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A GENERALIZATION OF THE INDIVIDUAL ERGODIC THEOREM

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The Individual Ergodic Theorem deals with Cesaro averages of a stationary sequence of integrable random variables on a given probability space (Ω, α, P). Such a sequence can always be canonically represented in the form $\{X \circ T^n\}$, where X is an integrable random variable on a suitable probability space, T is a measure preserving transformation.

Let a sequence $\{X_n\}$ of integrable random variables satisfying certain properties be given. We ask: what can we say about Cesaro averages of the sequence $\{X_n \circ T^n\}$?

Throughout this paper let (Ω, α, P) be a given probability space, let X, Y, X_1, $X_2, \ldots, X_n, \ldots, Y_1, Y_2, \ldots, Y_n, \ldots$ be integrable random variables on it. Let T: $\Omega \to \Omega$ be a measure preserving transformation, $K > 0$ a real constant.

Lemma 1 [3, Theorem 1.8]. *If $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of real numbers, then the following statements are equivalent:*

(1) $\frac{1}{n}\sum_{i=0}^{n-1}|a_i| \to 0.$

(2) $\exists J \subset Z^+$, *J of density zero, i.e.,*

$$\left(\frac{\text{card}\,\{J \cap \{0, 1, \ldots, n-1\}\}}{n}\right) \to 0,$$

such that $\lim_n a_n = 0$ *provided* $n \notin J$.

Theorem 1. *Let $X_n \to 0$ a.e., $|X_n| \leq K$, for $n = 1, 2, \ldots$ Then*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \circ T^i \to 0 \quad \text{a.e.}$$

Proof. By the Egoroff Theorem there exists a decreasing sequence of measurable sets $\{B_m\}_{m=1}^{\infty}$ such that $P(B_m) \searrow 0$ and on $B'_m = \Omega - B_m$ we have $X_n \to 0$ uniformly, $m = 1, 2, \ldots$ It is easy to see that there exists an increasing sequence of positive integers $\{N_i\}_{i=1}^{\infty}$ such that for all $n \geq N_i$, $\omega \in B'_i$ we have $|X_n(\omega)| < \left(\frac{1}{2}\right)^i$.

Let $N_0=1$, $B_0=\Omega$. Denote $C_n=B_j$ for $N_j \leq n < N_{j+1}$. We have $C_n\searrow$, $P(C_n)\searrow 0$, $|X_n| < \left(\frac{1}{2}\right)^j$ on C_n' for $N_j \leq n < N_{j+1}$.

Put $J(\omega)=\{i \in Z^+,\ T^i(\omega) \in C_i\}$. By Lemma 1 it follows that for every $J(\omega)$ of density zero there holds

$$\frac{1}{n}\sum_{i=1}^{n} X_i \circ T^i(\omega) \to 0,$$

as $\lim_n X_n \circ T^n(\omega)=0$, provided $n \notin J(\omega)$ holds. Thus we will prove that for a.e. $\omega \in \Omega$, $J(\omega)$ is of density zero. As

$$\text{card}\,\{J(\omega) \cap \{0, 1, \ldots, n-1\}\} = \sum_{i=0}^{n-1} \chi_{C_i} \circ T^i(\omega),$$

it is in fact the same as proving that

$$\frac{1}{n}\sum_{i=0}^{n-1} \chi_{C_i} \circ T^i \to 0 \quad \text{a.e.}$$

It is easy to see that $\chi_{C_n}\searrow 0$ a.e. We have

$$0 \leq \liminf_n \frac{1}{n}\sum_{i=0}^{n-1}\chi_{C_i}\circ T^i \leq \limsup_n \frac{1}{n}\sum_{i=0}^{n-1}\chi_{C_i}\circ T^i \leq \limsup_n \frac{1}{n}\sum_{i=0}^{n-1}\chi_{C_k}\circ T^i$$
$$\text{for } k=1,2,\ldots$$

Denote $f_k=\limsup_n \frac{i}{n}\sum_{i=0}^{n-1}\chi_{C_k}\circ T^i$, $k=1, 2, \ldots$ From the Individual Ergodic Theorem we have $f_k=\lim_n \frac{1}{n}\sum_{i=0}^{n-1}\chi_{C_k}\circ T^i$ a.e., $\int_\Omega f_k dP = \int_\Omega \chi_{C_k} dP = P(C_k)$. It is easy to see that $f_k\searrow$ and $\int_\Omega f_k dP\searrow 0$ so that $f_k\searrow 0$ a.e. This fact proves Theorem 1.

Remark. In Theorem 1 we need some type of stationarity. This is in fact implied by T being measure preserving. Next Example 1 shows that the stationarity is essential.

Example 1. There exist two sequences $\{X_n\}$ and $\{Y_n\}$ such that $X_n \to 0$ a.e., $|X_n| \leq K$, X_n and Y_n have the same probability distributions for $n=1, 2, \ldots$, but $\lim_n \frac{1}{n}\sum_{i=1}^{n} Y_i$ does not exist for any $\omega \in \Omega$.

Let $\Omega=\langle 0, 1)$, let α be the Borel σ-field on Ω, let P be the Lebesque measure on Ω. Let $X_1=X_2=\chi_{\langle 0, 1/2)}$, $X_3=\ldots=X_{10}=\chi_{\langle 0, 1/4)}$, $\ldots$, $Y_1=X_1$, $Y_2=\chi_{\langle 1/2, 1)}$, $Y_3=X_3$, $Y_4=\chi_{\langle 1/4, 1/2)}$, $Y_5=Y_6=\chi_{\langle 1/2, 3/4)}$, $Y_7=\ldots=Y_{10}=\chi_{\langle 3/4, 1)}$, $\ldots$ so that $X_n \to 0$ a.e., $|X_n| \leq 1$ for $n=1, 2, \ldots$ but

$$\liminf_n \frac{1}{n}\sum_{i=1}^n Y_i = 0 \neq \frac{1}{2} = \limsup_n \frac{1}{n}\sum_{i=1}^n Y_i.$$

Theorem 2. *Let $X_n \to 0$ a.e., $0 \leq X_n \leq Y$ for $n = 1, 2, \ldots$ Then*

$$\frac{1}{n}\sum_{i=1}^n X_i \circ T^i \to 0 \quad \text{a.e.}$$

Proof. Denote $Y^{(k)} = \min(Y, k)$ for $k = 1, 2, \ldots$ similarly denote $X_n^{(k)}$ for $n = 1, 2, \ldots, k = 1, 2, \ldots$ It is easy to see that

$$0 \leq X_n - X_n^{(k)} \leq Y - Y^{(k)}$$

so that

$$0 \leq \frac{1}{n}\sum_{i=1}^n X_i \circ T^i - \frac{1}{n}\sum_{i=1}^n X_i^{(k)} \circ T^i \leq \frac{1}{n}\sum_{i=1}^n (Y - Y^{(k)}) \circ T^i$$

$$\text{for} \quad n = 1, 2, \ldots, \quad k = 1, 2, \ldots$$

Since by Theorem 1 we have for $k = 1, 2, \ldots$

$$\frac{1}{n}\sum_{i=1}^n X_i^{(k)} \circ T^i \to 0 \quad \text{a.e.,}$$

it follows that

$$0 \leq \limsup_n \frac{1}{n}\sum_{i=1}^n X_i \circ T^i \leq \limsup_n \frac{1}{n}\sum_{i=1}^n (Y - Y^{(k)}) \circ T^i \quad \text{a.e.}$$

Denote

$$g_k = \limsup_n \frac{1}{n}\sum_{i=1}^n (Y - Y^{(k)}) \circ T^i, \quad k = 1, 2, \ldots$$

From the Individual Ergodic Theorem it follows that

$$\int_\Omega g_k dP = \int_\Omega (Y - Y^{(k)}) dP.$$

It easy to see that $g_k \searrow$. From the Beppo—Levi Theorem it follows

$$\int_\Omega (Y - Y^{(k)}) dP \searrow 0$$

so that $g_k \searrow 0$ a.e. This proves Theorem 2.

Corollary 1. *Let $X_n \to 0$ a.e., $|X_n| \leq Y$ for $n = 1, 2, \ldots$ Then*

$$\frac{1}{n}\sum_{i=1}^n X_i \circ T^i \to 0 \quad \text{a.e.}$$

Proof. Applying Theorem 2 to both $\{X_n^+\}$ and $\{X_n^-\}$ we get what was claimed.

Corollary 2. *Let $X_n \to X$ a.e., $|X_n| \leqslant Y$ for $n = 1, 2, \ldots$ Then*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \circ T^i \to \lim_n \frac{1}{n}\sum_{i=1}^{n} X \circ T^i \quad \text{a.e.}$$

Proof. Applying Corollary 1 to the sequence $\{X_n - X\}$ and using the Individual Ergodic Theorem we get what was claimed.

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ОБОЩЕНИЕ ИНДИВИДУАЛЬНОЙ ЭРГОДИЧЕСКОЙ ТЕОРЕМЫ

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Резюме

Индивидуальная эргодическая теорема касается сходимостьи средних Чезара последовательности $\{X \circ T^n\}$, где X интегрируемая случайная величина и T меру сохраняющая трансформация. Мы занимаемся сходимостью средних Чезара последовательности $\{X_n \circ T^n\}$, где X_n последовательность интегрируемых случайных величин.

Теорема. *Пусть $X_n \to X$ п.в., $|X_n| \leq Y$, $n = 1, 2, \ldots$ где Y интегрируемая случайная величина. Тогда*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \circ T^i \to \lim_n \frac{1}{n}\sum_{i=1}^{n} X \circ T^i \quad \text{п.в.}$$