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ENLARGING THE CONVERGENCE ON THE REAL LINE VIA METRIZABLE GROUP TOPOLOGIES

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ABSTRACT. We prove that there exists a metrizable group topology on the real numbers preserving the bounded part of the usual convergence and adding convergent sequences to the unbounded part so that the nth power of 2 converges to zero. No such topology is compatible with the multiplication by constants.

Studying the relationships between algebraic operations and sequential convergence, R. Frič in [4] proved that the usual sequential convergence on the real numbers $\mathbb{R}$ can be enlarged to an $\mathcal{L}_0^*$-group convergence such that, for each $a \in \mathbb{R}$, the sequence $\{a2^n : n = 1, 2, \dots\}$ converges to 0. He formulated (oral communication) the following question: Is there a topology $\mathcal{T}$ on $\mathbb{R}$ such the following four conditions

(1) $(\mathbb{R}, \mathcal{T})$ is a metrizable topological group with respect to the usual addition;
(2) If a sequence $\{x_n : n = 1, 2, \dots\}$ in $\mathbb{R}$ converges to $x \in \mathbb{R}$ with respect to the usual topology, then $\{x_n : n = 1, 2, \dots\}$ converges to x with respect to $\mathcal{T}$, too;
(3) The sequence $\{2^n : n = 1, 2, \dots\}$ converges to 0 with respect to $\mathcal{T}$;
(4) For each $a \in \mathbb{R}$ and each sequence $\{x_n : n = 1, 2, \dots\}$ which converges to x with respect to $\mathcal{T}$, the sequence $\{ax_n : n = 1, 2, \dots\}$ converges to ax with respect to $\mathcal{T}$, too;

are satisfied?

In the present note, we show that there is a topology $\mathcal{T}$ on $\mathbb{R}$ satisfying conditions (1), (2) and (3), and prove that no topology can satisfy all four conditions.

Independently, J. Doboš in [2] explicitly constructed a metric on $\mathbb{R}$ such that the induced topology satisfies conditions (1), (2) and (3).

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THEOREM 1. *There is a topology $\mathcal{T}$ on $\mathbb{R}$ satisfying conditions* (1), (2) *and* (3).

Proof. Let $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers and integers, respectively. For each $n \in \mathbb{N}$, put

$$U_n = \bigcup_{m \in \mathbb{Z}} \left(m2^n - \tfrac{1}{n}\,,\ m2^n + \tfrac{1}{n}\right).$$

Furthermore, for each $x \in \mathbb{R}$, put

$$U_n(x) = x + U_n \ \left(= \bigcup_{m \in \mathbb{Z}} (x + m2^n - \tfrac{1}{n}\,,\ x + m2^n + \tfrac{1}{n})\right).$$

Let $\mathcal{U}(x) = \{U_n(x) :\ n \in \mathbb{N}\}$. Then it is easy to see that $\{\mathcal{U}(x) :\ x \in \mathbb{R}\}$ satisfies the axioms of a basic neighborhood system. Hence there is a topology $\mathcal{T}$ generated by $\{\mathcal{U}(x) :\ x \in \mathbb{R}\}$ (see [1; Proposition 1.2.3]).

Then $(\mathbb{R}, \mathcal{T})$ is a T_0-space. Indeed, for each two distinct points x and y in $\mathbb{R}$ there is $n \in \mathbb{N}$ such that $|x - y| < 2^n$. Clearly $y \notin U_n(x)$. We shall show that $(\mathbb{R}, \mathcal{T})$ is a topological group. First, we shall show that the mapping $(x, y) \mapsto x + y$ is $\mathcal{T}$-continuous. Let $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $n \in \mathbb{N}$. For each $(u, v) \in U_{2n}(x) \times U_{2n}(y)$ there are $\ell, m \in \mathbb{Z}$ such that $u \in \left(x + \ell 2^{2n} - \frac{1}{2n}\,,\ x + \ell 2^{2n} + \frac{1}{2n}\right)$ and $v \in \left(y + m2^{2n} - \frac{1}{2n}\,,\ y + m2^{2n} + \frac{1}{2n}\right)$. Then we have

$$|x + y + (\ell + m)2^{2n} - (u + v)| \le |x + \ell 2^{2n} - u| + |y + m2^{2n} - v| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}\,.$$

Therefore, $u + v \in \left(x + y + (\ell + m)2^{2n} - \frac{1}{n}\,,\ x + y + (\ell + m)2^{2n} + \frac{1}{n}\right) \subset U_n(x + y)$. Hence, the mapping $(x, y) \mapsto x + y$ is $\mathcal{T}$-continuous. It is easy to see that $U_n(-x) = -U_n(x)$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus the mapping $x \mapsto -x$ is $\mathcal{T}$-continuous. Hence $(\mathbb{R}, \mathcal{T})$ is a topological group.

Since $(\mathbb{R}, \mathcal{T})$ is first countable, it follows that $(\mathbb{R}, \mathcal{T})$ is metrizable (see [1], [4] and [3; Theorem 8.3]). By the definition of $\mathcal{U}(x)$, it is also clear that the conditions (2) and (3) are satisfied. Thus the proof is complete. □

THEOREM 2. *There is no topology $\mathcal{T}$ on $\mathbb{R}$ which satisfies conditions* (1), (2), (3) *and* (4).

Proof. Let $\operatorname{Cl} A$ and $\operatorname{Cl}_{\mathcal{T}} A$ denote the closures of a subset A of $\mathbb{R}$ taken by the usual topology and $\mathcal{T}$, respectively. Suppose that there is a topology $\mathcal{T}$ of $\mathbb{R}$ which satisfies the conditions (1), (2), (3) and (4). Let $\{U_k :\ k = 1, 2, \dots\}$ be a countable $\mathcal{T}$-neighborhood base at 0 such that $\operatorname{Cl}_{\mathcal{T}} U_{k+1} \subset U_k$. For each natural number m we put

$$A_m = \left\{a \in [0, 1] :\ \{a2^n :\ n \ge m\} \subset U_2\right\}.$$

Since for each $a \in [0,1]$ the sequence $\{a2^n : n = 1,2,\dots\}$ converges to 0 with respect to $\mathcal{T}$, it follows that $\bigcup_{m=1}^{\infty} A_m = [0,1]$. By the Baire's category theorem ([3]), there are an m_1 and a non-degenerate closed interval $[\alpha_1, \beta_1] \subset [0,1]$ such that $[\alpha_1, \beta_1] \subset \operatorname{Cl} A_{m_1}$. For each $a \in [\alpha_1, \beta_1]$ and each natural number $n \geq m_1$ we have $a2^n \in \operatorname{Cl} U_2$. Hence, it follows from the condition (2) that $a2^n \in \operatorname{Cl}_{\mathcal{T}} U_2 \subset U_1$ and hence $\{a2^n : n \geq m_1\} \subset U_1$.

By continuing this process, for each natural number k we have a non-degenerate closed interval $[\alpha_k, \beta_k]$ and a natural number m_k such that for each k

(a) $[\alpha_k, \beta_k] \supset [\alpha_{k+1}, \beta_{k+1}]$,
(b) $m_k \leq m_{k+1}$, and
(c) for each $a \in [\alpha_k, \beta_k]$, $\{a2^n : n \geq m_k\} \subset U_k$.

Let $a_0 \in \bigcap_{k=1}^{\infty} [\alpha_k, \beta_k]$. We shall see that the sequence $\{a_0 2^n + 1 : n = 1,2,\dots\}$ converges to 0 with respect to $\mathcal{T}$. Let k be a natural number. Since $a_0 \in [\alpha_k, \beta_k]$, there is a natural number $n(k) \geq m_k$ such that $\left[a_0 - \frac{1}{2^{n(k)}}, a_0\right] \subset [\alpha_k, \beta_k]$ or $\left[a_0, a_0 + \frac{1}{2^{n(k)}}\right] \subset [\alpha_k, \beta_k]$. Without loss of generality, we can assume that $\left[a_0, a_0 + \frac{1}{2^{n(k)}}\right] \subset [\alpha_k, \beta_k]$ for each k. Then for each k and $n \geq n(k)$ it follows that $a_0 + \frac{1}{2^n} \in \left[a_0, a_0 + \frac{1}{2^{n(k)}}\right] \subset [\alpha_k, \beta_k]$. Hence $a_0 2^n + 1 = \left(a_0 + \frac{1}{2^n}\right) 2^n \in U_k$. Therefore, the sequence $\{a_0 2^n + 1 : n = 1,2,\dots\}$ converges to 0 with respect to $\mathcal{T}$. On the other hand, since $(\mathbb{R}, \mathcal{T})$ is a topological group, the sequence $\{a_0 2^n + 1 : n = 1,2,\dots\}$ converges to $\lim_{n\to\infty} a_0 2^n + 1 = 0 + 1 = 1$ with respect to $\mathcal{T}$. This is a contradiction. □

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