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FLEXIBLE NIVELOIDS

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ABSTRACT. Niveloids are isotone functionals that commute with the addition of finite constants. A niveloid is flexible if it commutes with lattice isomorphisms. Limitoids are isotone functionals that commute with lattice homomorphisms. Limitoids are characterized using carriers within the class of flexible niveloids. This characterization contains an illuminating new proof of the Greco representation theorem.

1. Introduction

Denote by $\overline{\mathbb{R}} = [-\infty, +\infty]$ the extended real line. A functional $T\colon \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ is called a *flexible niveloid* if it is isotone (i.e., $f \le g$ implies $T(f) \le T(g)$ for each $f, g \in \overline{\mathbb{R}}^X$) and if

$$T(\varphi f) = \varphi(T(f)) \tag{1.1}$$

for every lattice isomorphism φ of $\overline{\mathbb{R}}$ onto $\overline{\mathbb{R}}$.

Every flexible niveloid is a *niveloid*, i.e., an isotone functional such that

$$\underset{r\in\mathbb{R}}{\forall} \quad T(f+r) = T(f) + r\,. \tag{1.2}$$

An isotone functional is a *limitoid* if (1.1) holds for every lattice homomorphism φ of $\overline{\mathbb{R}}$ onto $\overline{\mathbb{R}}$.

Niveloids were introduced in [4]. Numerous regularization functionals, such as convexification, quasi-convexification, bi-conjugation, lower semicontinuous regularization, upper semicontinuous regularization, are niveloids. In particular, lower and upper limits along families of sets are niveloids; actually, they are examples of flexible niveloids. Lower and upper limits have been characterized by Greco [7] as limitoids.

Within the class of all niveloids flexible niveloids are distinguished by a degree of insensitivity to vertical stretching and squeezing (of the functions on which

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they act). In other words, flexible niveloids depend primarily on the levels of functions. Complete dependence on the levels of functions characterizes the limitoids. The aim of this paper is to investigate that subtle difference that makes a limitoid of a flexible niveloid.

If $\mathcal{A}$ is a family of subsets of X, then the functionals $\liminf_{\mathcal{A}}$, $\limsup_{\mathcal{A}}$ on $\overline{\mathbb{R}}^X$ (*lower* and the *upper limits along* $\mathcal{A}$) given by[1]

$$\begin{aligned} \liminf_{\mathcal{A}} f &= \sup_{A\in\mathcal{A}} \inf_{A} f\,, \\ \limsup_{\mathcal{A}} f &= \inf_{A\in\mathcal{A}} \sup_{A} f \end{aligned} \tag{1.3}$$

are limitoids. Recall that for a given family $\mathcal{A}$ of subsets of X the *grill* $\mathcal{A}^{\#}$ consists of all the subsets of X that intersect every element of $\mathcal{A}$. Notice that $\limsup_{\mathcal{A}} = \liminf_{\mathcal{A}^{\#}}$, and vice versa, because $\mathcal{A}^{\#\#}$ is the least family stable for supsets that includes $\mathcal{A}$. Denote by χ_H^{∞} the *characteristic function* of H valued in the lattice $\overline{\mathbb{R}}$:

$$\chi_H^{\infty}(x) = \begin{cases} +\infty & \text{if } x \in H\,, \\ -\infty & \text{if } x \notin H\,. \end{cases} \tag{1.4}$$

G. H. Greco gives in [7] the following representation theorem:

THEOREM 1.1. (Greco) *For every limitoid L on $\overline{\mathbb{R}}^X$, there exists a family $\mathcal{A}$ of subsets of X such that for every $f \in \overline{\mathbb{R}}^X$,*

$$L(f) = \liminf_{\mathcal{A}} f\,. \tag{1.5}$$

Moreover, the largest family fulfilling (1.5) *is*

$$\updownarrow L = \{A:\ L(\chi_A^{\infty}) = +\infty\}\,. \tag{1.6}$$

In particular, it follows from Theorem 1.1 that all the Γ-functionals ([2]) are limitoids. In [6] G. H. Greco gives another characterization of limitoids, namely, in terms of the *monotone integral.* A function $\beta\colon 2^X \to \{0,1\}$ is called increasing if $\beta(\emptyset) = 0$, $\beta(1) = 1$ and $\beta(A) \le \beta(B)$ provided that $A \subset B$. If $f\colon X \to \overline{\mathbb{R}}_+$, then the integral of f with respect to β (G. Vitali [8], G. Choquet [1]) is defined by

$$\int_X f\ \mathrm{d}\beta = \int_0^{\infty} \beta\{f > t\}\ \mathrm{d}t$$

and $\int_X f\ \mathrm{d}\beta = \int_X f_+\ \mathrm{d}\beta - \int_X f_-\ \mathrm{d}\beta$ for $f\colon X \to \overline{\mathbb{R}}$.

[1]Of course, $\inf_A f = \inf_{x\in A} f(x)$ and $\sup_A f = \sup_{x\in A} f(x)$.

THEOREM 1.2. (Greco) *For every limitoid L on $\overline{\mathbb{R}}^X$, the function $\beta(A) = L(\chi_A)$ is increasing and for every $f \in \overline{\mathbb{R}}^X$,*

$$L(f) = \int_X f \,\mathrm{d}\beta .$$

In what follows we use the upper and lower extensions to $\overline{\mathbb{R}}$ of the addition and the subtraction; namely, $+\infty \dot{+} (-\infty) = +\infty$ and $+\infty \underset{\cdot}{+} (-\infty) = -\infty$, while $\dot{-} r = \dot{+}(-r)$ and $\underset{\cdot}{-} r = \underset{\cdot}{+}(-r)$.

If $\mathcal{P} \subset \overline{\mathbb{R}}^X$, then the functionals $\nabla_{\mathcal{P}}$, $\Delta_{\mathcal{P}}$ defined by

$$\begin{aligned} \nabla_{\mathcal{P}} f &= \sup_{p \in \mathcal{P}} \inf (f \dot{-} p) , \\ \Delta_{\mathcal{P}} f &= \inf_{p \in \mathcal{P}} \sup (f \underset{\cdot}{-} p) \end{aligned} \tag{1.7}$$

are niveloids. They play a distinguished role in the study of flexible niveloids. The functionals (1.3) constitute a special case of (1.7) with

$$\mathcal{P} = \{\chi_A^{\infty} : A \in \mathcal{A}\} .$$

Denote by Φ the set of all lattice isomorphisms of $\overline{\mathbb{R}}$ and by Φ_0 the subset of Φ of those φ for which $\varphi(0) = 0$. Of course, $\varphi \in \Phi$ if and only if φ is a strictly increasing continuous function with $\varphi(-\infty) = -\infty$ and $\varphi(+\infty) = +\infty$. A niveloid T is a flexible niveloid if and only if (1.1) holds for every $\varphi \in \Phi_0$. In fact, every $\rho \in \Phi$ is of the form $\rho = \varphi + r$, where $\varphi \in \Phi_0$ and $r \in \mathbb{R}$. Therefore, it is sufficient to use (1.2) to conclude the proof.

2. Normality

A niveloid T is called *normal* if $T(0) = 0$ ([4]).

PROPOSITION 2.1. *A niveloid T on X is normal if and only if, for each $f \in \overline{\mathbb{R}}^X$,*

$$\inf f \le T(f) \le \sup f . \tag{2.1}$$

Proof. Clearly, (2.1) implies normality. Suppose that T is normal and $\inf f > -\infty$. Then $\inf f = T(\inf f + 0) = T(\inf f) + 0 = T(\inf f)$. Thus, by isotonicity, $\inf f \le T(f)$. The second inequality follows from duality. □

A niveloid is *normalizable* if $T(0) \in \mathbb{R}$. If T is normalizable, then, for each bounded function b, one has $T(b) \in \mathbb{R}$. The converse is even more obvious. If b is bounded and $T(b) \in \mathbb{R}$, then the niveloid T_b defined by

$$T_b(f) = T(f + b) \tag{2.2}$$

is normal. This explains the term "normalizable".

Here is an example of biconjugation which is not a normalizable niveloid.

EXAMPLE 2.2. Let $w = \{\lambda|\cdot - y| : y \in \mathbb{R}, \lambda \geq 1\}$. Then $T(f) = f^{**}(0) = \sup\limits_{\lambda|\cdot-y|-r\leq f} \big(\lambda|y| - r\big)$ is a proper niveloid which is not normalizable, because $T(0) = -\infty$.

Every limitoid is normal. Indeed, if $L(0) \neq 0$, then $L(0) \in \{-\infty, +\infty\}$, because a flexible niveloid is normal if and only if it is normalizable. By choosing $\varphi \equiv 0$, we get the following contradiction

$$0 \neq L(0) = L\big(\varphi(0)\big) = \varphi\big(L(0)\big) = 0\,.$$

PROPOSITION 2.3. *Every normalized flexible niveloid is normal.*

Proof. Let T be normalized and flexible but $T(0) \neq 0$. Let $\varphi \in \Phi_0$ be such that $\varphi\big(T(0)\big) \neq T(0)$. Then $T(0) = T\big(\varphi(0)\big) = \varphi\big(T(0)\big)$, a contradiction. □

Denote by $\overline{A}$, the closure in the natural topology of a subset A of $\overline{\mathbb{R}}$. Clearly, $\overline{A}$ is the least complete sublattice of $\overline{\mathbb{R}}$ that includes A.

PROPOSITION 2.4. *If T is a normal flexible niveloid, then for every $f \in \overline{\mathbb{R}}^X$,*

$$T(f) \in \overline{f(X)}\,. \tag{2.3}$$

Proof. Suppose that, on the contrary, $T(f) \notin \overline{f(X)}$. If $T(f) \in \mathbb{R}$, then there exists $\delta > 0$ such that $\big[T(f) - \delta, T(f) + \delta\big] \cap f(X) = \emptyset$. There exists $\varphi \in \Phi$ such that $\varphi(r) = r$ if $r \notin \big[T(f) - \delta, T(f) + \delta\big]$ and for which $\varphi\big(T(f)\big) \neq T(f)$. Then $\varphi f = f$ and thus, by flexibility, $T(f) = T(\varphi f) = \varphi\big(T(f)\big)$, a contradiction.

If $T(f)$ is infinite, say, $T(f) = +\infty$, then, by normality, $\sup f = +\infty$ showing that $+\infty \in \overline{f(X)}$. □

3. Flexible families of functions

A family $\mathcal{B} \subset \overline{\mathbb{R}}^X$ is said to be *flexible* if $\Phi_0(\mathcal{B}) \subset \mathcal{B}$. For a flexible niveloid T, the families $\{T = 0\}$, $\{T = +\infty\}$ and $\{T = -\infty\}$ are flexible. We shall see that the converse also holds (Theorem 3.3).

The family $\mathcal{B}$ is called 0-*admissible* if there exists a niveloid T such that $T(\mathcal{B}) = \{0\}$.

PROPOSITION 3.1. *If a flexible family $\mathcal{B}$ is 0-admissible, then for every $b \in \mathcal{B}$, $\inf |b| = 0$.*

P r o o f. Suppose that, for same $b \in \mathcal{B}$, $\inf|b| = r_0 > 0$. Let $\varphi\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be defined by

$$\varphi(r) = \begin{cases} \frac{r}{2}, & \text{for } r \geq 0 \\ 2r, & \text{for } r < 0. \end{cases} \tag{3.1}$$

Then, since b is proper, $r_0 < +\infty$ and

$$\inf(b - \varphi(b)) = \inf_{\{b>0\}} \frac{1}{2} b \wedge \inf_{\{-\infty<b<0\}} (-b) \geq \frac{r_0}{2} > 0,$$

contrary to 0-admissibility. □

Let $p \in \overline{\mathbb{R}}^X$. We shall consider niveloids of the form (1.7) generated by $\Phi_0(p)$, the least flexible family that contains p. It is clear that $q \in \Phi_0(p)$ if and only if there exists a lattice isomorphism $v\colon \overline{p(X)} \to \overline{q(X)}$ such that $r \geq 0$ entails $v(r) \geq 0$, $s \leq 0$ entails $v(0) \leq 0$, $v(-\infty) = -\infty$ provided that $-\infty \in \overline{p(X)}$ and $v(+\infty) = +\infty$ provided that $+\infty \in \overline{p(X)}$. In particular, if $q \in \Phi_0(p)$, then $\sup_{\{p<+\infty\}} p = +\infty$ if and only if $\sup_{\{p<+\infty\}} q = +\infty$; as well, $\sup_{\{p<0\}} p = 0$ if and only if $\sup_{\{p<0\}} q = 0$. Of course, dual formulæ hold too.

We have seen in Proposition 3.1 that in order that $\Phi_0(p)$ be 0-admissible it is necessary that $\inf |p| = 0$. This condition is also sufficient. Indeed, suppose, for example, that $\sup_{\{p\leq 0\}} p = 0$. Then there exists a sequence $(x_n)_n \subset \{p \leq 0\}$ with $\lim_n p(x_n) = 0$. Consequently, for each $q \in \Phi_0(p)$, $q(x_n) \leq 0$ and $\lim_n q(x_n) = 0$. Therefore, for each $\varepsilon > 0$, there exists n such that $|p(x_n)| \leq \varepsilon$, $|q(x_n)| \leq \varepsilon$ so that $\inf(p \dot{-} q) \leq 0$ and $\inf(q \dot{-} p) \leq 0$.

THEOREM 3.2. *If $\inf |p| = 0$, then $\nabla_{\Phi_0(p)}$ is a flexible niveloid. Moreover, $\nabla_{\Phi_0(p)} f = 0$ if and only if*

$$\sup_{r\in\mathbb{R}} \inf_{\{r\leq p\}} f = +\infty, \tag{3.2}$$

$$\forall_{r\in\mathbb{R}} \inf_{\{r\leq p\}} f > -\infty, \tag{3.3}$$

$$\sup_{\varepsilon>0} \inf_{\{-\varepsilon\leq p\}} f = 0. \tag{3.4}$$

P r o o f. Observe first that every $q \in \Phi_0(p)$ satisfies (3.2), (3.3) and (3.4). Indeed, if $\sup_{r\in\mathbb{R}} \inf_{\{r\leq p\}} q < +\infty$, then there would exist a sequence $(x_n)_n$ such that $\lim_n p(x_n) = +\infty$ and $\sup_n q(x_n) < +\infty$ which is impossible. If there existed $r \in \mathbb{R}$

for which $\inf_{\{r\le p\}} q = -\infty$, then there would exist a sequence $(r_n)_n \subset \{r \le p\}$ for which $\lim_n q(x_n) = -\infty$, which contradicts $q \in \Phi_0(p)$. Finally to see that q fulfills (3.4) note, that in the case where $\inf_{\{0\le p\}} p = 0$, $\inf_{\{0\le p\}} q = 0$ as well. Otherwise, $\sup_{\varepsilon>0} \inf_{\{-\varepsilon\le p<0\}} q = 0$.

Let $\nabla_{\Phi_0(p)} f \ge 0$. Then, for each $\delta > 0$, there exists $q \in \Phi_0(p)$ such that $f \ge q - \delta$. Since q satisfies (3.2), (3.3) and (3.4), f fulfills (3.2) (3.3) and $\sup_{\varepsilon>0} \inf_{\{-\varepsilon\le p\}} f \ge -\delta$ (for each $\delta > 0$), hence

$$\sup_{\varepsilon>0} \inf_{\{-\varepsilon\le p\}} f \ge 0 .$$

If $\nabla_{\Phi_0(p)} f \le 0$, then the inverse inequality also holds, hence we have (3.4). Assume that, on the contrary, there exists $r > 0$ such that $\inf_{\{-\varepsilon\le p\}} f \ge r$ for some $\varepsilon > 0$. As, on the other hand, $\nabla_{\Phi_0(p)} f \ge 0$, there exists $q \in \Phi_0(p)$ such that $f \ge q - \frac{r}{2}$. But there exists $\delta > 0$ such that $\inf_{\{-\delta\le q\}} f \ge r$. We define

$$\varphi(x) = \begin{cases} \frac{t}{2} & \text{for } t \ge 0 , \\ \left(\frac{r}{\delta} + 1\right)t & \text{for } t < 0 . \end{cases} \tag{3.5}$$

Then $\varphi \in \Phi_0$ and $\varphi(q) \in \Phi_0(p)$. Now, on $\{q \ge 0\}$, $f = \frac{f}{2} + \frac{f}{2} \ge \frac{r}{2} + \frac{q}{2} - \frac{r}{4} = \frac{q}{2} + \frac{r}{4} = \varphi(q) + \frac{r}{4}$; on $\{-\delta \le q < 0\}$, $\varphi(q) \le 0$ and $f \ge r$, hence, a fortiori $f \ge \varphi(q) + \frac{r}{4}$; on $\{q < -\delta\}$, $\varphi(q) = \left(\frac{r}{\delta} + 1\right)q$. Therefore, on $\{q < -\delta\}$, one has $f \dot- \varphi(q) \ge q - \frac{r}{2} \dot- r\left(\frac{q}{\delta}\right) \dot- q \ge \frac{r}{2}$, so that $\inf\left(f \dot- \varphi(q)\right) \ge \frac{r}{4} > 0$, contrary to the assumption.

Suppose now that f satisfies (3.2), (3.3) and (3.4). We have seen that, for every $q \in \Phi_0(p)$ (3.4) amounts to $\sup_{\delta>0} \inf_{\{-\delta\le q\}} f = 0$. In particular, for every $\delta > 0$, there exists $x \in \{-\delta \le q\}$ for which $f(x) \le \delta$. It follows that $\inf(f - q) \le f(x) - q(x) \le 2\delta$ which proves that $\nabla_{\Phi_0(p)} f \le 0$.

Let $\varepsilon > 0$. We shall find $q \in \Phi_0(p)$ for which $f \ge q - \varepsilon$. In view of (3.2), for each $k \in \mathbb{N}$, there exists $n(k) \in \mathbb{N}$ such that $\inf_{\{n(k)\le p\}} f \ge k\varepsilon$ and, besides, $n(k+1) > n(k)$. As a consequence of (3.4), $\inf_{\{0\le p\}} f \ge 0$. Define now for $n(k) \le r \le n(k+1)$,

$$\varphi(r) = \frac{\varepsilon}{n(k+1) - n(k)}\big(r - n(k)\big) + k_\varepsilon , \tag{3.6}$$

where $k = 0, 1, 2, \ldots$ and $n(0) = 0$.

By virtue of (3.3), for every $k \in \mathbb{N}$, there exists an $m(k) > 0$ such that $\inf_{\{-k\le p\}} f \ge -m(k)$ and we may assume that $m(k+1) \ge m(k) > \varepsilon$. By (3.4), there exists $1 > \delta > 0$ such that $\inf\limits_{\{-\delta \le p\}} f \ge -\varepsilon$. We complete now the definition of φ for negative r.

$$\varphi(r) = \begin{cases} \frac{m(1)}{\delta} r & \text{if } -\delta \le r \le 0\,, \\ \frac{m(2)-m(1)}{1-\delta}(r+1) - m(2) & \text{if } -1 \le r \le -\delta\,, \\ \big[m(k+1) - m(k)\big](r+k) - m(k+1) & \text{if } -k \le r \le -k+1\,,\ \ k \ge 2\,. \end{cases}$$

So defined φ belongs to $\Phi_0(p)$ and enjoys the property $f \ge \varphi(p) - \varepsilon$ proving that $\nabla_{\Phi_0(p)} = 0$. It is clear that if f satisfies (3.2), (3.3) and (3.4) and $\varphi \in \Phi_0$, then $\varphi(f)$ also satisfies (3.2), (3.3) and (3.4).

If $\nabla_{\Phi_0(p)} g = t \in \mathbb{R}$, then $f = g - t$ satisfies (3.2), (3.3) and (3.4), hence $\sup\limits_{\varepsilon>0} \inf\limits_{\{-\varepsilon \le p\}} g = t$. As a consequence, for every $\varphi \in \Phi_0$, one has

$$\sup_{\varepsilon>0} \inf_{\{-\varepsilon \le p\}} \varphi(g) = \varphi(t)$$

and $\varphi(g)$ satisfies (3.2) and (3.3), proving that $\nabla_{\Phi_0(p)} \varphi(g) = \varphi(t)$. If $\nabla_{\Phi_0(p)} g = +\infty$, then there is a sequence $(f_n)_n$ of functions satisfying $\nabla_{\Phi_0(p)} f_n = 0$ such that $g \ge f_n + n$. As we have shown $\nabla_{\Phi_0(p)} \varphi(f_n + n) = \varphi(n)$. Hence, $\varphi(g) \ge \varphi(f_n + n)$ and $\nabla_{\Phi_0(p)} \varphi(g) = +\infty$. Finally, if $\nabla_{\Phi_0(p)} g = -\infty$, then $\nabla_{\Phi_0(p)} \varphi(g) = -\infty$ for each $\varphi \in \Phi_0$, because otherwise there would be $\nabla_{\Phi_0(p)} \varphi(g) > -\infty$ and $\nabla_{\Phi_0(p)} g = \nabla_{\Phi_0(p)} \varphi^-\big(\varphi(g)\big) = -\infty$ which is impossible, because of the discussion above. □

If T_i is a flexible niveloid on $\overline{\mathbb{R}}^X$ for each $i \in I$, then $\bigvee\limits_{i\in I} T_i$ and $\bigwedge\limits_{i\in I} T_i$ are also flexible niveloids (the bounds being considered in the complete lattice of all the functionals on $\overline{\mathbb{R}}^X$). The above holds also for the empty set of indices I; in other words, the degenerate niveloids ($T \equiv +\infty$ and $T \equiv -\infty$) are flexible. Therefore to every functional T on $\overline{\mathbb{R}}^X$, there corresponds a least flexible niveloid $\mathbb{F}^+ T$ that majorizes T and a greatest flexible niveloid $\mathbb{F}^- T$ that minorizes T. In fact, the upper and lower projections on the class of flexible niveloids in the sublattice of all niveloids are

$$\begin{aligned} (\mathbb{F}^+ T)(f) &= \sup_{\varphi\in\Phi_0} \varphi^- T\big(\varphi(f)\big)\,, \\ (\mathbb{F}^- T)(f) &= \inf_{\varphi\in\Phi_0} \varphi^- T\big(\varphi(f)\big)\,. \end{aligned} \tag{3.7}$$

THEOREM 3.3. *If T is a niveloid for which $\{T = 0\}$ and $\{T = +\infty\}$ are flexible families (or $\{T = 0\}$ and $\{T = -\infty\}$ are flexible), then T is flexible.*

P r o o f . By the Second Representation Theorem [4; Theorem 2.3], $T = \nabla_{\{T=0\}} \vee \nabla_{\{T=+\infty\}}$. As $\{T = 0\}$ is a flexible 0-admissible family, for every $p \in \{T = 0\}$, $\nabla_{\Phi_0(p)}$ is a flexible niveloid by virtue of Theorem 3.2.

Consequently $\nabla_{\{T=0\}} = \bigvee_{T(p)=0} \nabla_{\Phi_0(p)}$ is flexible as the least upper bound of flexible niveloids. To see that $\nabla_{\{T=+\infty\}}$ is flexible, consider f such that $\nabla_{\{T=+\infty\}} f = +\infty$. As $\{T = +\infty\}$ is an $(+\infty)$ family, this amounts to $f \in \{T = +\infty\}$, hence $\varphi(f) \in \{T = +\infty\}$ for each $\varphi \in \Phi_0$ in view of flexibility so that $\nabla_{\{T=+\infty\}} \varphi(f) = +\infty$. Finally, T is flexible being the supremum of two flexible niveloids. □

EXAMPLE 3.4. Let $i: \mathbb{R} \to \mathbb{R}$ be the identity on $\mathbb{R}$. Of course, $\Phi_0(i)$ consists of all the strictly increasing continuous finite functions that vanish at the origin and tend to $+\infty$ and to $-\infty$ with the argument. By Theorem 3.2, $\nabla_{\Phi_0(i)} f = 0$ if and only if $\lim_{x \to +\infty} f(x) = +\infty$, $\inf_{x \geq r} f(x) > -\infty$ for each $r \in \mathbb{R}$ and $\sup_{\varepsilon > 0} \inf_{-\varepsilon \leq x} f(x) = 0$. In particular, f may admit $(+\infty)$ values. Now $f \in \{\nabla_{\Phi_0(i)} = 0\} \cap \{\nabla_{\Phi_0(i)} < +\infty\}$ whenever the preceding conditions hold and when f is bounded from above by a vertical translate of a function from $\Phi_0(i)$. In particular, such an f admits only finite values and must tend to $-\infty$ when the argument does.

Observe that the family $\mathcal{B} = \{\nabla_{\Phi_0(i)} = 0\} \cap \{\nabla_{\Phi_0(i)} < +\infty\}$ is a flexible 0-family. The flexibility follows from the fact that $\{\nabla_{\Phi_0(i)} = 0\}$ is flexible and that, if for some $h \in \Phi_0(i)$ and $r \in \mathbb{R}$, $f \leq h + r$, then, for every $\varphi \in \Phi_0$, $\varphi(f) \leq \varphi(h + r) = [\varphi(h + r) - \varphi(h(0) + r)] + \varphi(h(0) + r)$ and the term under the brackets belongs to $\Phi_0(i)$. Since $\Phi_0(i) \subset \mathcal{B} \subset \{\nabla_{\Phi_0(i)} = 0\}$, $\{\nabla_{\mathcal{B}} = 0\} = \{\nabla_{\Phi_0(i)} = 0\}$. Now if $\nabla_{\mathcal{B}} f < +\infty$, then there exists $b \in \mathcal{D}$ for which $f \sqsubseteq b$. But now $\triangle_{\Phi_0(i)} b < +\infty$ so that $\triangle_{\Phi_0(i)} f < +\infty$ proving that $\{\triangle_{\mathcal{B}} < +\infty\} = \{\triangle_{\Phi_0(i)} < +\infty\}$. Consequently, $\mathcal{B}$ is a 0-family.

We note that $\mathcal{B}$ is strictly greater than $\{\nabla_{\Phi_0(i)} = 0\} \cap \{\triangle_{\Phi_0(i)} = 0\}$. For instance, the function $f(x) = (x + 1)(x - 1)^2$ belongs to the former but not to the latter.

EXAMPLE 3.5. Let $\mathcal{G} = \{\nabla_{\Phi_0(i)} = 0\}$, that is, the family of those functions g for which $\lim_{x \to \infty} g(x) = +\infty$, and $\inf_{x \geq r} g(x) > -\infty$ for each $r \in \mathbb{R}$ and

$$\sup_{\varepsilon > 0} \inf_{-\varepsilon \leq x} g(x) = 0. \tag{3.8}$$

We shall see that $\{\triangle_{\mathcal{G}} = 0\}$ consists of all the functions fulfilling (3.8). Therefore, in view of section 12, $\{\triangle_{\mathcal{G}} = 0\}$ is maximal 0-admissible.

Let $\triangle_{\mathcal{G}} f = 0$. In particular, for every $\delta > 0$, there exists $g \in \mathcal{G}$ such that $f \leq g + \delta$, so that, by (3.8), $\sup_{\varepsilon > 0} \inf_{-\varepsilon \leq x} f(x) \leq 0$. If this inequality were strict, then there would exist $r < 0$ and a sequence $(x_n)_n$ such that $-\frac{1}{n} \leq x_n$ and

$f(x_n) \leq r$. Then either $x_{n_0} \geq 0$ for some $n_0 \in \mathbb{N}$, so that $\psi_{\{x_{n_0}\}} \in \mathcal{G}$ and $\sup(f \dot{-} \psi_{\{x_{n_0}\}}) = f(x_{n_0}) \leq r < 0$, or $x_n < 0$, for each $n \in \mathbb{N}$. In the latter case, $\psi_{\{x_n :\, n \in \mathbb{N}\}} \in \mathcal{G}$ and $\sup(f \dot{-} \psi_{\{x_n :\, n\in\mathbb{N}\}}) \leq r < 0$. Consequently, the 0-admissibility of $\{\Delta_{\mathcal{G}} = 0\}$ is contradicted.

Let f satisfy (3.8). As every $g \in \mathcal{G}$ also satisfies (3.8), we have, by the 0-admissibility of the set of those functions that fulfil (3.8), that $\Delta_{\mathcal{G}} f \geq 0$. On the other hand, for each $n \in \mathbb{N}$, there exists $x_n \geq -\frac{1}{n}$ for which $f(x_n) \leq \frac{1}{n}$. If there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with $x_{n_k} < 0$, then $f \leq \psi_{\{x_{n_k} :\, k \geq k_0\}} + \frac{1}{n_{k_0}}$ and $\psi_{\{x_{n_k} :\, k\geq k_0\}} \in \mathcal{G}$ for each k_0; otherwise, there exists n_0 such that $x_n \geq 0$ for each $n \geq n_0$. Now $\psi_{\{x_n\}} \in \mathcal{G}$ and $f \leq \psi_{\{x_n\}} + \frac{1}{n}$. Anyway, $\Delta_{\mathcal{G}} f \leq 0$.

In view of [4; Theorem 6.4], a maximal 0-admissible family need not contain 0. However,

THEOREM 3.6. *Every maximal 0-admissible flexible family contains 0.*

P r o o f. If $\mathcal{B}$ is 0-admissible and flexible, then

$$\nabla_{\mathcal{B}} 0 \leq 0 \leq \nabla_{\mathcal{B}} 0 .$$

In fact, $\inf(-b) \leq \inf|b| = 0$ in view of Proposition 3.1, hence $\nabla_{\mathcal{B}} 0 = \sup_{b\in\mathcal{B}} \inf(-\mathcal{B}) \leq 0$. As well, $\inf(b) \leq \inf|b| = 0$ so that $\Delta_{\mathcal{B}} 0 = \inf_{b\in\mathcal{B}} \sup(-b) \geq 0$. If $\mathcal{B}$ is maximal 0-admissible then $\nabla_{\mathcal{B}} 0 = \Delta_{\mathcal{B}} 0$, thus $0 \in \mathcal{B}$. □

PROPOSITION 3.7. *Let $\mathcal{B}$ be a 0-admissible flexible family. Then $\nabla_{\mathcal{B}} f = 0$ if and only if there exists a sequence $(b_n)_n \subset \mathcal{B}$ such that*

$$\sup_{r\in\mathbb{R}} \inf_{\{r \leq b_n\}} f = +\infty , \tag{3.9}$$

$$\forall_r \inf_{\{r\leq b_n\}} f > -\infty , \tag{3.10}$$

$$\sup_{\varepsilon>0} \inf_{\{-\varepsilon \leq b_n\}} f \geq -\frac{1}{n} , \tag{3.11}$$

$$\sup_{b\in\mathcal{B}} \sup_{\varepsilon > 0} \inf_{\{-\varepsilon\leq b\}} f \leq 0 . \tag{3.12}$$

P r o o f. We have that $\mathcal{B} = \bigcup_{b\in B_0} \Phi_0(b)$, for some $\mathcal{B}_0$ (in particular for $\mathcal{B}_0 = \mathcal{B}$). Therefore, $\nabla_{\mathcal{B}} f \geq 0$ whenever there exists a sequence $(b_n)_n \subset \mathcal{B}_0$ such that $\nabla_{\Phi_0(b_n)} f \geq \frac{1}{n}$ which, in view of Theorem 3.2, amounts to (3.9), (3.10). Now, $\nabla_{\mathcal{B}} f \leq 0$ is equivalent to (3.11) in view of Theorem 3.2. □

A niveloid T is *inf-convolutive* (resp. *sup-convolutive*) if for every function f,

$$T(f) = \nabla_{\{T=0\}} f \qquad (\text{resp. } T(f) = \Delta_{\{T=0\}} f) .$$

Here is an example of a flexible niveloid which is neither inf- nor sup- convolutive.

EXAMPLE 3.8. Consider the following niveloid

$$T(f) = \begin{cases} \inf f: & \sup f < +\infty\,, \\ -\infty: & \inf f = -\infty\,,\ \sup f = +\infty\,, \\ +\infty: & \inf f > -\infty\,,\ \sup f = +\infty\,. \end{cases}$$

We observe that $T(f) = 0$ whenever $\inf f = 0$ and $\sup f < +\infty$, $T(f) = +\infty$ if and only if $\sup f = +\infty$, and $\inf f > -\infty$ and $T(f) = -\infty$ if and only if $\inf f = -\infty$. In view of Theorem 3.3, it is sufficient to define $\{T = 0\}$ and $\{T = +\infty\}$ or $(T = 0)$ and $\{T = -\infty\}$.

$$\nabla_{\{T=0\}} f = \sup_{\{b:\, \inf b = 0,\ \sup b < +\infty\}} \inf(f - b)\,,$$

$$\Delta_{\{T=0\}} f = \inf_{\{b:\, \inf b = 0,\ \sup b < +\infty\}} \sup(f - b)\,.$$

Now $\nabla_{\{T=0\}} f = +\infty$ whenever there exists a sequence of functions $(b_n)_n \subset \{T = 0\}$ such that $f \geq b_n + n$. Therefore, in particular $f \geq n$ so that $f \equiv +\infty$. We have seen that there exist functions f for which $T(f) = +\infty$ but $\nabla_{\{T=0\}} f < +\infty$: T is not inf-convolutive.

Now if $\Delta_{\{T=0\}} f = -\infty$, then in particular, $\sup f < f_0$. Consequently, there exists a function f for which $T(f) = -\infty$ but $\Delta_{\{T=0\}} f > -\infty$: T is not sup-convolutive, but $\{T = 0\}$ is sup-convolutive.

THEOREM 3.9. *If $\mathcal{G}$ is an inf-convolutive flexible family, then $\Delta_{\mathcal{G}}$ is a limitoid.*

P r o o f. Let $\mathcal{A} = \bigcup_{b \in \mathcal{B}} \{\{-\varepsilon \leq b\} : \varepsilon > 0\}$. If $\Delta_{\mathcal{G}} f \leq 0$, then, for every $n \in \mathbb{N}$, there exists $g_n \in \mathcal{G}$ such that $f \leq g_n + \frac{1}{n}$ and since g_n fulfills (3.11), $\liminf_{\mathcal{A}} f \leq 0$. □

4. Limitoids and carriers

We shall denote by $\mathcal{L}(X)$ the set of all limitoids together with the functionals $-\infty$ and $+\infty$. The latter will be referred to as *degenerate* limitoids. We shall see later that $\mathcal{L}(X)$ is closed under arbitrary least upper bounds and greatest lower bounds.

The notions of (lower and upper) carriers will enable us to characterize the limitoids, recovering Theorem 1.1 via a simple alternative proof.

Denote by ψ_H the *indicator function* of H:

$$\psi_H(x) = \begin{cases} 0 & \text{if } x \in H\,, \\ +\infty & \text{if } x \notin H\,, \end{cases} \tag{4.1}$$

and by ψ_H^∞ the *indicator function* of H valued in the lattice $\overline{\mathbb{R}}$:

$$\psi_H^\infty(x) = \begin{cases} -\infty & \text{if } x \in H, \\ +\infty & \text{if } x \notin H. \end{cases} \tag{4.2}$$

Of course, $\psi_H^\infty = -\chi_H^\infty$.

Let T be an arbitrary niveloid. The *lower carrier* of T is defined by

$$\downarrow T = \{A : T(-\psi_A) \geq 0\} \tag{4.3}$$

and its *upper carrier* by

$$\uparrow T = (\downarrow T^*)^\# . \tag{4.4}$$

Here $T^*(f) = -T(-f)$ is the conjugate functional of T. Of course, $\uparrow T = \{H : T(\psi_H) \leq 0\}^\#$.

PROPOSITION 4.1. *If T is a flexible normal niveloid, then $A \in \downarrow T$ if and only if $T(-\psi_A) > -\infty$ and $A \in \uparrow T$ if and only if $T(\psi_A) < \infty$.*

Proof. Let $T(-\psi_A) > -\infty$. By normality, $T(-\psi_A) \leq 0$ and, in view of Proposition 2.4, $T(-\psi_A) = 0$. The second statement follows from duality. □

The *pseudocarrier* is defined by

$$\updownarrow T = \{A : T(\chi_A^\infty) = +\infty\} . \tag{4.5}$$

Recall that pseudocarriers have already appeared in the Greco representation theorem (Theorem 1.1). In view of [4; Corollary 1.3], if T is a niveloid, then

$$A \in \downarrow T \iff \forall_f \ \inf_A f \leq T(f) , \tag{4.6}$$

$$H \in (\uparrow T)^\# \iff \forall_f \ T(f) \leq \sup_H f . \tag{4.7}$$

Now it is clear that $\emptyset \in \downarrow T$ if and only if $T \equiv +\infty$; also $\downarrow T \neq \emptyset$ if and only if $\inf \leq T$. Dually, $\emptyset \in \uparrow T$ if and only if $T \equiv -\infty$; as well, $\uparrow T \neq \emptyset$ if and only if $T \leq \sup$. Consequently, T is normal if and only if $\downarrow T$ and $\uparrow T$ are nondegenerate semifilters.

We observe that $\mathbb{F}_+(T^*) = (\mathbb{F}_- T)^*$. Moreover,

$$\updownarrow(\mathbb{F}_+ T) = \updownarrow T = \updownarrow(\mathbb{F}_- T) . \tag{4.8}$$

Formulæ (4.6) and (4.7) imply that, for every niveloid T, and for each f,

$$\liminf_{\downarrow T} f \leq T(f) \leq \limsup_{(\uparrow T)^\#} f$$

and because $\liminf_{\mathcal{A}} = \limsup_{\mathcal{A}^\#}$,

$$\liminf_{\downarrow T} \leq T \leq \liminf_{\uparrow T} . \tag{4.9}$$

Moreover, these are the greatest lower limit that minorizes T, and the least lower limit that majorizes T. It follows from the definitions that, for every niveloid T,

$$\downarrow T \subset \updownarrow T \subset \uparrow T . \tag{4.10}$$

THEOREM 4.2. *A niveloid T is a limitoid (possibly degenerate) if and only if $\downarrow T = \uparrow T$ (and thus equal to $\updownarrow T$).*

P r o o f. It is straightforward that $\liminf_{\mathcal{A}}$ (and $\limsup_{\mathcal{A}}$) is a limitoid (proper, whenever $\mathcal{A}$ is a base of a proper semifilter). Consequently if $\uparrow T = \downarrow T$, then by (4.9) T is a limitoid.

Suppose now that T is a limitoid. Let $A \in \updownarrow T$. By setting $\varphi(r) = r \wedge 0$, we get from (1.1),

$$T(-\psi_A) = T(\chi_A^\infty \wedge 0) = T(\chi_A^\infty) \wedge 0 = 0,$$

thus $A \in \downarrow T$. If now $H \in (\updownarrow T)^\#$, then by setting $\varphi(r) = r \vee 0$, we derive from (1.1),

$$T(\psi_H) = T(\psi_H^\infty \vee 0) = T(\psi_H^\infty) \vee 0 = 0$$

so that $H \in (\uparrow T)^\#$. We have proved that $\uparrow T \subset \downarrow T$. □

In view of (4.9), (4.10) and (4.5), Theorem 4.2 implies Theorem 1.1.

EXAMPLE 4.3. Let C be the functional of convexification at a given point of $\mathbb{R}$, for instance, $C(f) = (\operatorname{co} f)(0)$. We have that

$$\uparrow C \neq \updownarrow C = \downarrow C.$$

In fact, $A \in \downarrow C$ whenever $\operatorname{co}(-\psi_A)(0) = 0$ and this happens only when $A = \mathbb{R}$. On the other hand, $\operatorname{co}(\chi_A^\infty)(0) = \chi_{(\operatorname{co} A^c)^c}^\infty(0)$. Now, $A \in (\uparrow C)^\#$ if and only if $0 = \operatorname{co}(\psi_A)(0) = \psi_{\operatorname{co} A}(0)$ and $A \in (\updownarrow C)^\#$ if and only if $\operatorname{co}(\psi_A^\infty)(0) = -0$, equivalently $-\infty = \psi_{\operatorname{co} A}^\infty(0)$.

It follows from our considerations that the lower projection of C on the class of limitoids is equal to inf. On the other hand, $H \in \uparrow C$ if and only if $0 \notin \operatorname{co} H^c$. Therefore the upper projection of C on the class of limitoids is equal to the quasi-convexification at 0. Summarizing,

$$(\mathbb{L}^- \operatorname{co})(f) = \inf f, \qquad (\mathbb{L}^+ \operatorname{co})f = \sup_{\{H:\, 0 \notin \operatorname{co} H^c\}} \inf_H f.$$

EXAMPLE 4.4. Let $D(f) = f^{**}(0)$, the classical biconjugation on the line $\mathbb{R}$. As $(-\psi_A)^{**} \not\equiv -\infty$ if only if $A = \mathbb{R}$ and alike $(\chi_A^\infty)^{**} \not\equiv -\infty$ if and only if $A = \mathbb{R}$, $\downarrow D = \updownarrow D = \{\mathbb{R}\}$. Now $(\psi_A^\infty)^{**} \equiv -\infty$ if and only if $A \neq \emptyset$, while $\psi_A^{**} = \psi_{\operatorname{cl}\operatorname{co} A}$ so that

$$\downarrow D = \updownarrow D \neq \uparrow D.$$

The lower projection on limitoids is also inf and the upper the quasi-convex lower semicontinuous hull.

EXAMPLE 4.5. (A flexible normal inf-convolutive niveloid which is not a limitoid) Let $\mathcal{B} = \{\nabla_{\Phi_0(i)} = 0\}$ from Example 3.4. We shall set $\mathcal{F} = \left\{\nabla_{\mathcal{B}\cup\{0\}} = 0\right\}$. We have that $f \in \mathcal{F}$ if and only if

$$\nabla_{\mathcal{B}} f \vee \inf f = 0\,,$$

equivalently if $\nabla_{\mathcal{B}} f \leq 0$ and $\inf f \leq 0$ and either $\nabla_{\mathcal{B}} f \geq 0$ or $\inf \geq 0$. This is tantamount to: $f \in \mathcal{B}$ (because $\nabla_{\mathcal{B}} f \leq 0$ implies $\inf f \leq 0$) or $\nabla_{\mathcal{B}} f \leq 0$ and $\inf f = 0$.

Another characterization given in [7] describes limitoids as those isotone functionals L for which

$$L(\chi_A^\infty) \in \{-\infty, +\infty\}\,, \tag{4.11}$$

$$\forall_{r\in\overline{\mathbb{R}}} \ \forall_{f} \quad L(f \wedge r) = L(f) \wedge r\,, \tag{4.12}$$

$$\forall_{r\in\overline{\mathbb{R}}} \ \forall_{f} \quad L(f \vee r) = L(f) \vee r\,. \tag{4.13}$$

Of course, (4.11) is satisfied by every niveloid L. We note that a weaker version of (4.12) and (4.13) was used in the proof of Theorem 4.2.

As it was shown in [7], $\mathcal{L}(X)$ is isomorphic to the complete lattice of semifilters on X (degenerate ones comprised). Denote by $\mathbb{L}^+T$ and $\mathbb{L}^-T$ respectively the upper and the lower projection on the class of limitoids. We note that

$$\downarrow(\mathbb{L}^-T) = \downarrow T\,, \qquad \uparrow(\mathbb{L}^+T) = \uparrow T\,;$$

the latter may be obtained from the former by duality.

THEOREM 4.6. *The lower (resp. upper) projection of a niveloid T on $\mathcal{L}(X)$ is given by*

$$(\mathbb{L}^-T)(f) = \liminf_{\downarrow T} f \qquad (\textit{resp. } (\mathbb{L}^+T)(f) = \liminf_{\uparrow T} f\,)\,. \tag{4.14}$$

Moreover, for every family $\{T_i\}_{i\in I}$ of niveloids, one has

$$\mathbb{L}^-\Big(\bigwedge_{i\in I} T_i\Big) = \bigwedge_{i\in I} \mathbb{L}^-(T_i) \qquad \textit{and} \qquad \mathbb{L}^+\Big(\bigvee_{i\in I} T_i\Big) = \bigvee_{i\in I} \mathbb{L}^+ T_i\,. \tag{4.15}$$

This property corresponds to the following rule concerning carriers

$$\downarrow\Big(\bigwedge_{i\in I} T_i\Big) = \bigcap_{i\in I} \downarrow T_i \qquad \text{and} \qquad \uparrow\Big(\bigvee_{i\in I} T_i\Big) = \bigcup_{i\in I} \uparrow T_i\,. \tag{4.16}$$

On the other hand, if I is finite then

$$\downarrow\Big(\bigvee_{i\in I} T_i\Big) = \bigcup_{i\in I} \downarrow T_i \qquad \text{and} \qquad \uparrow\Big(\bigwedge_{i\in I} T_i\Big) = \bigcap_{i\in I} \uparrow T_i\,. \tag{4.17}$$

In particular, as a consequence of Proposition 4.1 and Theorem 4.2 we have:

PROPOSITION 4.7. *If* $\{T_i\}_{i\in I}$ *are flexible normal niveloids, then* (4.16) *and* (4.17) *hold.*

Therefore, if $\mathcal{B}$ is a flexible 0-admissible family containing 0, then

$$\begin{aligned}(\mathbb{L}^{-}\nabla_{\mathcal{B}})f &= \sup_{\{b\in\mathcal{B}:\ \sup b<+\infty\}}\ \inf_{\{-\infty<b\}} f\,,\\ (\mathbb{L}^{+}\nabla_{\mathcal{B}})f &= \inf_{\{H:\ \underset{B\in\mathcal{B}}{\exists}\ \sup\limits_{H} b\geq 0\}}\ \sup_{H} f\,.\end{aligned}\tag{4.18}$$

THEOREM 4.8. *Every limitoid is a convolutive niveloid.*

P r o o f. Let L be a limitoid. By Theorem 4.2 and by (4.9), $L(f)=0$ if and only if $\sup\limits_{A\in\updownarrow L}\inf\limits_{A} f=0$ and thus $L=\nabla_{\{L=0\}}$. Dually, $L(f)=0$ if and only if $\inf\limits_{H\in(\updownarrow L)^{\#}}\sup\limits_{H} f=0$, hence $L=\Delta_{\{L=0\}}$. □

5. Flexible niveloids on bounded functions

Let T be a normal niveloid. A set A belongs to the *lower quasi-carrier* $\twoheaddownarrow T$ of T if there exist $-\infty<r\leq s<+\infty$ such that the function

$$b(x)=\begin{cases} s & \text{if } x\in A\,,\\ r & \text{if } x\notin A\end{cases}\tag{5.1}$$

satisfies $T(b)=s$. On the other hand, $H\in\twoheaduparrow T$ if $T(b)=r$ with $A=H^c$. We have that

$$\downarrow T\subset\twoheaddownarrow T \qquad\text{and}\qquad \twoheaduparrow T\subset\uparrow T\,.\tag{5.2}$$

Indeed, if $A\in\downarrow T$, then $0=T(-\psi_A)\leq T(\chi_A-1)\leq T(\chi_A)-1\leq\sup\chi_A-1\leq 0$, by normality. Similarly, $-\psi_A\leq b-s$. On the other hand, if $H\in\uparrow T$ and b is given by (5.1), then $\psi_H\geq b-s$ so that $0\geq T(\psi_H)\geq T(b)-s\geq\inf b-s=0$.

PROPOSITION 5.1. *If* T *is a normal niveloid and* f *is a bounded function, then*

$$\liminf_{\twoheaddownarrow T} f\leq T(f)\leq\limsup_{\twoheaduparrow T} f\,.\tag{5.3}$$

P r o o f. Let $A\in\twoheaddownarrow T$ and let $s=\inf\limits_{A} f$ and $r=\inf f$. Then $b=(-\psi_A+s)+r\vee\inf f$ is of the type (5.1) and fulfills $b\leq f$. Thus $\inf\limits_{A} f=T(b)\leq T(f)$ so that the first inequality holds. The second is similar. □

PROPOSITION 5.2. *If T is normal and flexible, then*

$$A \in \downarrow T \iff T(\chi_A) = 1 \iff A^c \notin \uparrow T .$$

P r o o f. Let $A \in \downarrow T$ and b be as in (5.1). Then $T(b-r) = s-r$ and $T(\chi_A) = T\left(\frac{1}{s-r}(b-r)\right) = 1$. If $A \notin \downarrow T$, then $T(\chi_A) = 0$, because $T(f) \in \overline{f(X)}$. By definition $A^c \in \uparrow T$. □

COROLLARY 5.3. *If T is normal and flexible, then $\downarrow T = (\uparrow T)^{\#}$ and hence for each bounded function f,*

$$\liminf_{\downarrow T} f = T(f) .$$

This approach that enables us to exhibit the limitoid coinciding with T on the bounded functions is akin to the method of carriers in the characterization of limitoids.

6. Application to Moreau-Yosida approximation

Approximations of the Moreau-Yosida [DOLECKI, S: *Fuzzy Γ-operators and convolutive approximations.* In: Nonsmooth Optimization and Related Topics (F. H. Clarke, V. F. Demyanov, F. Giannessi, eds.), Plenum, New York 1989, 109–131] type hinge on the coincidence of certain niveloids on the sets of functions of the form

$$\sqsupseteq h = \left\{ f : \exists_{r \in \mathbb{R}} \; f \geq h + r \right\} .$$

We say that Q *approximates* $\mathcal{P}$ *(from below) over* h whenever, for each $p \in \mathcal{P}$, $\varepsilon > 0$ and $s \in \mathbb{R}$ there exists $q \in Q$ such that

$$p \vee (h+s) \geq q - \varepsilon . \tag{6.1}$$

THEOREM 6.1. *If Q approximates $\mathcal{P}$ from below over h, then for each $f \sqsupseteq h$, $\nabla_{\mathcal{P}} f \leq \nabla_Q f$.*

P r o o f. Let $r < \nabla_{\mathcal{P}} f$. Then there exists $p \in \mathcal{P}$ such that $f \geq p+r$. On the other hand, by assumption, there exists $t \in \mathbb{R}$ for which $f \geq h+t$. Therefore, $f \geq \left[p \vee (h+(t-r))\right] + r$ and, since Q approximates $\mathcal{P}$ over h, for every $\varepsilon > 0$, there exists $q \in Q$ such that $f \geq q - \varepsilon + r$, hence $\nabla_Q f \geq r - \varepsilon$ for each $\varepsilon > 0$. □

REFERENCES

[1] CHOQUET, G.: *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1953), 131–295.

[2] DE GIORGI, E.: *Generalized limits in calculus of variations.* In: Topics in Functional Analysis, Quaderni Scuola Normale Superiore, Pisa, 1980-1981, pp. 117–148.

[3] DOLECKI, S.—GUILLERME, J.—LIGNOLA, M. B.—MALIVERT, C.: *Méthode des* Γ*-inégalités*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 659–662.

[4] DOLECKI, S.—GRECO, G. H.: *Niveloids*, Topol. Methods Nonlinear Anal. **5** (1995), 1–22.

[5] DOLECKI, S.—GRECO, G. H.: *Niveloides: fonctionnelles isotones commutant avec l'addition de constantes finies*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), 113–116.

[6] GRECO, G. H.: *Integrale monotono*, Rend. Sem. Mat. Univ. Padova **57** (1977), 149–166.

[7] GRECO, G. H.: *Limitoidi e reticoli completi*, Ann. Univ. Ferrara Sez. VII (N.S.) **29** (1983), 153–164.

[8] VITALI, G.: *Sulla definizione d'integrale delle funzioni di una variabile reale*, Ann. Mat. Pura Appl. (4) **2** (1925), 111–121.

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