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ON A HORIZONTAL STRUKTURE ON DIFFERENTIABLE MANIFOLDS

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Let M be a differentiable manifold, $\dim M = n$. Let p or β be the fibre projection of the tangent bundle $T(M)$ over M or of the bundle $J^1T(M)$ over $T(M)$, respectively. A linear connection on M can be introduced as a vector bundle morphism ${}^1\Gamma$: $T(M) \to J^1T(M)$ over M such that

$$\beta\Gamma = \mathrm{id}_{T(M)}\,.$$

In this paper we find some properties of the structure on M determined by a global cross-section Γ: $T(M) \to J^1T(M)$ which is not a linear connection. Standard terminology and notations of the theory of jets are used throughout the paper, see, e.g. [2]. Our considerations are in the category C^∞.

1. Let $E(M, p)$ be a fibred manifold over M. A tangent subspace $\Gamma_u \subset T_u(E)$ will be called horizontal if $T_u(E) = \Gamma_u \oplus T_u(E_x)$, $pu = x$. Let J^1E be the first prolongation of E, i.e. the space of all 1-jets of all local cross-sections of E. Every $h \in J^1E$ determines a horizontal tangent subspace $\Gamma_u \subset T_u(E)$, $\beta h = u$, and conversely. Then every distribution of horizontal tangent subspaces Γ_u on E can be identified with a global cross-section Γ: $E \to J^1E$.

Let X be a projectable vector field on E. Let ${}^1X$ be the first prolongation of X on J^1E. The distribution Γ determines a submanifold $\Gamma(E) \subset J^1E$. Hence $\Gamma_* X$ is a vector field on $\Gamma(E)$.

Definition 1. *A projectable field X on E will be said to be conjugate with Γ if*

$$\Gamma_*(X)_{(h)} = {}^1X_{(h)} \tag{1}$$

for every $h \in \Gamma(E)$.

It will be useful to write down the coordinate form of (1). Let (x^i, y^α) or $(x^i, y^\alpha, y_i^\alpha)$ be local coordinates on E or J^1E, respectively. Let $X = a^i(x, y)\partial x_i + b^\alpha(x, y)\partial y_\alpha$. Then

$${}^1X = a^i\, \partial x_i + b^\alpha\, \partial y_\alpha + \left(\frac{\partial b^\alpha}{\partial x^i} + \frac{\partial b^\alpha}{\partial y^\beta}\, y_i^\beta - y_j^\alpha\, \frac{\partial a^j}{\partial x^i}\right) \partial y_i^\alpha, \quad \text{see [5].} \tag{2}$$

Let Γ: $E \to J^1E$ be determined by $y_i^\alpha = a_i^\alpha(x, y)$. Then

$$\Gamma_*(X) = a^i\, \partial x_i + b^\alpha\, \partial y_\alpha + \left(\frac{\partial a_i^\alpha}{\partial x^j}\, a^j + \frac{\partial a_i^\alpha}{\partial y^\beta}\, b^\beta\right) \partial y_i^\alpha$$

and (1) yields

$$\frac{\partial b^\alpha}{\partial x^i} + \frac{\partial b^\alpha}{\partial y^\beta}\, a_i^\beta - a_j^\alpha \frac{\partial a^j}{\partial x^i} - \frac{\partial a_i^\alpha}{\partial x^j}\, a^j - \frac{\partial a_i^\alpha}{\partial y^\beta}\, b^\beta = 0\,. \tag{3}$$

Further, if $Y \in T_x(M)$ and $u \in E$, $pu = x$, then there is a unique vector $\bar{Y} \in \Gamma_u$ such that $P_* \bar{Y} = Y$. The field $\bar{Y}$ obtained in this way will be called the Γ-lift of the vector field Y on M. Locally, $Y = a^i(x)\, \partial x_i$, Γ: $y_i^\alpha = a_i^\alpha(x, y)$, then $\bar{Y} = a^i(x)\, \partial x_i + a_i^\alpha(x, y) \times a^i(x)\, \partial y_\alpha$.

2. Let E be a vector bundle. Denote by V the Liouville field on E determined by the 1-parametric group of all homothetic transformations on E. Locally,

$$V = y^\alpha\, \partial y_\alpha\,. \tag{4}$$

Let $k \geqq 0$ be an integer. A vector field X on E will be called *k-homogeneous*, or *V-vertical*, if $[V, X] = kX$ or $[V, X]$ is vertical on E, respektively. In coordinates, the field

$$X = a^i(x, y)\, \partial x_i + b^\alpha(x, y)\, \partial y_\alpha$$

is k-homogeneous or V-vertical and only if

$$\frac{\partial a^i}{\partial y^\alpha}\, y^\alpha = k a^i\,, \qquad \frac{\partial b^\alpha}{\partial y^\beta}\, y^\beta = (k+1) b^\alpha\,, \tag{5}$$

i.e. if the functions $a^i(x, y)$ ($b^\alpha(x, y)$) are homogeneous of degree k (of degree $k+1$) with respect to the variables y^γ or if

$$\frac{\partial a^i}{\partial y^\alpha}\, y^\alpha = 0\,, \tag{6}$$

i.e., if the functions $a^i(x, y)$ are homogeneous of degree 0 with respect to the variables y^γ.

Example 1. Every projectable field on E is V-vertical.

Example 2. The projectable field X on E will be said to be linear if the local transformations of its local 1-parametric group are linear fibre isomorphisms. Locally, the field $X = a^i(x)\, \partial x_i + b^\alpha(x, y)\, \partial y_\alpha$ is linear if and only if

$$b^\alpha(x, y) = b_\beta^\alpha(x) y^\beta\,. \tag{7}$$

Now (7) and (5) yield

Lemma 1. *Every linear field X on E is* 0*-homogeneous, i.e.*

$$[V, X]=0 .$$

Example 3. The Γ-lift $\bar{Y}$ of the field $Y=a^i(x)\, \partial x_i$ is 0-homogeneous if and only if

$$\left(\frac{\partial a_i^\alpha}{\partial y^\beta}\, y^\beta - a_i^\alpha\right)a^i=0 . \tag{8}$$

In the case of the vector bundle J^1E ever X we can easy deduce from (2) the following assertion.

Lemma 2. *If V is the Liouville field on E, then its prolongation ^1V is the Liouville field on J^1E.*

Locally, we recall that $^1V=y^\alpha\, \partial y_\alpha + y_i^\alpha\, \partial y_i^\alpha$.

Lemma 3. *If the projectable field X on E is* 0*-homogeneous, then ^1X is* 0*-homogeneous on J^1E.*

Proof. The operator $X \to {}^1X$ is R-linear and $^1[X, Y]=[^1X, {}^1Y]$.

Remark 1. Let $T_k^1(M)$ be the set of all k^1-velocities on M. Let Y be a vector field on M. We recall that the prolongation of Y on $T_k^1(M)$ is given by

$$^1\xi(h)=j_0^1({}^t\Phi\, .\, h), \qquad h \in T_k^1(M) , \tag{9}$$

where $'\varphi$ is the transformation of the local 1-parametric group of the fiels Y and the dot denotes the jet composition. In particular, if $k=1$, then (9) yields locally

$$^1\xi=a^i(x)\, \partial x_i + \frac{\partial a^i}{\partial x_j}\, y^j\, \partial y_i . \tag{10}$$

It means that $^1\xi$ is a linear field on $T(M)$ and thus is 0-homogeneous.

3. Further we will study a special case. Denote by p the fibre projection of the tangent bundle $T(M)$ over M. We recall some structure properties of the space $T(T, (M))$. Denote by π the fibre projection of the tangent bundle $T(T(M))$. Let $\mathcal{T}$ signify the vector bundle strucure of $T(T(M))$ over $T(M)$ with the fibre projection P_*. Further ^2T be the fibre structure of $T(T(M))$ over M, with the fibre projection $\beta=p\pi$. It is known that ^2T can be identified with the fibre structure of all the non-holonomic 1^2-velocities on M. Then the subset $\mathcal{S}$ of all the semi-holonomic 1^2-velocities on M is the set of such elements $z \in T(T(M))$ that

$$\pi(z)=P_*(z) .$$

Let $u \in T(M)$, $p(u)=x$. Denote by $\mathcal{S}_u$ the subset $s_x \cap T_u(T(M))$, where $\mathcal{S}_x$ is the fibre of $\mathcal{S}$ over x.

Proposition 1. *Let $u \in T(M)$, $p(u)=x$. Then $\mathcal{S}_u$ is a class of the factor-space $T_u(T(M)) \mid T_u(T_x(M))$. Further $s_u = T_u(T_x(M))$ if and only if $u=O \in T_x(M)$.*

Proof. Let $z_1, z_2 \in \mathscr{S}_u$. Then $P_*(z_1) = u = P_*(z_2)$. Therefore $O = P_*(z_1) - P_*(z_2) = P_*(z_1 - z_2)$, i.e. $z_1 - z_2 \in T_u(t_x(M))$. Conversely let $z_1 \in S_u$ and z_2 be an element of the class determined by z_1. Then $z_1 - z_2 \in T_u(T_x(M))$, i.e. $P_*(z_2)$. Since $u = \pi_{(}z_1) = P_*(z_1)$ and $P(z_2) = u$, therefore $z_2 \in S_u$. The equivalence $S_u = T_u(T_x(M)) \Leftrightarrow u = O$ is obvious.

Corollary. *Let Γ_u be a horizontal subspace of $T_u(T(M))$. Then the set $_u \cap S_u$ has just one element, which will be called the h-element of Γ_u.*

Further we recall (see [3]) that a differential equation of second order on M is a vector field X on $T(M)$ satisfying

$$\pi(X) = P_*(X) .$$

Locally, $X = a^i(x, y)\, \partial x_i + b^i(x, y)\, \partial y_i$ is a differential equation of the second order on M if and only if $a^i(x, y) = y^i$. Thus if X_1 and X_2 are two differential equations of the second order on M, then the field $X_1 - X_2$ is vertical on $T(M)$.

Proposition 2. *Let X_1 and X_2 be two differential equations of the second order on M. Then $[X_1, X_2]$ is a differential equation of the second order on M if and only if $X_1 - X_2 = V$ is the Liouville field on $T(M)$.*

Proof. If $X_1 = y^i\, \partial x_i + a^i(x, y)\, \partial y_i$, $X_2 = y^i\, \partial x_i + b^i(x, y)\, \partial y_i$, then $[X_1, X_2] = (a^i - b^i)\, \partial x_i + B^i(x, y)\, \partial y_i$. This proves our assertion.

Definition 2. *Let Γ: $T(M) \to J^1 T(M)$ be a global cross-section. The pair (M, Γ) will be called an H-structure on M.*

Locally, Γ is given by the equations

$$x^i = x^i, \qquad y^i = y^i ,$$

$$y^i_j = a^i_j(k^k, y^k) \equiv a^i_j(x, y)$$

and the H-structure (M, Γ) is a linear connection on M if and only if the functions $a^i_j(x, y)$ are linear with respect to the variables y^k i.e. if and only if

$$y^i_j = \Gamma^i_{jk}(x) y^k .$$

Proposition 3. *Let Y be a vector field on M. Then $[\bar{Y}, {}^1\xi]$, where $\bar{y}$ is the Γ-lift of Y and ${}^1\xi$ is the first prolongation of Y on $T(M)$, is vertical.*

The proof is obvious.

One can also easily prove the following assertion.

Proposition 4. *Let $(M, \Gamma_1$ and (M, Γ_2) be two H-structures. Let f_1 and f_2 be two functions on $T(M)$. Then $(M, f_1\Gamma_1 + f_2\Gamma_2)$ is an H-structure if and only if $f_1 + f_2 = 1$.*

Denote by $D(M)$ the module of all vector fields on M over the ring $F(M)$ of all real functions on M.

Definition 3. *A mapping D_X: $D(M) \to D(M)$ determined by $X \in D(M)$ will be called a d-mapping if*

$$D_X(Y+Z) = D_X(Y) + D_X(Z) ,$$

$$D_X(fY) = X(f)Y + fD_X(Y) .$$

Let X, $Y \in D(M)$. Considering Y as a cross-section Y: $M \to T(M)$, denote by $Y_*(X)$ the field on the submanifold $Y(M) \subset T(M)$ determined by the differential of Y. We recall that $\bar{X}$ or ${}^1X$ denotes the Γ-lift of X or the first prolongation of X on $T(M)$. Every $X \in D(M)$ determines the following transformations of $D(M)$:

$$\begin{aligned} \Omega_X &: y \to i[Y_*(X) - \bar{X}|_{Y(M)}] , \\ \omega_X &: Y \mapsto i[X_*(Y) - \bar{Y}|_{X(M)}] , \\ \Theta_X &: Y \mapsto i[{}^1Y - \bar{Y})|_{X(M)}] , \\ \delta_X &: Y \mapsto i[({}^1X - \bar{X})|_{Y(M)}] , \end{aligned}$$

where i indicates the canonical identification of $T_u(T_x(M))$ with $T_x(M)$. Obviously $\omega_X(Y) = \Omega_Y(X)$, $\delta_X(Y) = \Theta_Y(X)$. In coordinates, if $X = a^i(x)\, \partial x_i$, $Y = b^i(x)\, \partial x_i$, then $Y_*(X) = a^i\, \partial x_i + \frac{\partial b^i}{\partial x^j} a^j\, \partial y_i$ and

(11)
$$\begin{aligned} \Omega_X(Y) &= \left[\frac{\partial b^i}{\partial x^j} a^j - a^i_j(x, b)a^j\right] \partial x_i \\ \omega_X(Y) &= \left[\frac{\partial a^i}{\partial x^j} b^j - a^i_j(x, a)b^j\right] \partial x_i \\ \Theta_X(Y) &= \left[\frac{\partial b^i}{\partial x^j} a^j - a^i_j(x, a)b^j\right] \partial x_i , \\ \delta_X(Y) &= \left[\frac{\partial a^i}{\partial x^j} b^j - a^i_j(x, b)a^j\right] \partial x_i . \end{aligned}$$

Proposition 5. *Let X, $Y \in D(M)$. Then*

$$\Omega_X(Y) - \delta_X(Y) = [X, Y] = \Theta_X(Y) - \omega_X(Y) .$$

Proof. Using (11), Proposition 5 can be easily proved by direct evaluation.

Assertion. *The transformation Θ_X is a d-mapping for every $X \in D(M)$.*

Proof is obvious from (11).

Every H-structure (M, Γ) determines on $T(M)$ the vector field of h-elements (see Corollary of the proposition 1), which will be called the h-field of (M, Γ) and denoted by H. In coordinates,

(12)
$$H = y^i\, \partial x_i + a^i_j(x, y) y^j\, \partial x_i .$$

The h-field of (M, Γ) is a differential equation of the second order. We recall that a differential equation Z of the second order on M is a spray on M if and only if Z is 1-homogeneous on TM.

Definition 4. *The H-structure* (M, Γ) *will be said to be homogeneous if the mapping* $\delta_X: D(M) \to D(M)$ *is homogeneous for every* X, *i.e. if* $\delta_X(fY) = f\delta_X(Y)$ *for any* $Y, X \in D(M)$ *and* $f \in F(M)$.

Proposition 6. *Let the H-structure* (M, Γ) *be homogeneous. Then the h-field of* (M, Γ) *is a spray on* M.

Proof. The equations (11_4) yield that (M, Γ) is homogeneous if and only if the functions $a_j^i(x, y)$ are homogeneous of the first degree with respect to the variables y^k. Then comparing (12) with (5) we complete our proof.

In the first part of this paper we have introduced a field conjugate with Γ.

Proposition 7. *The Liouville field* V *on* $T(M)$ *is conjugate with* Γ: $T(M) \to J^1T(M)$ *if and only if the H-structure* (M, Γ) *is homogeneous.*

Proof. In this case the conditions (3) give

$$\frac{\partial a_j^i(x^k, y^k)}{\partial y^k} y^k = a_j^i(x, y) .$$

It is a sufficient and necessary condition for $a_j^i(x, y)$ to be homogeneous of the first degree with respect to the variables y^k. It proves our assertion.

The relations (8) immediatly yield

Proposition 8. *The H-structure* (M, Γ) *is homogeneous if and only if the* Γ*-lift* $\bar{Y}$ *of the field* Y *is O-homogeneous for any* $Y \in D(M)$.

Let (M, Γ_1) and (M, Γ_2) be two H-structures. Let H_s be the H-field of (M, Γ_s), $s = 1, 2$. The $H-$structure (M, Γ_2) will be said to be conjugate with (M, Γ_1) if $[H_1, H_2]$ is a differential equation of the second order.

It is known (see [4]) that the space J^1E is an affine bundle over E associated with the vector bundle $T^*(M) \otimes T(E|X)$ over E, where $T(E|X)$ denotes the vector bundle of all vertical tangent vectors on E. Two cross-sections $\Gamma_1: E \to J^1E$, $\Gamma_2: E \to J^1E$ determine a cross-section $E \to T^*(M) \otimes T(E|X)$ which will be denoted by $(\Gamma_1 - \Gamma_2)$. In the case of $E = T(M)$ using the canonical identification i: $T_u(T_x(M)) = T_x(M)$ we get $(\Gamma_1 - \Gamma_2): T(M) \to \mathrm{Hom}\,(T(M), T(M))$ over M.

Proposition 9. *The H-structure* (M, Γ_2) *is conjugate with* (M, Γ_1) *if*

$$(\Gamma_1 - \Gamma_2)\,(u) = \mathrm{id}\,|_{T_x(M)},\; p(u) = x ,$$

for any $u \in T(M)$.

Proof. In coordinates, $(\Gamma_1 - \Gamma_2)\,(u)$ is determined by the matric

$${}^1a_j^i(x, y) - {}^2a_j^i(x, y) = c_j^i(x, y) .$$

If $c^i_j = \delta^i_j$, then $c^i_j y^j = y^i$. It means that $^1H - ^2H = V$. Now Proposition 2 completes our assertion.

Remark 2. In the case of the linear connection Γ the operator Ω is the covariant derivative determined by Γ. Localy, $a^i_j(x, y) = \Gamma^i_{jk}(x) y^k$ and then the connection $\bar{\Gamma}$ transposed to Γ is given by $\bar{a}^i_j(x, y) = \Gamma^i_{jk} y^j$. Now it is easy to see that the covariant derivative of $\bar{\Gamma}$ is determined by the operator Θ_X. Thus we get the interesting relation of the first prolongation ^1Y to the connection $\mathcal{C}$

. Quite similarly to the case of the linear connection in the more general case of the H-structure we can introduce a parallelism over a curve on M. Let $\gamma(t)$ or $\{Y(t)\}$ be a curve in M or in $T(M)$ over γ, $(pY(t) = \gamma(t))$, respectively. The set $\{Y(t)\}$ will be said to be H-parallel over $\gamma(t)$ if $Y_*(t) \in \Gamma_{Y(t)}$. Locally, the set $\{Y(t)\} = \{x^i = x^i(t),\ y^i(t) = b^i(t)\}$ is H-parallel over $\gamma: x^i = x^i(t)$ if and only if

$$\frac{\mathrm{d}b^i}{\mathrm{d}t} = a^i_j[x^k(t), b^k(t)] \frac{\mathrm{d}x^j}{\mathrm{d}t} .$$

Let $Y = b^i\, \partial x_i$ or $X = a^i\, \partial x_i$ be such a vector field on M that $Y|_\gamma = \{Y(t)\}$ or $X|_\gamma = \{\gamma_*(t)\}$, respectively. Then

$$\Omega_{\gamma(Y(t))} = \Omega_X(Y)|_\gamma = \left\{\frac{\mathrm{d}b^i}{\mathrm{d}t} - a^i_j\left[x(t), b(t)\right] \frac{\mathrm{d}x^j}{\mathrm{d}t}\right\} \partial x_i$$

does not depend on the choice of X and Y. Hence the set $\{Y(t)\}$ is parallel over γ if and only if

$$Y(t) = 0 .$$

Analogously, the set $Y(t)$ will be called H-transp-parallel over a curve γ if

$$\Theta_\gamma(Y(t)) = 0 .$$

Locally, the set $\{Y(t)\}$ is H-transp-parallel over γ if and only if

$$\frac{\mathrm{d}b^i}{\mathrm{d}t} = a^i_j\left(x(t), \frac{\mathrm{d}x^k}{\mathrm{d}t}\right) b^j(t) . \tag{13}$$

Let $c, d \in \gamma(t)$. We deduce from (13) that the H-transp-parallelism over γ determines an isomorphism $T_c(M) \to T_d(M)$.

We can also introduce geodesic of the H-structure (M, Γ). The curve γ can be said to be a geodesic of the H-structure (M, Γ) if the set $\{\gamma_*(t)\}$ is H-parallel over γ. In coordinates, γ is a H-geodesic if and only if

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = a^i_j\left[x(t), \frac{\mathrm{d}x}{\mathrm{d}t}\right] \frac{\mathrm{d}x^j}{\mathrm{d}t} . \tag{14}$$

Comparing (14) with (13) we get: The curve γ is a H-geodesic if and only if the set

$\{\gamma_*(t)\}$ is H-transp-parallel over γ. In coordinates, the curve $\xi(t)=(x(t), y(t))$ is the integral curve of the H-field if and only if

$$y^i(t)=\frac{dx^i}{dt}, \qquad \frac{d^2x^i}{dt^2}=a^i_j\left[x(t), \frac{dx^k}{dt}\right]\frac{dx^j}{dt}$$

It means that the curve $\xi(t)$ on $T(M)$ is the integral curve of the H-field if and only if $\xi(t)=\gamma_*(t)$, where γ is the H-geodesic.

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О ГОРИЗОНТАЛЬНОЙ СТРУКТУРЕ НА ДИФФЕРЕНЦИАЛЬНОМ МНОГООБРАЗИИ

Антон Декрет

Резюме

Пусть E векторное расслоение и V поле Лиувилля на E. В первой части статьи описаны некоторые свойства скобки $[V, X]$, В первой части статьи описаны некоторые свойства скобки $[V, X]$, где X векторное поле на E. Пусть M дифференциальное многообразие. Пусть Γ: $TM \to J^1TM$ сечение расслоения 1-струей локальных сечений расслоения TM касательных пространств. Γ определяет на TM n-мерное распределение ($n=\dim M$) и векторное поле H на TM, которое является дифференциальным уравнением второго рода на M. Пусть $'X(\bar{X})$ означает продолжение (Γ-подъем) векторного поля X на M на пространство TM. С помощью этих полей определена однородность сечения Γ. В теореме 6 доказывается, что если Γ однородно, то поле N пульверзация. В определении 1 вводится понятие векторного поля сопряженного с Γ. В теореме 7 доказано, что поле V сопряжено с Γ тогда и только тогда, когда Γ однородно. В статье показано, что с помощью Γ можно вводить параллельный перенос и геодезические, которые в локальных кординатах имеют вид

$$\frac{d^2x^i}{dt^2}=a^i_j\left(x^k(t), \frac{dx^k}{dt}\right)\frac{dx^j}{dt},$$

где a^i_j функции на TM.