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THE COVERING OF RINGS BY INTEGRALLY CLOSED RINGS

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Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

We introduce first some basic notions used in this paper.

Let K be a field. Let G be a linearly ordered abelian group extended by the element ∞, greater than any other element of G, and such that

$$\infty + a = a + \infty = \infty$$
$$\infty = \infty = \infty$$

for all $a \in G$.

A mapping v from K into G is called a valuation of the field K if

1) $v(a) = \infty \Leftrightarrow a = 0$;
2) $v(a \cdot b) = v(a) + v(b)$ for all $a, b \in K$;
3) $v(a + b) \geq \min \{v(a), v(b)\}$ for all $a, b \in K$.

From the definition of the valuation it follows that $v(a + b) = \min \{v(a), v(b)\}$ whenever $v(a) \neq v(b)$.

The set of all $x \in K$ such that $v(x) \geq 0$ forms a ring V. This ring V is called the valuation ring belonging to v. We shall say that $V \subset K$ is a valuation ring of K if V is a valuation ring for some valuation of K.

The following simple facts will be used. If $x \in K$ and V is a valuation ring of K, then at least one of the elements x, x^{-1} is contained in V. If both x and x^{-1} are contained in V, then x is called a unit of the ring V. If u is a unit and $y \in V$ is not a unit, then $u + y$ is a unit of the valuation ring V. In particular, if $y \in V$ is not a unit, then $1 + y$ is a unit. An element $u \in V$ is a unit if and only if $v(u) = 0$. In particular, $v(1) = v(-1) = 0$.

If V is a valuation ring of a field K, then $K - V$ is multiplicatively closed, i.e. $a \notin V$, $b \notin V$ imply $a \cdot b \notin V$.

If $a \notin V$ and u is a unit of V, then $a \cdot u \notin V$, since $a \cdot u \notin V$ would imply $a \cdot u \cdot u^{-1} \in V$, i.e. $a \in V$.

In the following we shall deal with covering a subring B of K by subrings $A_1, A_2, \ldots, A_n$ of K, i.e. we shall find conditions under which

$$B \subset A_1 \cup A_2 \cup \ldots \cup A_n$$

may hold.

We first show:

Lemma 1. *If B, A_1, A_2 are subrings of K and $B \subset A_1 \cup A_2$, then either $B \subset A_1$ or $B \subset A_2$.*

Proof. Suppose that there exist two elements a_1, a_2 such that A_1, $a_2 \in B$ and $a_1 \in A_1 - A_2$, $a_2 \in A_2 - A_1$. Since $a_1 + a_2 \in B \subset A_1 \cup A_2$ we have either $a_1 + a_2 \in A_1$ or $a_1 + a_2 \in A_2$. Suppose, e.g. that $a_1 + a_2 \in A_1$. Then $a_1 + a_2 - a_1 = a_2 \in A_1$, contrary to our assumption. Analogously if $a_1 + a_2 \in A_2$, then $a_1 + a_2 - a_2 = a_1 \in A_2$, contrary to the assumption. Hence either $B \subset A_1$ or $B \subset A_2$. This proves our statement.

Lemma 1 does not necessarily hold if the number of "summands" is larger than 2. In [4] an example of a field K and subrings B, A_1, A_2, A_3 of K has been given such that $B \subset A_1 \cup A_2 \cup A_3$, while B is contained in none of the A_1, A_2, A_3.

In the following suppose that B, A_1, ..., A_n is a finite number of subrings of K.

In [2] we have shown: If G is Archimedean and

$$B \subset A_1 \cup A_2 \cup \ldots \cup A_n,$$

where all A_i are valuation rings, then B is contained in one of the rings A_i.

In [4] has been shown: If $B \subset A_1 \cup A_2 \cup \ldots \cup A_n$, $n \geq 3$, and all A_i with the exception of at most two "summands" are valuation rings, then B is contained in one of the rings A_i.

Let K be a field and A a subring of K. An element $x \in K$ is called integral over the ring A if there are elements A_0, a_1, ..., $a_{n-1} \in A$, $n \geq 1$, such that

$$x^n + a_{n-1} x^{n-1} + \ldots + a_0 = 0.$$

A ring A is called integrally closed in the field K if each element of K which is integral over A is contained in A.

The purpose of the present paper is to show that the result of [4] (mentioned above) may be sharpened. Instead of supposing that A_1, A_2, ..., A_n with the exception of two of them are valuation rings it is sufficient to suppose that A_1, A_2, ..., A_n with the exception of at most two of them are integrally closed rings.

We shall use the following known result

Theorem 1. ([1], p. 70, Corollary 10.9)

A subring A of a field K is integrally closed in K if and only if A is an intersection of valuation rings of K.

Note that the intersection of two valuation rings need not be a valuation ring.

Lemma 2. *Let K be a field and B a subring of K containing the unity element of K. Let $W = \{V_1, V_2, \ldots, V_l\}$ be a finite set of valuation rings each of which contains* *a given element $b \in B$. Then there is an element $c \in B$ such that $c \in \bigcap_{i=1}^{l} V_i$ and c is a unit of each $V_i \in W$.*

Proof. Denote by v_i the valuation of K belonging to V_i. Put $b_0 = b$ and define the sequence $\beta = \{b_0, b_1, \ldots, b_l\}$ by the recurrent formula

$$b_k = 1 + \prod_{i=0}^{k-1} b_i.$$

The supposition

$$b_0 \in B \cap \bigcap_{i=1}^{l} V_i$$

implies

$$b_k \in B \cap \bigcap_{i=1}^{l} V_i$$

and therefore $v_i(b_k) \geq 0$ for all v_i, $i = 1, 2, \ldots, l$. We next show that for a fixed chosen v_j there is at most one element $b_{s(j)} \in \beta$ such that $v_j(b_{s(j)}) > 0$.

If for all $b_u \in \beta$ we have $v_j(b_u) = 0$, there is nothing to prove. Let $b_{s(j)}$ be the first term in β for which $v_j(b_{s(j)}) > 0$. Then for any $m > s(j)$ we have

$$v_j\left(\prod_{u=0}^{m-1} b_u\right) > 0,$$

hence $v_j(b_m) = 0$. Hence to any v_j there is at most one element $b_{s(j)}$ such that $v_j(b_{s(j)}) \neq 0$. Since card $\beta = l + 1$, there is necessarily an element, say $b_r = c$, such that $v_j(c) = 0$ for all $j \in \{1, 2, \ldots, l\}$. Hence c is a unit for all $V_i \in W$.

Remark. Let V be an arbitrary valuation ring not containing b. We shall prove that the element c chosen from the sequence β in the proof of Lemma 2 is not contained in V. We show by induction that any $b_i \in \beta$ is not contained in V. By the assumption $b_0 = b \notin V$. Let for $i < s$ $b_i \notin V$, then from the fact that the complement of a valuation ring in the field K is multiplicatively closed it follows that

$$\prod_{i=0}^{s-1} b_i \notin V$$

and so

$$b_s = 1 + \prod_{i=0}^{s-1} b_i \notin V,$$

hence $c \notin V$.

Theorem 2. *Let $B, A_1, A_2, \ldots, A_n$ be subrings of a field K containing the unity element of K. Suppose that the rings A_i, with the exception of at most two of them, are integrally closed in K. If $B \subset \bigcup_{i=1}^{n} A_i$, then there is an i $(1 \leq i \leq n)$ such that $B \subset A_i$.*

Proof. By Lemma 1 we may suppose that $n \geqslant 3$. Without loss of generality we may suppose that $A_3, \ldots, A_n$ are integrally closed, hence intersections of valuation rings of the field K. Suppose for an indirect proof that $B \cap A_j \neq B$ for all $j \in N$, $N = \{1, 2, \ldots, n\}$. Again without loss of generality we may suppose that $B \not\subset \bigcup_{i \neq j} A_i$ for any $j \in N$. We shall show that under this condition there is necessarily an $x \in B$ such that $x \notin \bigcup_{i=1}^{n} A_i$.

For any $j \in N$ there exists an element a_j such that

$$a_j \in B \cap A_j - \bigcup_{i \neq j} A_i.$$

Each of the rings $\{A_3, \ldots, A_n\}$ is an intersection of valuation rings of K. Hence to any a_j just chosen and to any ring A_i, $i \neq j$ and $i > 2$, there is a valuation ring $V_{j,i}$ such that $A_i \subset V_{j,i}$ and $a_j \notin V_{j,i}$. The set $M = \{V_{j,i} \mid i > 2,\ i \neq j\}$ is a finite set of valuation rings and, of course,

$$A_3 \cup A_4 \cup \ldots \cup A_n \subset \bigcup V_{j,i} \quad i > 2,\ i \neq j.$$

For a fixed a_j let $M^{(j)}$ be the set of all valuation rings $V \in M$ containing a_j. If $M^{(j)} \neq \emptyset$, by Lemma 2 there is a element $c_j \in B$, c_j with the property of Remark, that c_j is contained in all valuation rings $V \in M^{(j)}$ and c_j is a unit of each $V \in M^{(j)}$. If $M^{(j)} = \emptyset$ put $c_j = a_j$. Denote

$$d_0 = \prod_{j=1}^{n} c_j.$$

Clearly $d_0 \in B$. If $V = V_{j_0, i_0}$ is any valuation ring from M, then by definition $a_{j_0} \notin V = V_{j_0, i_0}$, hence $c_{j_0} \notin V$. Some other c_j may be contained in V. The product of those c_j which are not contained in V is not contained in V (since $K - V$ is multiplicatively closed). For those c_j which are contained in V the product is a unit of V. Since the product of an element not contained in V and a unit of V is not contained in V, we have the following result: The element d_0 constructed above is contained in none of the valuation rings $V \in M$.

We now consider the following subset of M:

$$M_0 = \{V_{1,i} \mid i > 2\} \cup \{V_{2,i} \mid i > 2\} \cup \{V_{3,i} \mid i > 3\}$$

(if $n = 3$, the last summand is empty) Clearly $d_0 \notin V_{j,i}$ for any $V_{j,i} \in M_0$ and $A_3 \cup \ldots \cup A_n$ is covered by $V_{j,i} \in M_0$.

We next show that there is a positive integer t such that none of the elements $a_1^{-1} d_0^t$, $a_2^{-1} d_0^t$, $a_3^{-1} d_0^t$ is a unit of any $V \in M_0$.

$d_0 \notin V$ for any $V \in M_0$; hence for any $v_{j,i}$, where j, i run through all admissible j, i, there is

$$v_{j,i}(d_0^{t_1}) \neq v_{j,i}(d_0^{t_2})$$

for positive integers $t_1 \neq t_2$. Hence for a fixed $l \in \{1, 2, 3\}$

$$v_{j,i}(a_l^{-1} d_0^{t_k}) = 0$$

for at most one positive integer t_k. Therefore there is a positive integer $t_{j,i,l}$ such that for any $t_k \geq t_{j,i,l}$ we have

$$v_{j,i}(a_l^{-1} d_0^{t_k}) \neq 0.$$

Now if $t = \max\{t_{j,i,l} \mid j \in \{1, 2, 3\}, i \in \{3, 4, \ldots, n\}, i > j, l \in \{1, 2, 3\}\}$, then none of the elements $a_1^{-1}d_0^t$, $a_2^{-1}d_0^t$, $a_3^{-1}d_0^t$ is a unit of any $V \in M_0$.

Put $d = d_0^t$. Clearly $d \notin \bigcup_{i=3}^{n} A_i$, since for each $V \in M$, $d \notin V$. From $d \in B$ it follows that $d \in A_1 \cup A_2$.

We now finally show that there is an element $x \in B$ such that $x \notin A_1 \cup \ldots \cup A_n$. There are three possibilities:

$$d \in A_1 \cap A_2, \quad d \in A_1 - A_2, \quad d \in A_2 - A_1.$$

1. Let $d \in A_1 \cap A_2$. Put $x = a_3 + d$. Then $x \notin A_1 \cup A_2 \cup A_3$ (For, $x \in A_1$ would imply $a_3 = x - d \in A_1$, $x \in A_2$ would imply $a_3 = x - d \in A_2$ and $x \in A_3$ would imply $d = x - a_3 \in A_3$, in all cases a contradiction). To show that $x \notin A_i$ for $i > 3$ write $x = a_3(1 + a_3^{-1}d)$.

α) Suppose $a_3^{-1}d \notin V_{3,i}$. Since $a_3 \notin V_{3,i}$ and $1 + a_3^{-1}d \notin V_{3,i}$ we have $x \notin V_{3,i}$ whence $x \notin A_i$.

β) Suppose $a_3^{-1}d \in V_{3,i}$. Then $1 + a_3^{-1}d$ is a unit of $V_{3,i}$, since $a_3^{-1}d$ is not a unit. The product of $a_3 \notin V_{3,i}$ and the unit $1 + a_3^{-1}d$ does not belong to $V_{3,i}$. Hence $x \notin V_{3,i}$ and so $x \notin A_i$.

2. Let $d \in A_1 - A_2$. Put $x = a_2 + d$. Then $x \notin A_1 \cup A_2$ (For, $x \in A_1$ would imply $x - d = a_2 \in A_1$, $x \in A_2$ would imply $x - a_2 = d \in A_2$, in both cases a contradiction). To show $x \notin A_i$ for $i > 2$ write

$$x = a_2(1 + a_2^{-1}d)$$

α) Suppose $a_2^{-1}d \notin V_{2,i}$. Since $a_2 \notin V_{2,i}$ and $1 + a_2^{-1}d \notin V_{2,i}$ we have $x \notin V_{2,i}$, whence $x \notin A_i$.

β) Suppose $a_2^{-1}d \in V_{2,i}$, $a_2^{-1}d$ is not a unit of $V_{2,i}$, hence $1 + a_2^{-1}d$ is a unit of $V_{2,i}$. The product $a_2 \notin V_{2,i}$ and the unit $1 + a_2^{-1}d$ does not belong to $V_{2,i}$. Hence $x \notin V_{2,i}$ and $x \notin A_i$.

3. Let $d \in A_2 - A_1$. Put $x = a_1 + d$. Analogously to the case 2 we get

$$x \notin \bigcup_{i=1}^{n} A_i.$$

Theorem 2 is proved.

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ПОКРЫТИЕ КОЛЕЦ ЦЕЛОЗАМКНУТЫМИ КОЛЬЦАМИ

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Резюме

В работе доказана следующая теорема: Пусть B, A_1, ..., A_n подкольца с единицей поля K, такие что

$$B \subset \bigcup_{i=1}^{n} A$$

и все кольца A_i, $i = 1, \ldots, n$ кроме быть может двух являются целозамкнутыми кольцами. Тогда существует такое кольцо A_i, $i \in \{1, \ldots, n\}$, что $B \subset A_i$.