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LIMIT THEOREMS FOR B-LATTICE VALUED RANDOM VARIABLES

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ABSTRACT. Results on the convergence of weighted sums of sequences of real random variables are extended to Banach lattices. A general result on almost sure convergence to 0 with respect to the order is proved, from which many known results follow as special cases.

1. Introduction

Vector-lattice valued functions have been studied by Cristescu [2], Kantorovitch [3] and others. The original assumption of the regularity of a vector lattice under investigation has been removed by Potocký [9]. (See also Kelemenová [4].) It turns out that instead of regular lattices spaces the so-called σ-property can be considered to obtain a fruitful theory. The definition of the expected value and higher-order moments for vector-lattice valued random variables can be found, e.g. in [9] and [11].

DEFINITION 1.1. Let (Ω, S, P) be a probability space, E a vector lattice. A sequence $\{X_n\}$ of functions from Ω to E *converges to a function* $X\colon \Omega \to E$ *almost uniformly* (with respect to (r)-convergence) if for every $\varepsilon > 0$ there exist a set $A \in S$ such that $P(A) < \varepsilon$, a sequence $\{a_n\}$ of positive real numbers converging to zero and an element $r \in E$ such that $|X_n(\omega) - X(\omega)| \leqq a_n r$ for each $\omega \in \Omega - A$.

DEFINITION 1.2. A function $X\colon \Omega \to E$ is called a *random variable* if there exists a sequence $\{X_n\}$ of measurable E-valued functions with countable range such that $\{X_n\}$ converges to X almost uniformly.

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DEFINITION 1.3. A vector lattice E is said to be *σ-complete* if every nonempty at most countable subset of E which is bounded from above has a supremum.

DEFINITION 1.4. A vector lattice E is said to *have the σ-property* if for every sequence $\{e_n\}$ of elements from E there exist $u \in E$, $u > 0$, and a sequence $\{K_n\}$ of positive real numbers such that $|e_n| \leqq K_n u$ for all $n \in \mathbb{N}$.

DEFINITION 1.5. A vector lattice with a monotone norm which is complete with respect to this norm is called *Banach lattice.*

PROPOSITION 1.1. *Let E be a Banach lattice. Then each random variable is a measurable map from Ω to E.*

Proof. See [7]. □

2. Convergence of weighted sums of random variables in Banach lattices

A strong law of large numbers for independent and identically distributed random variables with values in a vector lattice has been given by Potocký [8]. In what follows we omit both of the above assumptions. As a result we establish a limit theorem for weighted sums of Banach lattice-valued random variables for which no assumptions on mutual relationships are made.

DEFINITION 2.1. A sequence $\{X_n\}$ of random variables *satisfies the strong law of large numbers* if there exists an element $a \in E^+$ such that for every $\varepsilon > 0$

$$\lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} \left\{\omega : \left|\frac{1}{k}\sum_{i=1}^{k} X_i(\omega) - \frac{1}{k}\sum_{i=1}^{k} EX_i\right| \leqq \varepsilon a\right\}\right) = 1 .$$

PROPOSITION 2.1. (Potocký [8]) *Let E be a σ-complete vector lattice with the σ-property equipped with a complete compatible metrizable locally solid linear topology. If f_n are independent, identically distributed, symmetric random variables in E, then the condition*

$$\exists(a \in E^+)\left(\sum_{n=1}^{\infty} P\{\omega : |f_1(\omega)| \leqq na\}^c < \infty\right)$$

is necessary and sufficient for $\{f_n\}$ to satisfy the strong law of large numbers.

PROPOSITION 2.2. (Loéve [5]) *Let X_n and X be real-valued random variables. If $P(|X_n| \geqq x) \leqq P(|X| \geqq x)$ for every x, where $E|X|^r < \infty$, $0 < r < 1$, then*

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} X_k \to 0 \qquad a.s..$$

PROPOSITION 2.3. (Wang & Bhaskara Rao [13]) *Let X_n, $n \geqq 1$, be a sequence of real-valued random variables. If for every n, $E|X_n|^r < \infty$ for some $r > 0$, then there exists a nonnegative random variable X on Ω such that $EX^s < \infty$ for every $0 < s < r$, and X_n, $n \geqq 1$, is stochastically dominated by X, i.e. $P(|X_n| \geqq \lambda) \leqq P(|X| \geqq \lambda)$ for each $n \geqq 1$ and each $\lambda \geqq 0$.*

THEOREM 2.1. *Let $\{X_n\}$ be a sequence of random variables taking values in a σ-complete Banach lattice E with the σ-property. Suppose that for each $n \in \mathbb{N}$*

$$\sum_{k=1}^{\infty} k^{r-1} P(|X_n| < ka)^c \leqq M$$

for some $r \geqq 1$, $a > 0$, $a \in E$ and $M > 0$, $M \in \mathbb{R}$. Let $p > r$ and $q > 1$ satisfy $1/p + 1/q = 1$. Let $\{a_{nk}\}$, $1 \leqq k \leqq n$, $n \geqq 1$, be a double sequence of real numbers satisfying

$$\lim_{n\to\infty} \sup \sum_{k=1}^{n} |a_{nk}|^q < \infty .$$

Then

$$\frac{1}{n^{1/s}} \sum_{k=1}^{n} a_{nk} X_k \to 0$$

for every s, $0 < s < r$, almost surely in Ω relatively uniformly.

Proof. For each n let $\{X_n^k\}$ be a sequence of measurable E-valued functions with countable range converging almost uniformly to X_n. Then there exists a set Ω_0 of probability 1 such that $\{X_n^k\}$ converge (pointwise) relatively uniformly on this set for each n. Consider now all X_n as functions defined on this set with values in E. From the inequality

$$|X_n| \leqq |X_n - X_n^k| + |X_n^k|$$

which holds for each natural number n and each natural number k and from the assumption that E has σ-property we obtain that all values of X_n belong to the principal ideal of E generated by a single element, say u, $u \geqq a$, $u \in E$. Let us denote this ideal by I_u. Since E is a σ-complete vector lattice, I_u equipped with the order-unit norm, i.e. the norm induced by u, is a Banach space. It will be denoted $(I_u, \| \|_u)$. In such a lattice the norm-convergence and the relatively uniform convergence are equivalent (see [10; p. 102]).

Let us denote the set of all values which the above mentioned variables X_{nk}, $n, k \in \mathbb{N}$, take on by $\{y_n\}_{n=1}^{\infty}$ and put $y_0 = u$. Consider the countable set Γ of all linear combinations of the elements y_n with the rational coefficients. The set

$$B = \bigcap_{r \in Q^+} \bigcup_{\gamma \in \Gamma} \{x \in I_u : \ |x - \gamma| \leqq ru\},$$

where $\mathbb{Q}^+$ stands for the set of all positive rationals, is a linear subspace of I_u. It is obvious that all X_n take on only values in B. Equipped with the order-unit norm inherited from I_u, B becomes a separable Banach space. Indeed for each $x \in B$ and $\varepsilon > 0$ there exists an element $\gamma \in \Gamma$ such that $\|x - \gamma\|_u < \varepsilon$. The completeness follows from the fact that B is closed in $(I_u, \| \|_u)$. From now on this space will be denoted by $(B, \| \|_u)$.

It follows from the assumption that

$$\begin{aligned} E\|X_n\|^r &\leqq 1 + 2^r r \sum_{k=1}^{\infty} k^{r-1} P(\|X_n\| > k) \\ &= 1 + 2^r r \sum_{k=1}^{\infty} k^{r-1} P(|X_n| \leqq ku)^c \\ &\leqq 1 + 2^r r \sum_{k=1}^{\infty} k^{r-1} P(|X_n| \leqq ka)^c \leqq M \end{aligned}$$

for each n and some $r \geqq 1$. Using Proposition 2.3 we have that there exists a nonnegative real-valued random variable X on Ω such that $EX^s < \infty$ for every $0 < s < r$ and $\{\|X_n\|\}$, $n \geqq 1$, is stochastically dominated by X.

By Hőlder's inequality, for every $n \geqq 1$

$$\begin{aligned} \left\| \frac{1}{n^{1/s}} \sum_{k=1}^{n} a_{nk} X_k \right\| &\leqq \sum_{k=1}^{n} |a_{nk}| \frac{\|X_k\|}{n^{1/s}} \\ &\leqq \left(\sum_{k=1}^{n} |a_{nk}|^q \right)^{1/q} \left(\sum_{k=1}^{n} \frac{\|X_k\|^p}{n^{p/s}} \right)^{1/p}. \end{aligned}$$

We note that $0 < s/p < 1$ and $\{\|X_n\|^p\}$ is stochastically dominated by X^p with $E(X^p)^{s/p} < \infty$. By Proposition 2.2,

$$\lim_{n \to \infty} \frac{1}{n^{p/s}} \sum_{k=1}^{n} \|X_k\|^p = 0 \qquad \text{with probability} \quad 1.$$

Since

$$\lim_{n\to\infty} \sup \sum_{k=1}^{n} |a_{nk}|^q < \infty ,$$

it follows that

$$\left\| \frac{1}{n^{1/s}} \sum_{k=1}^{n} a_{nk} X_k \right\| \to 0$$

with probability 1 with respect the order-unit norm and consequently

$$\frac{1}{n^{1/s}} \sum_{k=1}^{n} a_{nk} X_k \to 0$$

almost surely in Ω relatively uniformly. □

The above result extends the result of Padgett and Taylor [6; Theorem 3] to vector lattice-valued random variables. They assume that X_n, $n \geqq 1$, is independently identically distributed with $EX_1 = 0$. We underline that no assumptions on X_n are needed.

COROLLARY 2.1. *Let $\{X_n\}$ be a sequence of random variables taking values in a σ-complete Banach lattice E with the σ-property such that for each $n \in \mathbb{N}$*

$$\sum_{k=1}^{\infty} P(|X_n| \leqq ka)^c < M$$

for some $a > 0$, $a \in E$ and $M > 0$. Let $\{a_{nk}\}$, $1 \leqq k \leqq n$, $n \geqq 1$, be a double sequence of real numbers satisfying

$$\lim_{n\to\infty} \sup \sum_{k=1}^{n} a_{nk}^2 < \infty .$$

Then

$$\frac{1}{n^{1/s}} \sum_{k=1}^{n} a_{nk} X_k \to 0$$

almost surely in Ω relatively uniformly for every $0 < s < 1$.

The following result was established by Padgett and Taylor [6; Theorem 5] for Banach space valued random variables under the additional assumptions that X_n, $n \geqq 1$, are independent, EX_n exists and equals zero for every $n \geqq 1$ and $\beta > \alpha$.

COROLLARY 2.2. *Let $\{X_n\}$ be a sequence of random variables in a σ-complete Banach lattice E with the σ-property such that for every n*

$$\sum_{k=1}^{\infty} kP(|X_n| \leqq ka)^c \leqq M$$

for some $a \in E$, $a > 0$ and $M \in \mathbb{R}$, $M > 0$. Let $\{a_{nk}\}$, $1 \leqq k \leqq n$, $n \geqq 1$, be a double sequence of real numbers such that for $\alpha > 0$ and $\beta > 1$

(i) $|a_{nk}| \leqq C_1 n^{-\alpha}$ *for all* $1 \leqq k \leqq n$ *and* $n \geqq 1$,

(ii) $\sum_{k=1}^{n} |a_{nk}| \leqq C_2 n^{\alpha-\beta}$ *for every* $n \geqq 1$
for some positive constants C_1 and C_2.

Then

$$\sum_{k=1}^{n} a_{nk} X_k \xrightarrow{\text{a.s.}} 0$$

relatively uniformly.

P r o o f. By virtue of Proposition 2.3, there exists a real-valued random variable X such that $\{\|X_n\|\}$ is stochastically dominated by X and $EX^r < \infty$ for $0 \leqq r < 2$ since $\forall n \in \mathbb{N}$ we have

$$E\|X_n\|^2 \leqq 1 + 8\sum_{k=1}^{\infty} kP(|X_n| \leqq ka)^c < \infty .$$

Choose $p > 0$ and $1 < r < 2$ satisfying $r < p < 2$ and

$$\frac{p}{r} \cdot \frac{1}{p-1} < \beta .$$

If $q > 0$ satisfies $1/p + 1/q = 1$, then $q > 2$. From this it follows that $-\alpha q + 2\alpha < 0$ and $q/r = \frac{p}{r} \cdot \frac{1}{p-1} < \beta$. Finally choose s such that $0 < s < r$ and $q/s - \beta < 0$. For all n and k put $b_{nk} = a_{nk} n^{1/s}$. We have

$$\sum_{k=1}^{n} |b_{nk}|^q \leqq C_1^{q-1} C_2 n^{-\alpha q + 2\alpha} n^{q/s-\beta}$$

for each $n \geqq 1$, and consequently

$$\lim_{n\to\infty} \sum_{k=1}^{n} |b_{nk}|^q = 0 .$$

Since $\sum_{k=1}^{\infty} k^{r-1} P(|X_n| \leqq ka)^c \leqq \sum_{k=1}^{\infty} kP(|X_n| \leqq ka)^c \leqq M$, it follows from Theorem 2.1 that

$$1/n^{1/s} \sum_{k=1}^{n} b_{nk} X_k = \sum_{k=1}^{n} a_{nk} X_k \to 0$$

almost surely in Ω relatively uniformly. □

COROLLARY 2.3. *Let $\{X_n\}$ be a sequence of random variables taking values in a σ-complete Banach lattice E with the σ-property such that for every n*

$$\sum_{k=1}^{\infty} kP(|X_n| \leqq ka)^c \leqq M$$

for some $a \in E$, $a > 0$ and $M \in \mathbb{R}$, $M > 0$. Let $\{a_{nk}\}$, $1 \leqq k \leqq n$, $n \geqq 1$, be a double sequence of real-valued random variables satisfying

$$P\left(\limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk}^2 = C < \infty\right) = 1 .$$

Then for every $0 < r < 2$

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} a_{nk} X_k \xrightarrow{\text{a.s.}} 0$$

relatively uniformly.

Proof. Fix arbitrarily $0 < r < 2$. Similarly as in the proof of Theorem 2.1 we obtain that $\{X_n\}$ is a sequence of random elements in a separable Banach space with an order unit norm. From the assumptions it follows that $E\|X_n\|^2 < \infty$ for every n. Consequently, by Proposition 2.3, there exists a nonnegative real-valued random variable X, such that $\{\|X_n\|\}$ is stochastically dominated by X and $EX^r < \infty$ for $0 < r < 2$. Then the proof of Theorem 2.1 can be adapted taking $p = q = 2$. □

Chow and Lai [1; p. 823] have established the above result under stronger conditions. They assume that $\{X_n\}$ is a sequence of independently identically distributed real-valued random variables with $E|X_1|^r < \infty$ for some $1 \leqq r \leqq 2$.

COROLLARY 2.4. *Let $\{X_n\}$ be a sequence of random variables taking values in a σ-complete Banach lattice E with the σ-property. Let X be a random variable in E such that for every n and k*

$$P(|X_n| \leqq ka) \geqq P(|X| \leqq ka) \qquad \text{and} \qquad \sum_{n=1}^{\infty} n^{r-1} P(|X| \leqq na)^c < \infty$$

for some r, $1 < r \leqq 2$, $a > 0$, $a \in E$.

Let $\{a_{nk}\}$, $1 \leqq k \leqq n$, $n \geqq 1$, be a double sequence of real numbers satisfying

$$\max_{1\leqq k\leqq n} |a_{nk}| = 0(n^{-\alpha}) \qquad \text{as} \quad n \to \infty$$

for some $0 < \frac{1}{\alpha} < r - 1$. Then

$$\sum_{k=1}^{n} a_{nk} X_k \xrightarrow{\text{a.s.}} 0 \qquad \text{relatively uniformly.}$$

Proof. Put $b_{nk} = a_{nk}n^{1/s}$ for each $1 \leqq k \leqq n$ and $n \geqq 1$ where s is chosen in such a way that $0 < s < r$. Then

$$\sum_{k=1}^{n} b_{nk}^2 \leqq \max_{1\leqq k\leqq n} a_{nk}^2 \sum_{k=1}^{n} (n^{1/s})^2$$
$$\leqq Bn^{-2\alpha}n^{2/s} \cdot n \leqq Bn^{-2/(r-1)}n \cdot n^{2/s} = B$$

if s is chosen in such a way that $1 + \frac{2}{s} - \frac{2}{r-1} = 0$ (such a choice is possible) for $1 < r < 2$ and for some positive constant B.

Consequently

$$\lim_{n\to\infty} \sup \sum_{k=1}^{n} b_{nk}^2 < \infty .$$

Owing to the assumptions we have $\sum_{k=1}^{\infty} k^{r-1}P\big(|X_n| \leqq na\big)^c < M$ for each $n \in \mathbb{N}$.

Applying Theorem 2.1 we obtain

$$\frac{1}{n^{1/s}} \sum_{k=1}^{n} b_{nk}X_k = \sum_{k=1}^{n} a_{nk}X_k \xrightarrow{\text{a.s.}} 0$$

relatively uniformly. If $r = 2$, choose p and q such that $p > r = 2$, $1/p+1/q = 1$ and $1 < q < 2$. Finally we take an s, $1 < s < 2$ and $q/s < 1$. Then we have

$$\sum_{k=1}^{n} |b_{nk}|^q \leqq \max_{1\leqq k\leqq n} |a_{nk}|^q \sum_{k=1}^{n} (n^{1/s})^q$$
$$\leqq Bn^{q/s-\alpha q} \leqq Bn^{q/s-1}$$

from which it follows that

$$\lim_{n\to\infty} \sup \sum_{k=1}^{n} |b_{nk}|^q < \infty .$$

Applying Theorem 2.1 once more yields the required result. □

The above Corollary extends the result of Taylor [12; Theorem 5.3.1], in which, however, the following assumptions are necessary: E is a separable normed linear space which is Beck-Convex, X_n, $n \geqq 1$, are independent, $EX_n = 0$ for all $n \geqq 1$, $\sup_{n\geqq 1} E\|X_n\|^r < \infty$ for some $r > 1$, a_{nk} are nonnegative with sums over k less than or equal to unity for all $n \geqq 1$ and

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} a_{nk} - n \min_{1\leqq k\leqq n} a_{nk} \right) = 0 .$$

3. Conclusions

In the last thirty years many authors have devoted their attention to the convergence of weighted sums of random variables in one or another sense. While a number of interesting results has been produced for the norm topology, the theory is much less developed for the weak topology and almost completely neglected is the convergence with the respect to the order. It is the reason why we have studied convergence of weighted sums of random variables in Banach lattices. It is interesting that in many spaces the order convergence is stronger than the topological one. From Theorem 2.1 many results in the literature follow as special cases under much less restrictive conditions.

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