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HYPERELLIPTIC MAPS AND SURFACES

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ABSTRACT. Hyperellipticity is a classical concept in the theory of Riemann surfaces. The object of this note is to begin to explore the relationship between hyperelliptic maps and hyperelliptic Riemann surfaces.

Harvey showed that underlying any map $\mathcal{M}$ on an orientable surface $\mathcal{S}$, there is a unique Riemann surface $X = X(\mathcal{M})$ naturally associated with $\mathcal{M}$. It is clear from the definition that if $\mathcal{M}$ is hyperelliptic, then $X(\mathcal{M})$ is is hyperelliptic. We shall show that in the special case where $\mathcal{M}$ is a regular map, the converse of this result is also true.

1. Introduction

Hyperellipticity is a classical concept in the theory of Riemann surfaces. More recently ([1], [2]) this concept has been applied to maps and hypermaps. The object of this note is to begin to explore the relationship between hyperelliptic maps and hyperelliptic Riemann surfaces.

In [7], it was shown that underlying any map $\mathcal{M}$ on an orientable surface $\mathcal{S}$, there is a unique Riemann surface $X = X(\mathcal{M})$ naturally associated with $\mathcal{M}$. (See §2 for more details.) The same idea was discussed by Grothendieck [6] who also pointed out that Belyi's Theorem implies that in the case where S is compact, the algebraic curve that is associated with X can be defined over the field of algebraic numbers. (See the article by Gareth Jones in this volume.) It is clear from the definition that if $\mathcal{M}$ is hyperelliptic, then $X(\mathcal{M})$ is hyperelliptic. We shall show that in the special case where $\mathcal{M}$ is a regular map, the converse of this result is also true.

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2. Basic ideas

A compact Riemann surface X of genus $g \geq 2$ is said to be *hyperelliptic* if it is a two-sheeted branched covering of the Riemann sphere Σ. There is then a conformal automorphism J of X such that $J^2 = I$ and $X/\langle J\rangle \cong \Sigma$. Note that by the Riemann-Hurwitz formula, J has $2g+2$ fixed points. It then follows that J is unique with this property, and as a consequence, J is central in $\operatorname{Aut} X$. (See [5; §3.7.])

A map $\mathcal{M}$ of genus $g \geq 2$ is said to be *hyperelliptic* if there is a map automorphism j of $\mathcal{M}$ such that $j^2 = I$ and $\mathcal{M}/\langle j\rangle$ is isomorphic to a map on the sphere. For a combinatorial proof of the uniqueness of the hyperelliptic involution of a map (or hypermap), see [2; Corollary 5.22].

As we shall see below, there is a natural complex structure on the surface S underlying $\mathcal{M}$ such that every map automorphism becomes a Riemann surface automorphism. It follows that if $\mathcal{M}$ is hyperelliptic, then so is the Riemann surface X associated with $\mathcal{M}$.

Let $\mathcal{M}$ be a map of type (m, n). This means that m is the least common multiple of the vertex valencies of the map, and n is the least common multiple of the face valencies. As described in [8], associated with $\mathcal{M}$ there is an "algebraic map" which consists of a quadruple (G, Ω, x, y), where G is a permutation group acting transitively on Ω, and x and y are generators of G obeying the relations $x^2 = y^m = (y^{-1}x)^n = 1$. Here Ω is the set of darts of $\mathcal{M}$, x is the permutation that interchanges the darts of an edge, and y is the permutation that cyclically permutes the darts around each vertex, following the orientation of the underlying surface. It then follows that $y^{-1}x$ cyclically permutes the edges around each face. Let $\Gamma(m, n)$ be the group with presentation $\langle X, Y \mid X^2 = Y^m = (Y^{-1}X)^n = 1\rangle$. If $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, then $\Gamma(m, n)$ acts as a group of conformal automorphisms of the upper-half complex plane $\mathcal{H}$. ($\Gamma(m, n)$ is then called a Fuchsian triangle group.) Also $\Gamma(m, n)$ acts as a group of automorphisms of the universal map $\hat{\mathcal{M}}$ of type (m, n) that lies on $\mathcal{H}$ (see [8]). There is an obvious homomorphism from $\Gamma(m, n)$ onto G, and hence an action of $\Gamma(m, n)$ on Ω. Let M be the stabilizer of a dart in this action. Then, as shown in [8], $\mathcal{M}$ is isomorphic to the map $\hat{\mathcal{M}}/M$ on the Riemann surface $\mathcal{H}/M$. We can thus define $X(\mathcal{M})$ to be the Riemann surface $\mathcal{H}/M$.

3. Hyperellipticity

We shall now assume that $\mathcal{M}$ is a regular map in the sense of [4]. Every vertex then has valency m, and every face has valency n. As shown in [8], M is now a torsion-free normal subgroup of $\Gamma(m, n)$. By [10], a necessary and sufficient condition for the Riemann surface $\mathcal{H}/M$ to be hyperelliptic is that M is a subgroup of index 2 in a Fuchsian group H, where H has signature $(0; 2, 2, \dots, 2)$, and there are $2g+2$ elliptic periods equal to 2. (See [9; Chapter 5] for an elementary treatment of Fuchsian groups.)

LEMMA. *$\mathcal{M}$ is hyperelliptic if and only if there is a subgroup H of $\Gamma(m,n)$ with the above signature such that M is a subgroup of index 2 in H.*

Proof. By [8], the automorphism group G of $\mathcal{M}$ is isomorphic to $N(M)/M$, where $N(M)$ is the normalizer of M in $\Gamma(m,n)$. Thus there is a homomorphism $\theta\colon \Gamma(m,n) \to G$ with kernel M. Let H be the inverse image of $\langle j \rangle$ under θ. Then H contains M with index 2, and by the definition of j and the fact that there must be $2g+2$ branch points of order 2, H has signature $(0; 2, 2, \dots, 2)$ with $2g+2$ elliptic periods equal to 2. Conversely, if such a group H exists, then H/M acts as the hyperelliptic involution on M. $\square$

THEOREM. *Let $\mathcal{M}$ be a regular map with underlying Riemann surface X. Then $\mathcal{M}$ is hyperelliptic if and only if X is hyperelliptic.*

Proof. If $\mathcal{M}$ is hyperelliptic, then, with the above notation, $M \triangleleft H$ with index 2, so that $X(\mathcal{M}) = \mathcal{H}/M$ is a hyperelliptic Riemann surface. Conversely, let $X(\mathcal{M})$ be hyperelliptic, and suppose that $\mathcal{M}$ is not hyperelliptic. We then have $M \triangleleft H$ with index 2, and $M \triangleleft \Gamma := \Gamma(m,n)$, but $H \not< \Gamma$. Let $\underline{H} = H/M$ and $G = \Gamma/M$. Both G and $\underline{H}$ are automorphism groups of the Riemann surface X, and as J is central and $J^2 = 1$, $G^* := \langle \underline{H}, G \rangle \cong C_2 \times G$. Let Γ^* be the lift of G^* to $\mathcal{H}$. Then Γ^* is a Fuchsian group containing the triangle group $\Gamma(m,n)$ with index 2. By [10], we see that $m = n$ and $\Gamma^* \cong \Gamma(4,n)$. By the first isomorphism theorem, $\Gamma/M \cong \Gamma^*/H$ which is a group of automorphisms of $\mathcal{H}/H$, i.e., the Riemann sphere. The only such groups are the rotation groups of the sphere, which are either cyclic, dihedral, or isomorphic to A_4, S_4 or A_5. We thus need to examine homomorphisms from $\Gamma(n,n)$ onto these rotation groups. Using the condition that $\frac{1}{n} + \frac{1}{n} < \frac{1}{2}$ we see that $n > 4$. As A_4 and S_4 have no elements of order greater than 4, none of these groups can occur as an image of $\Gamma(n,n)$ with a torsion-free kernel. Similarly, a dihedral group cannot be generated by two elements of order $n > 4$. The group A_5 is an image of $\Gamma(5,5)$, but $C_2 \times A_5$ is not an image of $\Gamma(4,5)$, so this case cannot occur. The only remaining possibility is the cyclic group C_n as a homomorphic image of $\Gamma(n,n)$ with kernel a torsion-free subgroup of genus g, which is possible if n is divisible by 4. (By the Riemann-Hurwitz formula, we find that $n = 4g$.) Now, as above, C_n is also isomorphic to Γ^*/H, so that C_n is a homomorphic image of $\Gamma^* \cong \Gamma(4,n)$. Thus C_n is generated by elements of order 2 and 4, which is a contradiction as $n > 4$. $\square$

EXAMPLES.

(1) It is well-known that every Riemann surface of genus $g = 2$ is hyperelliptic ([5; §3.7]). Hence all the regular maps on an orientable surface of genus 2 are hyperelliptic. According to [4], there are 10 such regular maps and many of them are drawn there (see Table 9 and Chapter 8).

(2) For a simple family of examples valid for each genus $g \geq 2$, consider a regular $4g$-sided hyperbolic polygon, and label the sides $a_1, a_2, \dots, a_{2g}, a_1^{-1}, a_2^{-1}, \dots, a_{2g}^{-1}$. We obtain a surface of genus g by identifying the opposite edges of this polygon, and the rotation about the centre of the polygon through 180 degrees induces an involution whose fixed points correspond to the midpoints of the $2g$ edges, the unique vertex and the unique face centre. Thus the quotient is a hyperelliptic map. This map has type $(4g, 4g)$ and has a cyclic automorphism group of order $4g$ corresponding to the normal subgroup of index $4g$ in $\Gamma(4g, 4g)$ considered in the proof of the Theorem.

(3) Another example of a hyperelliptic map with a cyclic automorphism group is the map of type $(2g+1, 4g+2)$ formed by identifying opposite edges of a regular $4g+2$-gon. This map has automorphism group isomorphic to C_{4g+2}; this is the largest order for a cyclic group of automorphisms of both a map and a Riemann surface of genus g, the second largest order being $4g$ ([2], [7]). In fact, one can easily show (using the results of [7]) that any regular map with cyclic automorphism group must be one of the maps with automorphism group of order $4g$ or $4g+2$, and these are both hyperelliptic.

(4) An example of an infinite family of hyperelliptic regular maps with non-cyclic automorphism group is given by Cori and Machì on page 462 of [2].

We note that the Theorem above is not true if we omit the hypothesis that the map is regular. To construct an example of a non-hyperelliptic map on a hyperelliptic surface, take the map in Example 2 above with $g = 2$. Construct a new map by drawing a geodesic from the vertex to the face centre. This map has no non-trivial automorphisms and thus is not hyperelliptic. However, the Riemann surface, being of genus 2, is hyperelliptic.

The definition of hyperellipticity extends in an obvious way to hypermaps (see [2], [3]). Regular hypermaps correspond to normal subgroups K of $\Gamma(l, m, n) := \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle$, and $\mathcal{H}/K$ is hyperelliptic if and only if $K \lhd H$ with index 2 in the same way as above. Now let n be an odd integer. Then $H \not< \Gamma(n, n, n)$. We can define a homomorphism from $\Gamma(n, n, n)$ onto C_n, and then the kernel K corresponds to a regular non-hyperelliptic hypermap. However, $\Gamma(n, n, n) < \Gamma(2, n, 2n)$ with index 2 ([11]). Now K is also the kernel of a homomorphism from $\Gamma(2, n, 2n)$ onto C_{2n}. By using Proposition 4 of [11], we calculate that the unique element of order 2 in C_{2n} is the hyperelliptic involution of the Riemann surface $\mathcal{H}/K$. Thus we have a non-hyperelliptic regular hypermap on a hyperelliptic Riemann surface. In fact, by [3], we have a non-hyperelliptic hypermap whose Walsh map is hyperelliptic.

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