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ON FUNCTIONAL INTEGRABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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In papers [2] and [3] there are investigated asymptotic properties of functionally integrable solutions of some forms of differential equations. In this paper some results of the above papers are generalized for a more general differential equation.

Consider the differential equation of n-th order of the form

$$(G_{n-1}x(t))' + f(t, x(g(t))) = h(t), \tag{1}$$

where $n \geqq 2$ and G_{n-1} is a differential operator defined by

$$G_{n-1}x(t) = a_{n-1}(t)(a_{n-2}(t)(\ldots(a_1(t)x'(t))'\ldots)')',$$

where the functions

$$a_i\colon [t_0, \infty) \to R, \qquad h\colon [t_0, \infty) \to R,$$
$$f\colon [t_0, \infty) \times R \to R, \qquad g\colon [t_0, \infty) \to R_+$$

are continuous functions and

$$a_i(t) > 0, \quad i = 1, \ldots, n-1, \quad g'(t) \geqq 0, \quad \lim_{t\to\infty} g(t) = \infty.$$

We introduce the notation:

$$G_0x(t) = x(t), \quad G_ix(t) = a_i(t)(G_{i-1}x(t))'. \quad 1 \leqq i \leqq n-1, \tag{2}$$

$$J_{m-1}(t, s) = \int_s^t \frac{ds_2}{a_2(s_2)} \int_s^{s_2} \frac{ds_3}{a_3(s_3)} \cdots \int_s^{s_{m-1}} \frac{ds_m}{a_m(s_m)} \tag{3}$$

from $m = 2, 3, \ldots, n-1$

$$J_0(t, s) = 1.$$

We restrict our attention to non-trivial solutions of (1) which exist on the interval $[t_0, \infty)$.

A solution $x(t)$ is called oscillatory, if it has an infinite sequence of zeros tending to infinity. Otherwise, we call $x(t)$ a non-oscillatory solution.

Definition. *A solution $x(t)$ of (1) belongs to the class $L(m, W(\cdot))$, if*

$$0<\int_{t_0}^{\infty} s^m W(|x(s)|)\, ds<\infty, \quad m\text{-real number,}$$

where W: $R\to R$, $W(|u|)\geqq 0$ is a given continuous non-decreasing function.

If in the above definition we put $m=0$ and $W(u)=u^p$, $p>0$, then we obtain the well-known class $L(0, |\cdot|^p)=L_p(0, \infty)$, i.e.

$$0<\int_0^{\infty}|u(s)|^p\, ds<\infty.$$

Lemma 1. *Let $a_i(t)>0$ on $[t_0, \infty)$. Then there exist positive constants α_i such that*

$$J_i(t, s)\leqq \alpha_i J_{i+1}(t, s) \quad \text{for} \quad i=1, 2, \dots, n-3, \quad t>s\geqq t_0.$$

Proof. See in paper [1], Lemma 1.

Lemma 2. *Let $J_{m-1}(t, t_0)\leqq K\cdot t^{m-1}$, $K>0$, $a_1(t)>\varrho>0$. If the function $u(t)$ satisfies the following conditions*

$$|G_m u(t)|\leqq M \quad \text{for} \quad t\geqq t_0 \quad \text{and} \quad m\geqq 1, \tag{4}$$

$$u\in L(m-1, W(\cdot)), \tag{5}$$

then $\lim_{t\to\infty} u(t)=0$.

Proof. From definition and condition (5) it follows that $\liminf_{t\to\infty}|u(t)|=0$. We shall prove that also $\limsup_{t\to\infty}|u(t)|=0$. Suppose that $\limsup_{t\to\infty}|u(t)|>\varepsilon>0$. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ such that $t_n\to\infty$, $\alpha_n\to\infty$ for $n\to\infty$, $\alpha_n<t_n$, $|u(t_n)|>\varepsilon$, $|u(\alpha_n)|=\frac{\varepsilon}{2}$ and for every $t\in(\alpha_n, t_n)|u(t)|>\frac{\varepsilon}{2}$. Let $\alpha_1\geqq t_0>1$. In each interval (α_n, t_n) there exists a number ξ_n such that

$$u'(\xi_n)=\frac{u(t_n)-u(\alpha_n)}{t_n-\alpha_n},$$

hence

$$\frac{\varepsilon}{2(t_n-\alpha_n)}\leqq|u'(\xi_n)| \tag{6}$$

From relation (4), using Lemma 1 and the assumption about $J_{m-1}(t, t_0)$ we obtain

$$|a_1(t)u'(t)|\leqq K_1 t^{m-1},$$

whence

$$|u'(t)|\leqq K_2 t^{m-1}.$$

With regard to the stated, from relation (6) we obtain

$$\frac{\varepsilon}{2K_2} \leqq t_n^m - \alpha_n^m. \tag{7}$$

Since $|u(t)| > \frac{\varepsilon}{2}$ for every $t \in (\alpha_n, t_n)$, then from (5) and (7) we get

$$\infty > \int_0^\infty s^{m-1} W(|u(s)|) \, ds \geqq \sum_{n=1}^\infty \int_{\alpha_n}^{t_n} s^{m-1} W(|u(s)|) \, ds \geqq$$

$$\geqq W\left(\frac{\varepsilon}{2}\right) \sum_{n=1}^\infty \frac{t_n^m - \alpha_n^m}{m} = \infty.$$

This contradiction proves that the case of $\limsup_{t\to\infty} |u(t)| > \varepsilon > 0$ is impossible. Therefore $\lim_{t\to\infty} u(t) = 0$.

Remark 1. If $a_1(t) = a_2(t) = \ldots = a_{n-1}(t) = 1$, then we obtain Lemma 1 from [2].

Let us start with the assumptions:

$$|f(t, u)| \geqq a(t)|W(u)|, \quad u \cdot W(u) > 0 \quad \text{for} \quad u \neq 0 \tag{A_1}$$
$$|f(t, u)| \leqq b(t) W(|u|) \tag{A_2}$$
$$|f(t, u)| \leqq b(t)[W(|u|)]^{1/p}, \quad p > 1 \tag{A_3}$$

where the functions a, b: $R_+ \to R_+$, W: $R \to R$, $W(|u|) \geqq 0$ are continuous and $W(u)$ is a non-decreasing function.

Theorem 1. *Let* (A_2) *hold and moreover assume that*

$$J_{n-2}(t, t_0) \leqq Kt^{n-2}, \quad a_1(t) > \varrho > 0 \tag{8}$$

$$h \in L(0, |\cdot|), \quad b(t) \leqq Mg'(t) g^{n-2}(t), \tag{9}$$

for sufficiently large t, *where* M *is a positive constant.*

Then every solution $x \in L(n-2, W(\cdot))$ *of (1) satisfies*

$$\lim_{t\to\infty} x(t) = 0$$

Proof. Let $x(t)$ be a solution of (1) belonging to the class $L(n-2, W(\cdot))$. In view of Lemma 2 we shall prove that the function $G_{n-1}x(t)$ is bounded. Integrating (1) from t_0 to t we obtain

$$G_{n-1}x(t) = G_{n-1}x(t_0) + \int_{t_0}^t h(s) \, ds - \int_{t_0}^t f(s, x(g(s))) \, ds, \tag{10}$$

whence, taking into account the assumptions of the theorem,

$$|G_{n-1}x(t)| \leqq |G_{n-1}x(t_0)| + \int_{t_0}^t |h(s)| \, ds + \int_{t_0}^t b(s) W(|x(g(s))|) \, ds \leqq$$

$$\leqq |G_{n-1}x(t_0)| + \int_{t_0}^{t} |h(s)|\, ds + M\int_{t_0}^{t} g'(s)g^{n-2}(s)W(|x(g(s))|)\, ds \leqq$$

$$\leqq A + M\int_{g(t_0)}^{g(t)} s^{n-2}W(|x(s)|)\, ds \leqq B.$$

Applying Lemma 2 we have $\lim_{t\to\infty} x(t)=0$.

Remark 2. If $n=2$, we obtain theorem 4 from [3]. If $a_i(t)=1$ for $i=1, \ldots, n-1$, we obtain theorem 2 from [2] for $m=0$.

Theorem 2. *Let* (A_3), (8), (9) *hold and let moreover*

$$\frac{b^p(t)}{g'(t)g^{n-2}(t)} \in L(0, |\cdot|^{1/p-1}), \quad g'(t)>0, \quad g(t)>0 \quad \text{for} \quad t \geqq t_0.$$

Then every solution $x \in L(n-2, W(\cdot))$ *of* (1) *satisfies* $\lim_{t\to\infty} x(t)=0$.

Proof. Let $x(t)$ be an arbitrary solution of (1) belonging to the class $L(n-2, W(\cdot))$. From (A_3) and Holder's inequality we have

$$\int_{t_0}^{t} |f(s, x(g(s)))|\, ds \leqq \int_{t_0}^{t} b(s)[W(|x(g(s))|)]^{1/p}\, ds =$$

$$= \int_{t_0}^{t} \frac{b(s)}{[g'(s)g^{n-2}(s)]^{1/p}} \cdot [g'(s)g^{n-2}(s)W(|x(g(s))|)]^{1/p}\, ds \leqq$$

$$\leqq \left(\int_{t_0}^{t} \left[\frac{b^p(s)}{g'(s)g^{n-2}(s)}\right]^{1/p-1} ds\right)^{p-1/p} \cdot \left(\int_{t_0}^{t} g'(s)g^{n-2}(s)W(|x(g(s))|)\, ds\right)^{1/p} =$$

$$= \left(\int_{t_0}^{t} \left[\frac{b^p(s)}{g'(s)g^{n-2}(s)}\right]^{1/p-1} ds\right)^{p-1/p} \cdot \left(\int_{g(t_0)}^{g(t)} s^{n-2}W(|x(s)|)\, ds\right)^{1/p}.$$

From (10) we have

$$|G_{n-1}x(t)| \leqq |G_{n-1}x(t_0)| + \int_{t_0}^{t} |h(s)|\, ds + \int_{t_0}^{t} |f(s, x(g(s)))|\, ds,$$

Whence, utilizing the last inequality and the assumption of the theorem, we have

$$|G_{n-1}x(t)| \leqq B_1,$$

where B_1 is constant. In view of Lemma 2 it follows that $\lim_{t\to\infty} x(t)=0$.

Remark 3. If $n=2$, we obtain theorem 3 from [3]. If $a_i(t)=1$ for $i=1, \ldots, n-1$, we obtain theorem 3 from [2].

Theorem 3. *Let* (A_3) *be and moreover assume that*

$$\frac{b^p(t)}{g'(t)g^m(t)} \in L(0, |\cdot|^{1/p-1}) \quad \text{for} \quad m \in R, \quad g'(t)>0, \quad g(t)>0$$

for $t \geqq t_0$. *If* $\left|\int_{t_0}^{\infty} h(s)\,\mathrm{d}s\right| = \infty$, *then every oscillatory solution* $x(t)$ *of* (1) *does not belong to the class* $L(m, W(\cdot))$.

Proof. Let $x(t)$ be the arbitrary oscillatory solution of (1). Then the function $G_{n-1}x(t)$ is also oscillatory. Let $\{t_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of consecutive zeros of $G_{n-1}x(t)$. Integrating (1) between t_n and t_{n+1} we have

$$\int_{t_n}^{t_{n+1}} h(s)\,\mathrm{d}s = \int_{t_n}^{t_{n+1}} f(s, x(g(s)))\,\mathrm{d}s,$$

whence

$$\int_{t_1}^{\infty} h(s)\,\mathrm{d}s = \int_{t_1}^{\infty} f(s, x(g(s)))\,\mathrm{d}s.$$

Taking into account the assumptions of the theorem and Holder's inequality, we have

$$\left|\int_{t_1}^{\infty} h(s)\,\mathrm{d}s\right| \leqq \left(\int_{t_1}^{\infty} \left[\frac{b^p(s)}{g'(s)g^m(s)}\right]^{1/p-1} \mathrm{d}s\right)^{p-1/p} \cdot \left(\int_{g(t_1)}^{\infty} s^m W(|x(s)|)\,\mathrm{d}s\right)^{1/p},$$

whence it follows that $x \notin L(m, W(\cdot))$.

Remark 4. If $m = n-2$, $a_i(t) = 1$ for $i = 1, \ldots, n-1$, we obtain theorem 4 from [2].

Theorem 4. *Let* (A_1), (9) *hold and* $uf(t, u) > 0$ *for* $u \neq 0$. *Let moreover*

$$a(t) \geqq \gamma g'(t) g^m(t) > 0 \tag{12}$$

for sufficiently large t, *where* $\gamma > 0$, $m \in R$ *and*

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{a_i(s)} = \infty \quad \text{for} \quad i = 1, \ldots, n-1. \tag{13}$$

Then every non-oscillatory solution $x(t)$ *of* (1) *belongs to the class* $L(m, W(\cdot))$.

Proof. Let $x(t)$ be a non-oscillatory solution of (1) and let $x(t) > 0$ eventually (the case when $x(t) < 0$ can be treated similarly). Let $T \geqq t_0$ be sufficiently large so that $x(g(t)) > 0$ for $t \geqq T$. Integrating equation (1) from T to $t \geqq T$ we obtain

$$G_{n-1}x(t) - G_{n-1}x(T) + \int_T^t f(s, x(g(s)))\,\mathrm{d}s = \int_T^t h(s)\,\mathrm{d}s \tag{14}$$

Since (9) holds, the right-hand side of (14) is finite as $t \to \infty$. If

$$\int_T^{\infty} f(s, x(g(s)))\,\mathrm{d}s = \infty, \quad \text{then} \quad G_{n-1}x(t) \to -\infty \quad \text{for} \quad t \to \infty$$

and because (13) holds we easily obtain the contradiction with the assumption $x(t) > 0$.

Let

$$\int_T^\infty f(s, x(g(s)))\,ds < \infty.$$

Using (A_1), (12) successively we obtain

$$\infty > \int_T^\infty f(s, x(g(s)))\,ds \geqq \int_T^\infty a(s)W(x(g(s)))\,ds \geqq$$

$$\geqq \gamma \int_{g(T)}^\infty s^m W(x(s))\,ds, \quad \text{hence} \quad x \in L(m, W(\cdot)).$$

This completes the proof of the theorem.

Remark 5. If $a_i(t)=1$ for $i=1, \ldots, n-1$, we obtain theorem 1 from [2].

Theorem 5. *Let (A_2), (9) be satisfied and moreover assume that*

$$\int_{t_0}^\infty \frac{ds}{a_i(s)} < \infty \quad \text{for} \quad i=2, \ldots, n-1 \tag{15}$$

$$b(t) \leqq Mg'(t)g^m(t),$$

$$\int_{t_0}^\infty s^m a_1^2(s)\,ds = \infty \quad \text{for} \quad m \in R. \tag{16}$$

Then for arbitrary two solutions $x_1(t)$ and $x_2(t)$ of (1) such that

$$|\sqrt{W(|x_1(t)|)}x_2'(t) - x_1'(t)\sqrt{W(|x_2(t)|)}| \geqq k > 0 \tag{17}$$

for $t \geqq t_0 > 0$ we have

$$x_1 \in L(m, W(\cdot)) \Rightarrow x_2 \notin L(m, W(\cdot)).$$

Proof. Let there exist two solutions $x_1(t)$ and $x_2(t)$ of equation (1) for which (17) is true and $x_i \in L(m, W(\cdot))$ for $i=1, 2$.

From (1) we obtain

$$|G_{n-1}x_i(t)| \leqq |G_{n-1}x_i(t_0)| + M\int_{g(t_0)}^{g(t)} s^m W(|x_i(s)|)\,ds + \int_{t_0}^t |h(s)|\,ds,$$

thereby

$$|G_{n-1}x_i(t)| \leqq B \quad \text{for every} \quad t \geqq t_0.$$

In view of assumption (15), we have

$$|a_1(t)x_i'(t)| \leqq B_1 \quad \text{for} \quad t \geqq t_0 > 0,$$

where B_1 is a positive constant.

We estimate now

$$I(t)=\int_{t_0}^{t}[\sqrt{W(|x_1(s)|)}\,x_2'(s)-x_1'(s)\sqrt{W(|x_2(s)|)}]^2\cdot a_1^2(s)s^m\,ds\leqq$$

$$\leqq B_1^2\int_{t_0}^{t}s^mW(|x_1(s)|)\,ds+B_1^2\int_{t_0}^{t}s^mW(|x_2(s)|)\,ds+$$

$$+2B_1^2\int_{t_0}^{t}\sqrt{W(|x_1(s)|)}\,s^{m/2}\cdot\sqrt{W(|x_2(s)|)}\,s^{m/2}\,ds$$

whence, utilizing Holder's inequality ($p=2$, $q=2$) we have

$$I(t)\leqq B_1^2\left[\sqrt{\int_{t_0}^{t}s^mW(|x_1(s)|)\,ds}+\sqrt{\int_{t_0}^{t}s^mW(|x_2(s)|)\,ds}\right]^2.$$

Since $x_i\in L(m, W(\cdot))$, we have

$$I(t)\leqq C \tag{18}$$

where C is a real constant.

Since (17) holds, we have

$$I(t)\geqq k^2\int_{t_0}^{t}s^ma_1^2(s)\,ds,$$

whence in view of (16), $\lim_{t\to\infty}I(t)=\infty$, which contradicts (18). This completes the proof of the theorem.

Remark 6. If $n=2$, $h(t)=0$, we obtain theorem 1 from [3].

Theorem 6. *Let the assumptions of theorem 5 be satisfied. Then any solution* $x(t)$ *of* (1) *such that*

$$W(|x(t)|)x'^2(t)>k, \tag{19}$$

does not belong to the class $L(m, W(\cdot))$.

Proof. Let $x(t)$ be a solution of (1) for which (19) holds and let $x(t)$ belong to the class $L(m, W(\cdot))$. Likewise as in theorem 5 we prove that there exists a constant B_1 such that

$$|a_1(t)x'(t)|\leqq B_1.$$

We estimate

$$I_1(t)=\int_{t_0}^{t}W(|x(s)|)x'^2(s)s^ma_1^2(s)\,ds\leqq B_1^2\int_{t_0}^{t}s^mW(|x(s)|)\,ds\leqq C \tag{20}$$

but in view of (19)

$$I_1(t)\geqq k\cdot\int_{t_0}^{t}s^ma_1^2(s)\,ds.$$

In view of (16) $\lim_{t\to\infty} I_1(t)=\infty$, which contradicts (20). The proof of Theorem 6 is complete.

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О ФУНКЦИОНАЛЬНОЙ ИНТЕГРИРУЕМОСТИ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАНИЕМ

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Резюме

В статье даются достаточные условия, при которых неколеблющиеся или колеблющиеся решения нелинейного дифференциального уравнения n-того порядка с запаздыванием (1) принадлежат или не принадлежат классу $L(m, W(\cdot))$, даются также условия стремления к нулю при $t\to\infty$ решений (1), принадлежающих классу $L(n-2, W(\cdot))$.