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# THE GRAPH OF LABELLINGS OF A GIVEN GRAPH

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At the Czechoslovak conference on Graph Theory and Combinatorics in the Raček Valley in May 1986, P. Tomasta [1] has proposed four problems concerning a certain graph $\Pi(H)$ assigned to a finite undirected graph $H$. Here we shall not solve these problems, but we shall treat simpler questions concerning $\Pi(H)$.

Let a finite undirected graph $H$ be given, let $V(H)$ be its vertex set, let $n = |V(H)|$. A labelling of $H$ is a bijection $\lambda$: $V(H) \to \{1, ..., n\}$. We shall consider ordered pairs $(H, \lambda)$, where $\lambda$ is a labelling of $H$. To the given graph $H$ there exist $n!$ such pairs.

Let $\lambda_1$, $\lambda_2$ be two labellings of $H$. The pairs $(H, \lambda_1)$, $(H, \lambda_2)$ are called isomorphic if they have the property that for any two numbers $i, j$ from the set $\{1, ..., n\}$ the vertices $\lambda_1^{-1}(i)$, $\lambda_1^{-1}(j)$ are adjacent in $H$ if and only if the vertices $\lambda_2^{-1}(i)$, $\lambda_2^{-1}(j)$ are adjacent in $H$.

These concepts may be interpreted in the following way. We consider the set $\{1, ..., n\}$ as a vertex set. To every pair $(H, \lambda)$ we assign a graph isomorphic to $H$ and having the vertex set $\{1, ..., n\}$; two vertices $i$, $j$ from this set will be adjacent if and only if the vertices $\lambda^{-1}(i)$, $\lambda^{-1}(j)$ are adjacent in $H$. To the isomorphic pairs $(H, \lambda_1)$, $(H, \lambda_2)$ the same graph corresponds; to non-isomorphic ones two distinct (but obviously isomorphic) graphs correspond. This isomorphism is an equivalence relation on the set of all pairs $(H, \lambda)$. Any class of this equivalence will be called a labelled graph $H$. Such a class will be considered as a certain graph isomorphic to $H$ with the vertex set $\{1, ..., n\}$.

Let $\Lambda(H)$ be the set of all labelled graphs $H$. Let $\Pi(H)$ be the graph whose vertex set is $\Lambda(H)$ and in which two verties $H_1$, $H_2$ are adjacent if and only if $|E(H_1) - E(H_2)| = |E(H_2) - E(H_1)| = 1$. (The symbol $E(G)$ for any graph $G$ denotes the edge set of $G$.)

There are four problems by P. Tomasta concerning $\Pi(H)$. The first two concern the neighbourhoods of vertices. The third asks about the characterization of graphs $\Pi(H)$ with a Hamiltonian circuit and the fourth asks an analogous question concerning a Hamiltonian path. These problems seem to be very difficult; note that the number of vertices of $\Pi(H)$ is usually much greater than that of $H$. We shall treat much simpler problems: the existence of edges in $\Pi(H)$ and its connestedness. Neither of these problems will be solved completely.

A graph will be called discrete if it has no edges.

**Theorem 1.** *Let $H$ be a regular graph. Then $\Pi(H)$ is discrete.*

Proof. Let $r$ be the regularity degree of $H$. If $r = 0$, then $H$ is discrete; therefore $(H, \lambda_1)$, $(H, \lambda_2)$ are isomorphic for any $\lambda_1$, $\lambda_2$ and $\Pi(H)$ consists of one vertex. Now let $r \geqq 1$. If $H$ is complete, then $\Pi(H)$ consists again of one vertex and thus it is discrete; otherwise it has at least two vertices. Suppose that $H$ is not complete and two vertices $H_1$, $H_2$ of $\Pi(H)$ are adjacent in $\Pi(H)$. Let $E(H_1) - E(H_2) = \{e_1\}$, $E(H_2) - E(H_1) = \{e_2\}$. Let $u$, $v$ be the end vertices of $e_1$. By deleting $e_1$ from $H_1$ we obtain a graph $H_0$ in which only the vertices $u$, $v$ have the degree $r - 1$ and all the others have the degree $r$. The graph $H_2$ is to be obtained by adding $e_2$ to $H_0$. If $e_2$ is adjacent to a vertex w distinct from $u$ and $v$, then w has the degree $r + 1$ in $H_2$, which is not possible, because $H_2$ is regular of degree $r$. Thus $e_2$ can join only $u$ and $v$ and thus $e_2 = e_1$, which is a contradiction. □

**Theorem 2.** *Let $H$ be a graph, let $\delta(H)$ (or $\Delta(H)$) be its minimum (or maximum respectively) degree. Let there exist an integer $r$ such that $\delta(H) < r < \Delta(H)$ and no vertex of $H$ has the degree $r$. Then $\Pi(H)$ is disconnected.*

Proof. Let $\Lambda_1$(or $\Lambda_2$) be the subset of $\Lambda(H)$ consisting of labelled graphs with the property that the degree of the vertex 1 is less (or greater respectively) than $r$. Evidently $\Lambda_1 \neq \emptyset$, $\Lambda_2 \neq \emptyset$, $\lambda_1 \cup \Lambda_2 = \Lambda(H)$, $\Lambda_1 \cap \Lambda_2 = 0$. Let $H_1 \in \Lambda_1$, $H_2 \in \Lambda_2$. The difference between the degrees of the vertex 1 in $H_2$ and in $H_2$ is at least 2 and thus also $|E(H_2) - E(H_1)| \geqq 2$ and $H_1$, $H_2$ are not adjacent in $\Pi(H)$. As $H_1$, $H_2$ were chosen arbitrarily, no vertex of $\Lambda_1$ is adjacent to a vertex of $\Lambda_2$ and $\Pi(H)$ is disconnected. □

**Theorem 3.** *Let $H$ be a disconnected graph in which all connected components have the same number of edges and are not all isomorphic. Then $\Pi(H)$ is disconnected.*

Proof. Let $D_1$, $D_2$ be two non-isomorphic connected components of $H$. Let $\Lambda_1$ (or $\Lambda_2$) be the subset of $\Lambda(H)$ consisting of all labelled graphs such that the vertex 1 is contained in a connected component of $H_1$ isomorphic to $D_1$ (or $D_2$ respectively). Let $H_1 \in \Lambda_1$, $H_2 \in \Lambda_2$. Let $E(H_1) - E(H_2) = \{e_1\}$, $E(H_2) - - E(H_1) = \{e_2\}$. Then $e_2$ joins two vertices of the connected component of $H_1$ which contains $e_1$; otherwise there would exist in $H_2$ connected components with different numbers of edges. If $e_1$ does not belong to the connected component of $H_1$ containing 1, then 1 in $H_2$ belongs again to a connected component isomorphic to $D_1$, which is a contradiction. If $e_1$ belongs to the connected component of $H_1$ containing 1 and this component after deleting $e_1$ and adding $e_2$ becomes isomorphic to $D_2$, then the number of connected components of $H_2$ isomorphic to $D_1$ (or to $D_2$) is less (or greater respectively) than that of $H_1$. Thus $H_1$ is not isomorphic to $H_2$, which is again a contradiction. □

**Theorem 4.** *Let $H$ be a disconnected graph consisting of 2-edge-connected* *components. Let there exist an integer r such that no connected component of H has r edges, but there exist those with less and those with more than r edges. Then $\Pi(H)$ is disconnected.*

Proof. Let $\Lambda_1$ (or $\Lambda_2$) be the subset of $\Pi(H)$ consisting of labelled graphs $H$ such that the vertex 1 is in a connected component with less (or more respectively) edges than $r$. We have $\Lambda_1 \neq \emptyset$, $\Lambda_2 \neq \emptyset$, $\Lambda_1 \cup \Lambda_2 = \Lambda(\mathrm{H})$, $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let $H_1 \in \Lambda_1$, $H_2 \in \Lambda_2$ and suppose that $H_1$, $H_2$ are adjacent in $\Pi(H)$. Let $E(H_1) - E(H_2) = \{e_1\}$, $E(H_2) - E(H_1) = \{e_2\}$. As each connected component of $H$ is 2-edge-connected, in the graph obtained from $H_1$ by deleting $e_1$ the vertex sets of the connected components are the same as in $H_1$. After adding $e_2$ to it the number of edges of the connected component containing 1 can increase at most by one, hence it cannot become greater than $r$, which is a contradiction. Hence no vertex of $\Lambda_1$ is adjacent to a vertex of $\Lambda_2$ and $\Pi(H)$ is disconnected. □

**Theorem 5.** *Let H be a 2-edge-connected bipartite graph with the bipartition classes A, B, let $|A| \neq |B|$. Then $\Pi(H)$ is disconnected.*

Proof. Let $\Lambda_1$ (or $\Lambda_2$) be the subset of $\Lambda(H)$ consisting of labelled graphs $H$ such that the vertex 1 is in the set corresponding to $A$ (or to $B$ respectively). As $H$ is 2-edge-connected, its bipartition classes remain the same after deleting an arbitrary edge. If $H_1 \in \Lambda_1$, then by deleting an edge $e_1$ and adding an edge $e_2$ we obtain again a graph from $\Lambda_1$ and thus analogously to the proof of Theorem 4 we may prove that $\Pi(H)$ is disconnected. □

**Theorem 6.** *Let H be a graph, let $\bar{H}$ be its complement. Then $\Pi(H) \cong \Pi(\bar{H})$.*

Proof. Let $H_1$, $H_2$ be two graphs from $\Lambda(H)$. The graph $H_2$ is obtained from $H_1$ by deleting an edge $e_1$ and adding an edge $e_2$ if and only if $\bar{H}_2$ is obtained from $\bar{H}_1$ by deleting $e_2$ and adding $e_1$. This imples the assertion. □

We have shown there exist wide classes of graphs $H$ such that $\Pi(H)$ is discrete and wide classes of graphs $H$ such that $\Pi(H)$ is disconnected. We may ask about two problems.

**Problem 1.** *Characterize the graphs H for which $\Pi(H)$ is discrete, i.e. graphs H with the property that by deleting an arbitrary edge $e_1$ and adding a new edge $e_2 \neq e_1$ always a graph non-isomorphic to H is obtained.*

**Problem 2.** *Characterize the graphs H for which $\Pi(H)$ is disconnected.*

## REFERENCE

[1] TOMASTA, P.: Problems 15—18. Czechoslovak Conference on Graph Theory and Combinatorics, Raček Valley, May 1986.

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# ГРАФ ПОМЕЧЕНИЙ ЗАДАННОГО ГРАФА

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Резюме

Пусть $H$ есть конечный неориентированный граф с $n$ вершинами. П. Томаста ввел граф $\Pi(H)$, вершинами которого являются все графы на множестве вершин $\{1, \ldots, n\}$, изоморфные графу $H$, и в котором две вершины $H_1$, $H_2$ смежны тогда и только тогда, когда $|E(H_1) - E(H_2)| = |E(H_2) - E(H_1)| = 1$. Показаны некоторые классы графов $H$, для которых $\Pi(H)$ является несвязным или дискретным.