \phi(p_2)$. For any $p \in L$, let $\overline{p} := \{x \in L : \phi(x) = \phi(p)\}$ and form $L/{\sim_\phi} := \{\overline{p} : p \in L\}$. Define a partial binary operation $\oplus$ on $L/{\sim_\phi}$ as follows:

$$\overline{p}_1 \oplus \overline{p}_2 := \overline{x_1 \oplus x_2} \qquad \text{if there exist} \quad x_1 \in \overline{p}_1 \text{ and } x_2 \in \overline{p}_2 \text{ with } x_1 \perp x_2 .$$

One can easily check that $\oplus$ as defined above is well-defined.

2.9. Theorem. *Let $\phi\colon L \to Q$ be a special morphism. Then $(L/{\sim_\phi}, \oplus, \overline{0}, \overline{1})$ is an OA, and $L/{\sim_\phi}$ is isomorphic to $\phi(L)$.*

Proof. It is not difficult to see that $L/{\sim_\phi}$ satisfies the orthoalgebra axioms (OA1), (OA3), and (OA4). To check that $L/{\sim_\phi}$ satisfies axiom (OA2), let

$\overline{p},\overline{q},\overline{r}\in L/\sim_\phi$ be such that $\overline{p}\oplus\overline{q}$ and $(\overline{p}\oplus\overline{q})\oplus\overline{r}$ are defined in $L/\sim_\phi$. We must show that $\overline{q}\oplus\overline{r}$ and $\overline{p}\oplus(\overline{q}\oplus\overline{r})$ are defined in $L/\sim_\phi$, and

$$(\overline{p}\oplus\overline{q})\oplus\overline{r}=\overline{p}\oplus(\overline{q}\oplus\overline{r}). \tag{1}$$

Since $\overline{p}\oplus\overline{q}$ is defined, there exist $x\in\overline{p}$ and $y\in\overline{q}$ with $x\perp y$ and $\overline{p}\oplus\overline{q}=\overline{x\oplus y}$. Since $(\overline{p}\oplus\overline{q})\oplus\overline{r}=\overline{x\oplus y}\oplus\overline{r}$ is defined, there exist $t\in\overline{x\oplus y}$ and $z\in\overline{r}$ with $t\perp z$ and $\overline{x\oplus y}\oplus\overline{r}=\overline{t\oplus z}$. It follows that $\phi(x)\oplus\phi(y)=\phi(x\oplus y)=\phi(t)\perp\phi(z)$; therefore, by the associativity of $\oplus$ in the OA $\phi(L)$, we have $\phi(x\oplus y)\oplus\phi(z)=\phi(x)\oplus\big(\phi(y)\oplus\phi(z)\big)$. Now, since $\phi(x\oplus y)\perp\phi(z)$ and ϕ is special, there exist $t_1,z_1\in L$ with $t_1\perp z_1$ and $\phi(t_1)=\phi(x\oplus y)$ and $\phi(z_1)=\phi(z)$. Thus $\phi(x\oplus y)\oplus\phi(z)=\phi(t_1\oplus z_1)$, and therefore

$$(\overline{p}\oplus\overline{q})\oplus\overline{r}=\overline{x\oplus y}\oplus\overline{z}=\overline{t_1\oplus z_1}. \tag{2}$$

Similarly, we have $\phi(y)\oplus\phi(z)=\phi(y_2\oplus z_2)$ for some $y_2,z_2\in L$ with $y_2\perp z_2$ and $\phi(y_2)=\phi(y)$ and $\phi(z_2)=\phi(z)$. Also, since $\phi(x)\perp\big(\phi(y)\oplus\phi(z)\big)$, there exist $x_3,s\in L$ with $x_3\perp s$ and $\phi(x_3)=\phi(x)$ and $\phi(s)=\phi(y_2\oplus z_2)$. It follows that $\phi(t_1\oplus z_1)=\phi(x_3\oplus s)$, and therefore

$$\overline{t_1\oplus z_1}=\overline{x_3\oplus s}=\overline{x}\oplus\overline{y_2\oplus z_2}=\overline{x}\oplus(\overline{y}\oplus\overline{z})=\overline{p}\oplus(\overline{q}\oplus\overline{r}). \tag{3}$$

Now (2) and (3) show that (1) holds, and therefore $\big(L/\sim_\phi,\oplus,\overline{0},\overline{1}\big)$ is an OA.

Next, define a mapping $\phi_*\colon L/\sim_\phi\to\phi(L)$ by $\phi_*(\overline{p}):=\phi(p)\quad\forall p\in L$. Since for $\overline{p},\overline{q}\in L/\sim_\phi$, $\overline{p}=\overline{q}\implies\phi(p)=\phi(q)$, ϕ_* is well-defined. Now it is not difficult to check that ϕ_* is an isomorphism. □

The following theorem characterizes epimorphisms of orthoalgebras.

2.10. Theorem. *Let $\phi\colon L\to Q$ be a morphism. Then ϕ is an epimorphism if and only if it satisfies the following conditions:*

1. *$\phi(L)=Q$,*
2. *ϕ is special,*
3. *there exists a morphism $\eta\colon L\to L$ with $\phi\eta=\phi$ and for $p_1,p_2\in L$, $\phi(p_1)=\phi(p_2)\implies\eta(p_1)=\eta(p_2)$.*

Proof.

$(\Longrightarrow)$: Assume that ϕ is an epimorphism. Then there exists a morphism $\psi\colon Q\to L$ with $\phi\psi=1_Q$. Hence $\phi(L)=Q$. Suppose that $\phi(p_1)\perp\phi(p_2)$ for some $p_1,p_2\in L$. Set $x_i:=\psi\phi(p_i)$ $(i=1,2)$. Then $x_1,x_2\in L$, $x_1\perp x_2$, and for $i=1,2$, $\phi(x_i)=\phi\psi\phi(p_i)=\phi(p_i)$. Thus ϕ is special. Finally, set $\eta:=\psi\phi$. Then $\eta\colon L\to L$ is a morphism, $\phi\eta=\phi$, and for $p_1,p_2\in L$, $\phi(p_1)=\phi(p_2)\implies\eta(p_1)=\eta(p_2)$.

$(\Longleftarrow)$: Assume that ϕ satisfies conditions 1, 2, and 3. Then, by Theorem 2.9, $L/\sim_\phi$ is an OA, and the mapping $\phi_*\colon L/\sim_\phi\to Q$ defined by $\phi_*(\overline{p}):=$

$\phi(p)$, $p \in L$, is an isomorphism. Define a mapping $\eta_*\colon L/{\sim_\phi} \to L$ by $\eta_*(\overline{p}) := \eta(p)$ $(p \in L)$. Then, by condition 3, η_* is well-defined. Moreover, $\eta_*(\overline{p}_1 \oplus \overline{p}_2) = \eta_*(\overline{x_1 \oplus x_2}) = \eta(x_1 \oplus x_2) = \eta(x_1) \oplus \eta(x_2) = \eta_*(\overline{x}_1) \oplus \eta_*(\overline{x}_2) = \eta_*(\overline{p}_1) \oplus \eta_*(\overline{p}_2)$, where $x_1 \in \overline{p}_1$ and $x_2 \in \overline{p}_2$ with $x_1 \perp x_2$. Also, $\eta_*(\overline{1}) = \eta(1) = 1$. Thus η_* is a morphism. Moreover, condition 3 implies that for every $\overline{p} \in L/{\sim_\phi}$, $\phi\eta_*(\overline{p}) = \phi\eta(p) = \phi(p) = \phi_*(\overline{p})$; therefore $\phi\eta_* = \phi_*$. Now set $\psi := \eta_*\phi_*^{-1}$. Then $\psi\colon Q \to L$ is a morphism, and $\phi\psi = \phi\eta_*\phi_*^{-1} = \phi_*\phi_*^{-1} = 1_Q$. Thus, by definition, ϕ is an epimorphism. □

Before we close this section, we make two remarks. Recall that in category theory, a morphism $\phi\colon L \to Q$ in a category $\mathcal{C}$ is called a monomorphism if

$$\phi\psi = \phi\eta \implies \psi = \eta$$

for all objects M and morphisms $\psi, \eta\colon M \to L$. Note that, in view of part (i) of Theorem 2.6, *such a categorical definition of a monomorphism is inadequate here.* Indeed, the example following Definition 2.3 provides a morphism $\phi\colon L \to Q$ of OAs that is a categorical monomorphism since for any OA M, and for all morphisms $\psi, \eta\colon M \to L$, we have $\phi\psi = \phi\eta \iff \psi = \eta$. However, ϕ is not even special.

Also, recall that in category theory, a morphism $\phi\colon L \to Q$ in a category $\mathcal{C}$ is called an epimorphism if

$$\psi\phi = \eta\phi \implies \psi = \eta$$

for all objects N and morphisms $\psi, \eta\colon Q \to N$. Note that, in view of Theorem 2.10, *such a categorical definition of an epimorphism is inadequate here.* Indeed, the example following Definition 2.4 provides a morphism $\phi\colon L \to Q$ of OAs that is a categorical epimorphism since for any OA N and for all morphisms $\psi, \eta\colon Q \to N$, $\psi\phi = \eta\phi \iff \psi = \eta$. However, ϕ is not even special.

3. Pasting Orthoalgebras

We begin this section with the following.

3.1. Definition. A *section* in an orthoalgebra L is a suborthoalgebra S such that $S = I \mathbin{\dot\cup} I'$, where I is an order ideal and $I' = \{a' : a \in I\}$. Two sections S_1 and S_2 of two orthoalgebras L_1 and L_2, respectively, are called *corresponding sections* in case $S_1 = I_1 \mathbin{\dot\cup} I_1'$ and $S_2 = I_2 \mathbin{\dot\cup} I_2^\sharp$ are such that there exists an (OA-) isomorphism $\theta\colon S_1 \to S_2$ with $\theta(I_1) = I_2$ and $\theta(I_1') = I_2^\sharp$.

Throughout this section, we assume that $(L_1, \oplus_1, 0_1, 1_1)$ and $(L_2, \oplus_2, 0_2, 1_2)$ are two disjoint orthoalgebras, and that S_1 and S_2 are corresponding sections of L_1 and L_2, respectively.

3.2. DEFINITION.

(1) Let $L_0 := L_1 \cup L_2$, $P_1 := \{(x,y) \in L_0 \times L_0 : \ y = \theta(x)\}$, and $\Delta := \{(x,x) : \ x \in L_0\}$.

(2) Let P be the equivalence relation defined on L_0 by $P := \Delta \cup P_1 \cup P_1^{-1}$, where $P_1^{-1} = \{(y,x) : \ (x,y) \in P_1\}$.

(3) Let $L := L_0/P$.

(4) For $i = 1,2$, let

$$\mathcal{O}_i = \{([a],[b]) \in L \times L : \ \exists a_i{\in}[a],\ b_i{\in}[b] \text{ with } a_i \perp_i b_i\},$$

and let

$$\begin{aligned}\perp := \{([a],[b]) \in L \times L : \ \exists [c]{\in}L \text{ with } &([a],[c^+]) \in \mathcal{O}_1 \cup \mathcal{O}_2 \\ &\text{and } ([c],[b]) \in \mathcal{O}_1 \cup \mathcal{O}_2, \quad \text{or equivalently,} \\ &([a],[c]) \in \mathcal{O}_1 \cup \mathcal{O}_2 \text{ and } ([c^+],[b]) \in \mathcal{O}_1 \cup \mathcal{O}_2\},\end{aligned}$$

where c^+ denotes $\{c_1'\}$ or $\{c_2^\sharp\}$ or $\{c_1', c_2^\sharp\}$ whenever $[c] = \{c_1\}$ or $\{c_2\}$ or $\{c_1, c_2\}$, respectively, where $c_i \in L_i$, $i = 1,2$.

The proof of the following lemma is not difficult and can be found in [4; Lemma 3.3] or [7; Lemma 1.47].

3.3. LEMMA. *Let S_1 and S_2 be corresponding sections of L_1 and L_2. If $[a],[b] \in L$ are such that $[a] = \{a_1\}$ and $[b] = \{b_2\}$ or $[a] = \{a_2\}$ and $[b] = \{b_1\}$, where $a_i, b_i \in L_i$ $(i = 1,2)$, then $[a] \not\perp [b]$. Consequently, for $[a],[b] \in L$, $[a] \perp [b]$ if and only if there exists $i \in \{1,2\}$ such that $[a] \cap L_i \neq \emptyset \neq [b] \cap L_i$, and the representatives of $[a]$ and $[b]$ in L_i are orthogonal.*

3.4. DEFINITION. Define $\oplus\colon \perp \to L$ as follows: For $([a],[b]) \in \perp$, set

$$[a] \oplus [b] := \begin{cases} [a_1 \oplus_1 b_1] & \text{if } [a] \cap L_1 = \{a_1\},\ [b] \cap L_1 = \{b_1\}, \\ [a_2 \oplus_2 b_2] & \text{if } [a] \cap L_2 = \{a_2\},\ [b] \cap L_2 = \{b_2\}. \end{cases}$$

3.5. PROPOSITION. *The mapping $\oplus$, as defined above, is well-defined.*

Proof. Let $([a],[b]) \in \perp$. By Lemma 3.3, there exists $i \in \{1,2\}$ such that $[a] \cap L_i = \{a_i\}$ and $[b] \cap L_i = \{b_i\}$, and so $[a] \oplus [b] = [a_i \oplus_i b_i]$. Note that in the cases $[a] = \{a_1, a_2\}$ and $[b] = \{b_1, b_2\}$ with $\theta(a_1) = a_2$, $\theta(b_1) = b_2$, θ being an isomorphism, $a_1 \perp_1 b_1$ and $a_2 \perp_2 b_2$, we have $\theta(a_1 \oplus_1 b_1) = \theta(a_1) \oplus_2 \theta(b_1) = a_2 \oplus_2 b_2$. Thus $[a_1 \oplus_1 b_1] = [a_2 \oplus_2 b_2]$ and $[a] \oplus [b]$ is unambiguous. □

Define $0, 1 \in L$ by $0 := [0_1] = [0_2]$ and $1 := [1_1] = [1_2]$. Using Lemma 3.3 and the orthoalgebra axioms of L_i, $i = 1,2$, it is not difficult to establish the following result.

3.6. THEOREM. *If S_1 and S_2 are corresponding sections of L_1 and L_2, then $(L, \oplus, 0, 1)$ is an orthoalgebra.*

The following lemma is known and its proof may be found in [3; Lemma 2.1.17].

3.7. LEMMA. *Suppose that L_1 and L_2 are OMPs. If for $i \in \{1,2\}$, $x_i \oplus_i y_i$ is defined in L_i, then $[x_i] \vee [y_i]$ exists in L and $[x_i] \vee [y_i] = [x_i \vee^i y_i]$.*

The following theorem asserts that the pasting of two OMPs along corresponding sections is another OMP.

3.8. THEOREM. *If L_1 and L_2 are OMPs, then L is an OMP.*

Proof. By Theorem 3.6, L is an OA. We need only show that

$$[x] \perp [y] \text{ in } L \implies [x] \vee [y] \text{ exists in } L \text{ and } [x] \vee [y] = [x] \oplus [y].$$

To this end, suppose that $[x] \perp [y]$ in L. Then, by Lemma 3.3, there exist $i \in \{1,2\}$ and $x_i \in [x] \cap L_i$, $y_i \in [y] \cap L_i$ such that $x_i \perp_i y_i$. Since L_i $(i = 1,2)$ is an OMP, Lemma 3.7 implies that $[x_i] \vee [y_i]$ exists in L and $[x_i] \vee [y_i] = [x_i \vee^i y_i]$. Thus

$$[x] \vee [y] = [x_i] \vee [y_i] = [x_i \vee^i y_i] = [x_i \oplus_i y_i] = [x] \oplus [y]$$

exists, as desired. □

We would like to point out that the hypothesis that S_1 and S_2 are corresponding sections in Theorem 3.6 cannot be dispensed with. This can easily be seen if we take L_1 and L_2 to be two disjoint copies of the eight-element Boolean algebra $\mathbf{2}^3$, and if we paste L_1 and L_2 along noncorresponding sections.

The next example shows how the results of this section can be used to construct examples of infinite orthomodular posets. This example will be used in an important way in a subsequent paper.

3.9. EXAMPLE. For $i = 1,2$, let $\mathbf{Z}_i := \mathbb{Z} \times \{i\} = \{(n,i) : n \in \mathbb{Z}\}$ so that $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are disjoint copies of the set $\mathbb{Z}$ of all integers. Write n^i for $(n,i) \in \mathbf{Z}_i$ and $2n^i$ for $(2n,i) \in \mathbf{Z}_i$. Let $L_i := \mathcal{P}(\mathbf{Z}_i)$, the power set of $\mathbf{Z}_i$. Then L_1 and L_2 are disjoint copies of $\mathcal{P}(\mathbb{Z})$. Let S_1 be the sub-OA of L_1 consisting of all the finite or cofinite subsets of L_1. Clearly, there is a natural (OA-) isomorphism $\phi\colon L_1 \to L_2$. Let $\theta := \phi|_{S_1}$ and $S_2 := \theta(S_1)$. Then $\theta\colon S_1 \to S_2$ is an isomorphism that maps the finite (resp., cofinite) subsets of L_1 to the finite (resp., cofinite) subsets of L_2. Hence S_1 and S_2 are corresponding sections (see Definition 3.1). Form L as in Definition 3.2. Then, by Theorem 3.6, L is an OA which consists of all equivalence classes $[a]$, where

$$[a] = \begin{cases} \{a, \theta(a)\} & \text{if } a \in S_1, \\ \{a, \theta^{-1}(a)\} & \text{if } a \in S_2, \\ \{a\} & \text{if } a \in L_1 \setminus S_1, \\ \{a\} & \text{if } a \in L_2 \setminus S_2. \end{cases}$$

Moreover, since L_1 and L_2 are OMPs, Theorem 3.8 ensures that L is an OMP.

Note that L is not a σ-orthocomplete OMP. This can be seen by observing that the pairwise orthogonal subset $\{[\{2n^1\}] = [\{2n^2\}] : n \in \mathbb{Z}\}$ of L does not have a supremum in L, since $[E_1]$ and $[E_2]$, where $E_1 := \{2n^1 : n \in \mathbb{Z}\}$ and $E_2 := \{2n^2 : n \in \mathbb{Z}\}$, are noncomparable upper bounds of $\{[\{2n^1\}] : n \in \mathbb{Z}\}$ in L.

3.10. Definition. A commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow{\delta} & D \end{array}$$

of morphisms of a category $\mathcal{C}$ is called a *pushout* for γ and δ if for every pair of morphisms $\phi\colon B \to E$, $\psi\colon C \to E$ such that $\phi\alpha = \psi\beta$ there is a unique morphism $\eta\colon D \to E$ such that $\eta\gamma = \phi$ and $\eta\delta = \psi$.

A *coproduct* for the family $\{A_i : i \in I\}$ of objects in a category $\mathcal{C}$ is an object A of $\mathcal{C}$, together with a family of morphisms $\{\iota_i\colon A_i \to A : i \in I\}$ such that for any object B and family of morphisms $\{\psi_i\colon A_i \to B : i \in I\}$, there is a unique morphism $\psi\colon A \to B$ such that $\psi\iota_i = \psi_i$ for all $i \in I$.

The first part of the following result characterizes the pasting of two OAs as being a pushout, and the second part characterizes the pasting of two OAs as being a coproduct in the following sense:

3.11. Theorem.

(i) *The following diagram*

$$\begin{array}{ccc} (S_2 \overset{\theta}{\cong})\ S_1 & \xrightarrow{1_{S_1}} & L_1 \\ {\scriptstyle 1_{S_2}\theta}\downarrow & & \downarrow{\scriptstyle\iota_1} \\ L_2 & \xrightarrow{\iota_2} & L \end{array}$$

is commutative and it is a pushout for ι_1 and ι_2, where $\iota_i\colon L_i \to L$ is given by $\iota_i(x_i) = [x_i]$ for $x_i \in L_i$, $i = 1, 2$.

(ii) *$(L; \iota_1, \iota_2)$ is a coproduct for L_1 and L_2.*

Proof.

(i) First, we check that the above square is commutative; i.e., we show that $\iota_1 1_{S_1} = \iota_2 1_{S_2}\theta$. Indeed, this follows from

$$\iota_1 1_{S_1}(s_1) = \iota_1(s_1) = [s_1] = [\theta(s_1)] = \iota_2(\theta(s_1)) = \iota_2 1_{S_2}(\theta(s_1)),$$

which holds for all $s_1 \in S_1$.

Second, we show that the above square is a pushout. Let $\phi\colon L_1 \to Q$ and $\psi\colon L_2 \to Q$ be morphisms with $\phi 1_{S_1} = \psi 1_{S_2}\theta$. Define a mapping $\eta\colon L \to Q$ by

$$\eta([x]) := \begin{cases} \phi(x) & \text{if } x \in L_1, \\ \psi(x) & \text{if } x \in L_2. \end{cases}$$

Then one can easily check that η is well-defined, and that η is a morphism. Next, note that, by definition of η, $\eta\iota_1(x_1) = \eta([x_1]) = \phi(x_1)\ \forall x_1 \in L_1$, so that $\eta\iota_1 = \phi$. Also, $\eta\iota_2(x_2) = \eta([x_2]) = \psi(x_2)\quad \forall x_2 \in L_2$, so that $\eta\iota_2 = \psi$. To complete the proof of (i), it remains to prove the uniqueness of η. To this end, suppose that there exists a morphism $\eta'\colon L \to Q$ such that $\eta'\iota_1 = \phi$, and $\eta'\iota_2 = \psi$. Then, for every $x \in L_1$, $\eta'([x]) = \eta'(\iota_1(x)) = \phi(x) = \eta([x])$ and, for every $x \in L_2$, $\eta'([x]) = \eta'(\iota_2(x)) = \psi(x) = \eta([x])$. Therefore $\eta' = \eta$ and η is unique.

(ii) The proof of this part is contained in the proof of part (i). □

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