Susumu Okada

Vector Daniell integrals

Mathematica Slovaca, Vol. 34 (1984), No. 1, 35--65

Persistent URL: http://dml.cz/dmlcz/131899

Math. Slovaca 34, 1984, No. 1, 35—65

VECTOR DANIELL INTEGRALS

SUSUMU OKADA

Consider a linear map from a Riesz space of functions on a set into a locally convex space. If it is continuous with respect to the monotone convergence of sequences, then it is called a vector Daniell integral. Thanks to the Dini theorem, every vector-valued Radon measure is a vector Daniell integral. Vector-valued Radon measures have been studied by Bourbaki [3] and Thomas [17]. Kluvánek [8] has investigated Banach space-valued Daniell integrals, adapting the method of F. Riesz from [15]. There has been an increasing interest recently in the theory of measures on non-locally compact spaces, and of Daniell integrals such as conical measures which are not generated by measures (set functions). Here we present a theory of Daniell integrals on abstract sets with values in locally convex spaces.

In § 1, we define a few variants of the Daniell integral and explore the relationships between them.

Among the essential themes which have to be developed in an integration theory are the Beppo Levi theorem, the Lebesgue convergence theorem and the completeness of the L_1-space. To have the Beppo Levi theorem, we extend the domain of a Daniell integral, applying the procedure of Stone [16] which has already been used in special cases by Bauer [2], Bourbaki [3], Kluvánek [9], Thomas [17] and others. The extension so obtained is called the Stone extension of the given Daniell integral. Its properties are studied in § 2 and § 4. The Beppo Levi theorem is proved in § 3.

To obtain the Lebesgue convergence theorem, we require an extra condition on a Daniell integral, namely that it maps order intervals into weakly compact sets. The details are discussed in § 5. This condition has been studied by Kluvánek [8] in the case of Banach space-valued Daniell integrals. Many authors have investigated linear maps from an abstract Riesz space into a Banach space which satisfy this condition. Thomas [17] has developed locally convex spacevalued Radon measures with this condition.

The L_1-space obtained from the Stone extension is not always quasi-complete with respect to the convergence in mean. In § 8, we therefore construct another extension guaranteeing quasi-completeness. We deal there with Daniell integrals which satisfy the well-known min $(f, 1)$-condition of Stone and for which the Lebesgue convergence theorem is valid.

In §6 we study the direct sum of Daniell integrals.

The relationship between Daniell integrals and vector measures is discussed in §7.

1. Daniell integral and Stone integral

Let Ω be a set. A subset of Ω and its characteristic function will be denoted by the same symbol.

Let X be a real quasi-complete locally convex Hausdorff space and X' its dual. Let $P(X)$ be the set of all continuous seminorms on X.

The set $\mathbf{R}^\Omega$ of all real-valued functions on Ω is a Riesz space (vector lattice) with respect to the pointwise order. Let $\mathbf{L}$ be a Riesz subspace of $\mathbf{R}^\Omega$ and $\mathbf{L}^+$ its positive cone, that is, $\mathbf{L}^+ = \{f \in \mathbf{L}: f \geq 0\}$. Let I: $\mathbf{L} \to X$ be a linear map. For every $p \in P(X)$ and every $f \in \mathbf{L}$ let

$$p(I)(f) = \sup \{p(I(g)): g \in \mathbf{L}, |g| \leq |f|\}.$$

In particular, if $X = \mathbf{R}$, then

$$|I|(f) = \sup \{|I(g)|: g \in \mathbf{L}, |g| \leq |f|\}$$

whenever $f \in \mathbf{L}$. Clearly, for every $p \in P(X)$ and every $f \in \mathbf{L}$, the equality

$$p(I)(f) = \sup \{|x' \circ I|(f): x' \in U'_p\} \tag{1}$$

holds, where U'_p is the set of all functionals $x' \in X'$ such that $|\langle x', x\rangle| \leq p(x)$ for each $x \in X$.

A linear map I: $\mathbf{L} \to X$ is called a Daniell integral if the sequence $\{I(f_n)\}_{n \in \mathbf{N}}$ is convergent to 0 in X for every sequence $\{f_n\}_{n \in \mathbf{N}}$ of functions in $\mathbf{L}$ which is decreasing and pointwise convergent to 0.

A linear map I: $\mathbf{L} \to X$ is said to be a weak Daniell integral if, for every $x' \in X'$, the functional $x' \circ I$: $\mathbf{L} \to \mathbf{R}$ is a real-valued Daniell integral.

A linear map I: $\mathbf{L} \to X$ is called a Stone integral if the following conditions are satisfied:

(i) If $p \in P(X)$ and $f \in \mathbf{L}$, then $p(I)(f) < \infty$;

(ii) If $p \in P(X)$ and if $f \in \mathbf{L}$ and $f_n \in \mathbf{L}$, $n \in \mathbf{N}$, are functions such that

$$|f| \leq \sum_{n=1}^{\infty} |f_n|, \tag{2}$$

then

$$p(I)(f) \leq \sum_{n=1}^{\infty} p(I)(f_n). \tag{3}$$

Proposition 1.1. *Every Daniell integral is a weak Daniell integral.*

Lemma 1.2. *A linear map $I: \mathbf{L} \to X$ is a Stone integral if and only if $x' \circ I$ is a real Stone integral for every $x' \in X'$.*

Proof. This follows from the equality (1).

Lemma 1.3 ([14: no. 11]). *If $I: \mathbf{L} \to \mathbf{R}$ is a Daniell integral, then there exist two non-negative Daniell integrals I^+ and $I^-: \mathbf{L} \to \mathbf{R}$ such that $I(f) = I^+(f) - I^-(f)$ for every $f \in \mathbf{L}$ and $|I|(g) = I^+(g) + I^-(g)$ for every $g \in \mathbf{L}^+$.*

Lemma 1.4. *If $I: \mathbf{L} \to \mathbf{R}$ is a linear map, then I is a Stone integral if and only if I is a Daniell integral.*

Proof. Suppose that I is a Stone integral. Let us take any sequence $\{f_n\}_{n \in \mathbf{N}}$ in $\mathbf{L}$ which is decreasing and pointwise convergent to 0. Then, since $0 \leq f_1 \leq \sum_{n=1}^{\infty} (f_n - f_{n+1})$, we have

$$|I|(f_1) \leq |I|(f_1) - \lim_{n \to \infty} |I|(f_n) \leq |I|(f_1).$$

Thus $|I|(f_n) \to 0$ as $n \to \infty$. That is, I is a Daniell integral.

On the other hand, by Lemma 1.3, if I is a Daniell integral, then I is a Stone integral.

From Lemmas 1.2 and 1.4 we have

Proposition 1.5. *If $I: \mathbf{L} \to X$ is a linear map, then I is a weak Daniell integral if and only if I is a Stone integral.*

Corollary 1.6. *Every Daniell integral is a Stone integral.*

Vector-valued Radon measures are Daniell integrals.

Example 1.7. Let $C_{\mathscr{C}}(T)$ be the linear space of all continuous functions with compact support on a locally compact Hausdorff space T. For a compact subset K of T the set of all functions in $C_{\mathscr{C}}(T)$ with vanish outside K is denoted by $C_{\mathscr{C}}(K)$ and equipped with the uniform norm. A linear map I from $C_{\mathscr{C}}(T)$ into a quasi-complete locally convex Hausdorff space is said to be a Radon measure if, for every compact subset K of T, the restriction of I to $C_{\mathscr{C}}(K)$ is continuous. From the Dini theorem it is clear that I is a Daniell integral.

2. Stone extension

Let Ω be an abstract set. Let I be a Stone integral from a Riesz subspace $\mathbf{L}$ of $\mathbf{R}^{\Omega}$ into a quasi-complete locally convex Hausdorff space X.

Given $p \in P(X)$, define

$$\sigma_p(f) = \inf \left\{ \sum_{n=1}^{\infty} p(I)(f_n) : f_n \in \mathbf{L},\ |f| \leq \sum_{n=1}^{\infty} |f_n| \right\}$$

for every $f \in \mathbf{R}^{\Omega}$, understanding that $\sigma_p(f) = \infty$ unless there exist functions $f_n \in \boldsymbol{L}$, $n \in \mathbf{N}$, such that (2) holds.

For every $p \in P(X)$ let $\boldsymbol{F}_p(I)$ be the space of functions f on Ω such that $\sigma_p(f) < \infty$. The intersection of the family $\{\boldsymbol{F}_p(I) : p \in P(X)\}$ is denoted by $\boldsymbol{F}(I)$ or simply by $\boldsymbol{F}$.

Proposition 2.1. *Let* $p \in P(X)$.

(i) *If* $f \in \mathbf{R}^{\Omega}$ *and* $f_n \in \mathbf{R}^{\Omega}$, $n \in \mathbf{N}$, *and if* (2) *holds, then*

$$\sigma_p(f) \leqslant \sum_{n=1}^{\infty} \sigma_p(f_n). \tag{4}$$

(ii) σ_p *is a seminorm on* $\boldsymbol{F}_p(I)$.

(iii) $\sigma_p = p(I)$ *on* $\boldsymbol{L}$.

Proof. To prove (i), we may assume that $\sigma_p(f_n) < \infty$ for every $n \in \mathbf{N}$. Given $\varepsilon > 0$ and $n \in \mathbf{N}$, there exist functions $f_{nm} \in \boldsymbol{L}$, $m \in \mathbf{N}$, such that

$$|f_n| \leqslant \sum_{m=1}^{\infty} |f_{nm}| \quad \text{and} \quad \sum_{m=1}^{\infty} p(I)(f_{nm}) \leqslant \sigma_p(f_n) + \varepsilon 2^{-n}.$$

Consequently,

$$\sigma_p(f) \leqslant \sum_{n,m=1}^{\infty} p(I)(f_{nm}) \leqslant \sum_{n=1}^{\infty} \sigma_p(f_n) + \varepsilon,$$

so that (i) holds.

Statement (ii) is now clear.

To prove (iii) it suffices to show that for every $f \in \boldsymbol{L}$ the inequality $\sigma_p(f) \geqslant p(I)(f)$ holds. For every $\varepsilon > 0$ choose functions $f_n \in \boldsymbol{L}$, $n \in \mathbf{N}$, such that (2) holds and

$$\sum_{n=1}^{\infty} p(I)(f_n) < \sigma_p(f) + \varepsilon.$$

Statement (iii) now follows from the inequality (3).

The family $\{\sigma_p : p \in P(X)\}$ of seminorms on $\boldsymbol{F}$ gives a locally convex topology. From now on, $\boldsymbol{F}$ will always be equipped with this topology. The space $\boldsymbol{F}$ is not always complete (see Example 2.12). However, if X is metrizable, then $\boldsymbol{F}$ is complete. More generally, we have

Proposition 2.2. *If* $\boldsymbol{F}(I)$ *has a countable basis* $\{\sigma_n\}_{n \in \mathbf{N}}$ *of continuous seminorms, then* $\boldsymbol{F}(I)$ *is complete.*

Proof. We may assume that $\sigma_1 \leqslant \sigma_2 \leqslant \ldots$. Take any Cauchy sequence $\{f_n\}_{n \in \mathbf{N}}$ in $\boldsymbol{F}$. Let $f_n^{(0)} = f_n$ for every $n \in \mathbf{N}$. Assume that for $m \in \mathbf{N}$ we have chosen functions $f_n^{(i)}$, $n \in \mathbf{N}$, $i = 1, 2, \ldots, m$, such that $\{f_n^{(i)}\}_{n \in \mathbf{N}}$ is a subsequence of $\{f_n^{(i-1)}\}_{n \in \mathbf{N}}$ and such that

$$\sigma_i(f_j^{(i)} - f_k^{(i)}) < 2^{-j} \tag{5}$$

whenever $j \in \mathbf{N}$, $k \in \mathbf{N}$, $k > j$. Then we select a subsequence $\{f_n^{(m+1)}\}_{n \in \mathbf{N}}$ of the sequence $\{f_n^{(m)}\}_{n \in \mathbf{N}}$ such that if $i = m+1$, then (5) is valid for all $j \in \mathbf{N}$ and $k \in \mathbf{N}$, $k > j$. Define the function f on Ω by

$$f(\omega) = f_1^{(1)}(\omega) + \sum_{n=1}^{\infty} (f_{n+1}^{(n+1)} - f_n^{(n)})(\omega) \tag{6}$$

for every $\omega \in \Omega$ such that the series (6) converges, and by $f(\omega) = 0$ otherwise. Let $i \in \mathbf{N}$. It follows from Proposition 2.1 that

$$\sigma_i(f - f_n^{(n)}) \leqslant \sum_{m=n}^{\infty} \sigma_i(f_{m+1}^{(m+1)} - f_m^{(m)}) \leqslant$$

$$\leqslant \sum_{m=n}^{\infty} \sigma_i(f_{m+1}^{(m+1)} - f_m^{(m+1)}) + \sum_{m=n}^{\infty} \sigma_i(f_m^{(m+1)} - f_m^{(m)}) \leqslant 2^{-n+2}$$

for every $n > i$. Hence, $f \in \boldsymbol{F}$ and $\{f_n\}_{n \in \mathbf{N}}$ converges to f in $\boldsymbol{F}$.

The closure of $\boldsymbol{L}$ in $\boldsymbol{F}$ will be denoted by $\boldsymbol{K}_1(I)$ or simply by $\boldsymbol{K}_1$. Clearly, $\boldsymbol{K}_1$ is a Riesz subspace of $\mathbf{R}^{\Omega}$.

For every $p \in P(X)$, let π_p be the natural map from X into the completion X_p of $X/p^{-1}(0)$. Since $\pi_p \circ I$: $\boldsymbol{L} \to X_p$ is a Stone integral, we have the following proposition (cf. [3: I, §4, no. 4]).

Proposition 2.3. *For the Stone integral I: $\boldsymbol{L} \to X$, the space $\boldsymbol{K}_1(I)$ coincides with the intersection of the family $\{\boldsymbol{K}_1(\pi_p \circ I) : p \in P(X)\}$ of spaces.*

Let $\hat{X}$ be the completion of X. Then every $p \in P(X)$ has a unique extension $\hat{p}$ to $\hat{X}$. Since $p(I(f)) \leqslant \sigma_p(f)$ for every $p \in P(X)$ and every $f \in \boldsymbol{L}$, the Stone integral I has a unique extension $I_1 : \boldsymbol{K}_1 \to \hat{X}$, which we call the Stone extension of I. For every $p \in P(X)$ and every $f \in \boldsymbol{K}_1$ let

$$\hat{p}(I_1)(f) = \sup\{\hat{p}(I_1(g)) : g \in \boldsymbol{K}_1,\ |g| \leqslant |f|\}.$$

Lemma 2.4. *Let $p \in P(X)$. Then $\hat{p}(I_1) = \sigma_p$ on $\boldsymbol{K}_1(I)$.*

Proof. Since $\hat{p}(I_1(f)) \leqslant \sigma_p(f)$ for every $f \in \boldsymbol{K}_1$, it follows that $\hat{p}(I_1) \leqslant \sigma_p$ on $\boldsymbol{K}_1$. Thus $\hat{p}(I_1)$ is continuous on $\boldsymbol{K}_1$ with respect to the topology induced from $\boldsymbol{F}$. On the other hand, $\sigma_p \leqslant \hat{p}(I_1)$ on $\boldsymbol{L}$ by Proposition 2.1; hence the same inequality holds on $\boldsymbol{K}_1$ since both σ_p and $\hat{p}(I_1)$ are continuous on $\boldsymbol{K}_1$.

Proposition 2.1 and Lemma 2.4 imply that the locally convex topology on $\boldsymbol{K}_1$ defined by the family $\{\hat{p}(I_1) : p \in P(X)\}$ of seminorms is equal to the topology induced from $\boldsymbol{F}$ and that the locally convex topology on $\boldsymbol{L}$ given by $\{p(I) : p \in P(X)\}$ coincides with the topology induced from $\boldsymbol{K}_1$. Hereafter, $\boldsymbol{L}$ and $\boldsymbol{K}_1$ will be endowed with these topologies, which we call the topologies of convergence in mean.

The following proposition is a direct consequence of Proposition 2.2.

Proposition 2.5. *If the space X is metrizable, then $\mathbf{K}_1(I)$ is a complete metrizable space.*

Theorem 2.6. *The Stone extension $I_1: \mathbf{K}_1(I) \to X$ is a Stone integral.*

Proof follows from Proposition 2.1 and Lemma 2.4

A Stone integral is called Stone-closed if its Stone extension coincides with itself.

As we shall see, the Stone extension of a Stone integral is Stone-closed.

Given $p \in P(X)$, define

$$\delta_p(f) = \inf\left\{\sum_n \hat{p}(I_1)(f_n) : f_n \in \mathbf{K}_1,\ |f| \leqslant \sum_n |f_n|\right\}$$

for every $f \in \mathbf{R}^\Omega$, understanding that $\delta_p(f) = \infty$ unless there exist functions $f_n \in \mathbf{K}_1$, $n \in \mathbf{N}$, such that (2) holds.

Lemma 2.7. *If $p \in P(X)$, then $\delta_p = \sigma_p$ on $\mathbf{R}^\Omega$.*

Proof. Since $\hat{p}(I_1) = p(I)$ on $\mathbf{L}$, we have $\delta_p \leqslant \sigma_p$. On the other hand, since $\hat{p}(I_1) = \sigma_p$ by Lemma 2.4, we have $\delta_p \geqslant \sigma_p$.

Theorem 2.8. *The Stone extension of a Stone integral is Stone-closed.*

A function $f \in \mathbf{R}^\Omega$ is said to be I-null if

$$\sigma_p(f) = 0 \tag{7}$$

for every $p \in P(X)$. The set of all I-null functions is denoted by $\mathbf{N}(I)$. Then $\mathbf{N}(I) \subset \mathbf{K}_1(I)$. Moreover $\mathbf{N}(I_1) = \mathbf{N}(I)$ by Theorem 2.8. A subset of Ω will be called I-null if its characteristic function belongs to $\mathbf{N}(I)$. A property which holds for all points of Ω outside some I-null set is said to hold almost everywhere (a. e.) in Ω or for almost all (a.a.) $\omega \in \Omega$.

For every function $f \in \mathbf{R}^\Omega$, let

$$S(f) = \{\omega \in \Omega : f(\omega) \neq 0\}$$

We omit the proof of the following

Proposition 2.9. (i) *if $f_n \in \mathbf{N}(I)$, $n \in \mathbf{N}$, and if a function $f \in \mathbf{R}^\Omega$ satisfies* (2), *then $f \in \mathbf{N}(I)$.*

(ii) *A function $f \in \mathbf{R}^\Omega$ is I-null if and only if $S(f)$ is an I-null set.*

(iii) *If $f \in \mathbf{R}^\Omega$ and if A is an I-null set, then $fA \in \mathbf{N}(I)$.*

(iv) *If $f \in \mathbf{K}_1(I)$, if $g \in \mathbf{R}^\Omega$ and if $f = g$ a.e., then $g \in \mathbf{K}_1(I)$.*

Propositions 1.5 and 2.3 imply the following

Proposition 2.10. *If Ψ is a continuous linear map from X into another quasi-complete locally convex Hausdorff space Y, then $\Psi \circ I: \mathbf{L} \to Y$ is also a Stone integral and*

$$\mathbf{K}_1(I) \subset \mathbf{K}_1(\Psi \circ I).$$

Furthermore, the natural injection from $\boldsymbol{K}_1(I)$ *into* $\boldsymbol{K}_1(\Psi \circ I)$ *is continuous.*

If I is a Daniell integral, then $\Psi \circ I$ *is also a Daniell integral.*

Proposition 2.11. *Let* $(X, \sigma(X, X'))$ *be the space* X *endowed with the weak topology* $\sigma(X, X')$. *Let* $\iota: X \to (X, \sigma(X, X'))$ *be the identity map. Then*

i) $\boldsymbol{K}_1 \subset \boldsymbol{K}_1(\iota \circ I) \subset \bigcap_{x' \in X'} \boldsymbol{K}_1(x' \circ I)$; *and*

(ii) *if* $p \in P(X)$ *and* $f \in \boldsymbol{K}_1(I)$, *then*

$$\hat{p}(I_1)(f) = \sup \{ |(x' \circ I)_1|(f(: x' \in U_p^{\prime}\}.$$

Proof. Statement (i) follows from Propositions 2.3 and 2.10. Since $x' \circ I_1 = (x' \circ I)_{1'}$ on $\boldsymbol{K}_1(I)$ for every $x' \in X'$, Statement (ii) follows from (1).

The space $\boldsymbol{K}_1$ is not always complete as the following example shows.

Example 2.12. For any uncountable set Ω, let $X = \mathbf{R}^{\Omega}$ be equipped with the product topology. Let

$$L = \{f \in \mathbf{R}^{\Omega}: S(f) \text{ is a finite subset of } \Omega\}.$$

Define $I: \boldsymbol{L} \to X$ to be the natural injection. Then I is a Daniell integral. Note that we can regard it as a Radon measure. In fact, let Ω be endowed with the discrete topology; then $C_{\mathscr{C}}(\Omega) = \boldsymbol{L}$ and I is a Radon measure.

Clearly,

$$\boldsymbol{F} = \boldsymbol{K}_1 = \{f \in \mathbf{R}^{\Omega}: S(f) \text{ is countable}\}.$$

The space $\boldsymbol{K}_1$ is not quasi-complete with respect to the topology of convergence in mean.

Even if the space X is not metrizable, $\boldsymbol{K}_1$ or $\boldsymbol{F}$ can be metrizable.

Example 2.13. Let $\boldsymbol{L} = l_1$, and let $X = (l_\infty)'$ equipped with the weak* topology. Then the natural injection from $\boldsymbol{L}$ into X is a Daniell integral for which $\boldsymbol{F} = \boldsymbol{K}_1 = l_1$. The topology of the convergence in mean on $\boldsymbol{L}$ coincides with the l_1-norm topology although X is not metrizable.

The Daniell integral in the following example does not satisfy the equality:

$$\boldsymbol{K}_1(I) = \bigcap_{x' \in X'} \boldsymbol{K}_1(x' \circ I).$$

Example 2.14. Let $\boldsymbol{L}$ denote the Riesz space $C\,[0, 1]$. Let X be the Banach space $C[0, 1]$ with the uniform norm. Then the identity map from $\boldsymbol{L}$ onto X is a Daniell integral.

3. Beppo Levi theorem

Let Ω be a set and $\boldsymbol{L}$ a Riesz subspace of $\mathbf{R}^{\Omega}$. Let X be a quasi-complete locally convex Hausdorff space.

Let Λ be an index set. Given subsets W_λ of X, $\lambda \in \Lambda$, we say that the series $\sum_{\lambda \in \Lambda} W_\lambda$ is convergent if the series $\sum_{\lambda \in \Lambda} x_\lambda$ is convergent in X for any choice of $x \in W_\lambda$, $\lambda \in \Lambda$.

For any subset W of X let

$$p(W) = \sup \{p(x) : x \in W\}.$$

Lemma 3.1. [8: Lemma 7.1]. *If W_n, $n \in \mathbf{N}$, are subsets of X and if the series $\sum_{n=1}^{\infty} W_n$ is convergent, then the sequence $\left\{p\left(\sum_{i=1}^{n} W_i\right)\right\}_{n \in \mathbf{N}}$ is convergent for every $p \in P(X)$.*

Suppose that $I: \mathbf{L} \to X$ is a Stone integral. For a subset $\mathbf{V}$ of $\mathbf{L}$ and a function $f \in \mathbf{L}$ put

$$I(\mathbf{V}, f) = \{I(g) : g \in \mathbf{V}, |g| \leq |f|\}.$$

Proposition 3.2. *If $I: \mathbf{L} \to X$ is a Stone integral and if $f_n \in \mathbf{L}$, $n \in \mathbf{N}$, are functions such that the series*

$$\sum_{n=1}^{\infty} I(\mathbf{L}, f_n) \tag{8}$$

is convergent, then the series

$$\sum_{n=1}^{\infty} |f_n(\omega)| \tag{9}$$

is convergent for a.a. $\omega \in \Omega$.

Proof. Let A denote the set of all $\omega \in \Omega$ for which the series (9) is divergent. Let $p \in P(X)$. Since the equality

$$p(I)\left(\sum_{i=m}^{n} |f_i|\right) = p\left(\sum_{i=m}^{n} I(\mathbf{L}, f_i)\right)$$

holds for all natural numbers m, n such that $m \leq n$, Lemma 3.1 implies that the sequence $\left\{\sum_{i=1}^{n} |f_i|\right\}_{n \in \mathbf{N}}$ is $p(I)$-Cauchy. If we let $g_n = \sum_{i=1}^{n} |f_i|$ for every $n \in \mathbf{N}$, then there exists an increasing sequence $\{n(k)\}_{k \in \mathbf{N}}$ such that

$$p(I)(g_{n(k+1)} - g_{n(k)}) < 2^{-k}.$$

Since

$$A \leq \sum_{k=i}^{\infty} (g_{n(k+1)} - g_{n(k)}),$$

Proposition 2.1 implies that $\sigma_p(A) < 2^{1-i}$ for every $i \in \mathbf{N}$. Hence A is an I-null set.

Theorem 3.3. *Let $I: \boldsymbol{L} \to X$ be a Stone-closed Stone integral. If $f_n \in \boldsymbol{L}$, $n \in \mathbf{N}$, are functions such that the series* (8) *is convergent and if $f \in \mathbf{R}^\Omega$ is a function such that*

$$f(\omega) = \sum_{n=1}^{\infty} f_n(\omega) \tag{10}$$

for a.a. $\omega \in \Omega$, then $f \in \boldsymbol{L}$ and

$$f = \sum_{n=1}^{\infty} f_n, \tag{11}$$

where (11) *is convergent in the mean convergent topology on $\boldsymbol{L}$.*

Proof. Let $p \in P(X)$. Following the notation in the proof of Proposition 3.2, we have

$$\sigma_p\left(f - \sum_{i=1}^{n} f_i\right) \leq \sum_{j=k}^{\infty} p(I)(g_{n(j+1)} - g_{n(j)}) < 2^{1-k}$$

for every natural number $n \geq n(k)$, $k \in \mathbf{N}$. Hence $f \in \boldsymbol{L}$, and (11) holds in mean.

Corollary 3.4. *Let $I: \boldsymbol{L} \to X$ be a Stone integral such that $\boldsymbol{L}$ is sequentially complete and includes all I-null functions. If $f_n \in \boldsymbol{L}$, $n \in \mathbf{N}$, are functions such that the series* (8) *is convergent, and if $f \in \mathbf{R}^\Omega$ is a function such that* (10) *holds for a.a. $\omega \in \Omega$, then $f \in \boldsymbol{L}$ and* (11) *holds in $\boldsymbol{L}$.*

Proof. Since the sequence $\left\{\sum_{i=1}^{n} f_i\right\}_{n \in \mathbf{N}}$ is Cauchy in $\boldsymbol{L}$, it is convergent to some function $g \in \boldsymbol{L}$. On the other hand, by Theorem 3.3, the function f belongs to $\boldsymbol{K}_1$ and (11) holds in $\boldsymbol{K}_1$. Thus $f = g$ a.e., which implies that $f = (f-g) + g \in \boldsymbol{L}$.

Consequently (11) holds in $\boldsymbol{L}$.

The conclusion of Theorem 3.3 does not always imply that I is Stoneclosed.

Example 3.5. Let $\Omega = [0, 1]$ and $\boldsymbol{B}$ the Borel field on Ω. Let ν be the Lebesgue measure on Ω. Let $\boldsymbol{L} = \boldsymbol{L}_1(\Omega, \boldsymbol{M}, \nu)$. Define the Daniell integral I: $\boldsymbol{L} \to \mathbf{R} \times \mathbf{R}^\Omega$ by

$$I(f) = \left(\int_\Omega f \, \mathrm{d}\nu, (f(\omega))_{\omega \in \Omega}\right)$$

for every $f \in \boldsymbol{L}$. Then $\boldsymbol{L}$ is sequentially complete with respect to the mean convergence topology. Clearly, $\boldsymbol{N}(I) = \{0\}$. Hence, the conclusion of Theorem 3.3 holds. But I is not Stone-closed; in fact, $\boldsymbol{K}_1(I) = \boldsymbol{L}_1(\Omega, \overline{\boldsymbol{B}}, \nu)$, where $\overline{\boldsymbol{B}}$ is the completion of $\boldsymbol{B}$ with respect to ν.

It seems to be open whether the conclusion of Theorem 3.3 implies the sequentially completeness of $\boldsymbol{L}$. Note that if the conclusion of Theorem 3.3 is valid and if the quotient space $\boldsymbol{L}/\boldsymbol{N}(I)$ is Dedekind complete, then $\boldsymbol{L}$ is sequentially complete (cf. (1: Exercise 7.9 and Theorem 13.2]).

4. Stone extension of the Daniell integral

Let Ω be a set and $\mathbf{L}$ a Riesz subspace of $\mathbf{R}^{\Omega}$. Let I be a Daniell integral from $\mathbf{L}$ into a quasi-complete locally convex Hausdorff space X.

The purpose of this section is to show that the Stone extension I_1 of I is a Daniell integral with values in X.

Lemma 4.1. *Let $p \in P(X)$ and $f \in \mathbf{R}^{\Omega}$. If there exist functions $f_n \in \mathbf{R}^{\Omega}$, $n \in \mathbf{N}$, such that $f(\omega) \leq \limsup |f_n(\omega)|$ for every $\omega \in \Omega$ and the series $\sum_{n=1} \sigma_p(f_n)$ is convergent, then* (7) *holds.*

Proof is immediate from the inequalities

$$f(\omega) \leq \limsup_{n\to\infty} |f_n(\omega)| \leq \sum_{n=N} |f_n(\omega)|$$

for every $\omega \in \Omega$ and every $N \in \mathbf{N}$.

Lemma 4.2. *Let $p \in P(X)$. Suppose that $\{f_n\}_{n\in\mathbf{N}}$ is a decreasing sequence of functions in $\mathbf{L}^+$. If $f \in \mathbf{R}^{\Omega}$ is the pointwise limit of $\{f_n\}_{n\in\mathbf{N}}$ and satisfies* (7), *then the sequence $\{p(I(f_n))\}_{n\in\mathbf{N}}$ is convergent to* 0.

Proof. The equality (7) implies that, for every $\varepsilon > 0$, there exist functions $g_n \in \mathbf{L}^+$, $n \in \mathbf{N}$, such that

$$f < \sum_{1}^{\infty} g_n \quad \text{and} \quad \sum_{n=1}^{\infty} p(I)(g_n) < \varepsilon .$$

Then, for every $n \in \mathbf{N}$, the inequality

$$f_n < \left(f_n - \sum_{i=1}^{n} g_i\right)^+ + 2\sum_{i=1}^{n} g$$

is valid. Since the sequence $\left\{f_n - \sum_{i=1}^{n} g_i\right\}_{n\in\mathbf{N}}$ is decreasing and pointwise convergent to 0, we have

$$\limsup_{n\to} p(I(f_n)) \leq 2\varepsilon .$$

Lemma 4.3. *If a sequence $\{f_n\}_{n\in\mathbf{N}}$ in $\mathbf{K}_1(I)$ is decreasing and pointwise convergent to* 0, *then the sequence $\{I_1(f_n)\}_{n\in\mathbf{N}}$ is convergent to* 0 *in X.*

Proof. Given $p \in P(X)$ and $\varepsilon > 0$, there exists a decreasing sequence $\{g_n\}_{n\in\mathbf{N}}$ in $\mathbf{L}^+$ such that $\sigma_p(f_n - g_n) < \varepsilon 2^{-n}$ for every $n \in \mathbf{N}$. For every $\omega \in \Omega$, let

$$f(\omega) = \inf\{g_n(\omega) : n \in \mathbf{N}\} .$$

Since $f(\omega) \leq \limsup |g_n(\omega) - f_n(\omega)|$ for every $\omega \in \Omega$, Lemma 4.1 and the inequality

$$\sum_{n=1}^{\infty} \sigma_p(g_n - f_n) < \varepsilon$$

imply that (7) holds. For every $n \in \mathbf{N}$ we have

$$\hat{p}(I_1(f_n)) \leq \sigma_p(f_n - g_n) + p(I(g_n)),$$

so that Lemma 4.2 applies.

Lemma 4.4. *Let I_1 be the Stone extension of I. Then the image $I_1(\boldsymbol{K}_1(I))$ is included in X.*

Proof. Let $f \in \boldsymbol{K}_1^+$. Then there exist functions $f_n \in \boldsymbol{L}^+$, $n \in \mathbf{N}$, such that

$$f \leq \sum_{n=1}^{\infty} f_n . \tag{12}$$

Let $g_n = \sum_{i=1}^{n} f_i$ for every $n \in \mathbf{N}$. Then the sequence $\{g_n \wedge f\}_{n \in \mathbf{N}}$ is increasing and pointwise convergent to f. Take a net $\{f_\gamma\}_{\gamma \in \Gamma}$ from $\boldsymbol{L}$ which is convergent to f in $\boldsymbol{K}$. Then, for every $n \in \mathbf{N}$, we have

$$\lim_{\gamma \in \Gamma} f_\gamma \wedge g_n = f \wedge g_n$$

in $\boldsymbol{K}_1$, so that the limit $I_1(f \wedge g_n)$ of the bounded net $\{I_1(f_\gamma \wedge g_n)\}_{\gamma \in \Gamma}$ lies in X. Lemma 4.3 implies that the sequence $\{I_1(f \wedge g_n)\}_{n \in \mathbf{N}}$ is convergent to $I_1(f)$ in $\hat{X}$; therefore, $I_1(f)$ belongs to X.

We can now write $p(I_1)$ instead of $\hat{p}(I_1)$ for every $p \in P(X)$ because I_1 maps $\boldsymbol{K}_1$ into X.

The results of thls section are sumarized in

Theorem 4.5. *Let $I: \boldsymbol{L} \to X$ be a Daniell integral. Then its Stone extension $I_1: \boldsymbol{K}_1(I) \to X$ is also a Daniell integral.*

5. Saturability

Let Ω be a set and $\boldsymbol{L}$ a Riesz subspace of $\mathbf{R}^{\Omega}$. Let X be a quasi-complete locally convex Hausdorff space.

A Stone integral $I: \boldsymbol{L} \to X$ is called saturable if for every decreasing sequence $\{f_n\}_{n \in \mathbf{N}}$ in $\boldsymbol{L}^+$ the sequence $\{I(f_n)\}_{n \in \mathbf{N}}$ converges weakly in X. The Orlicz—Pettis theorem ensures that the sequence $\{I(f_n)\}_{n \in \mathbf{N}}$ is then convergent in X with respect to the Mackey topology.

Proposition 5.1. *Every saturable Stone integral is a Daniell integral.*

Proof follows from Theorem 3.3.

The integrals in Examples 2.12, 2.13 and 3.5 are saturable; the integral in Example 2.14 is not saturable.

Lemma 5.2. *Let I: $\mathbf{L} \to X$ be a saturable Daniell integral, and let $\{f_n\}_{n \in \mathbf{N}}$ be an increasing sequence of non-negative function in $\mathbf{K}_1(I)$ with an upper bound $g \in \mathbf{L}$. Then the pointwise limit f of $\{f_n\}_{n \in \mathbf{N}}$ belongs to $\mathbf{K}_1(I)$ and the sequence $\{I_1(f_n)\}_{n \in \mathbf{N}}$ is convergent to $I_1(f)$ in X.*

Proof. Given $\varepsilon > 0$ and $p \in P(X)$, there exists an increasing sequence $\{g_n\}_{n \in \mathbf{N}}$ of functions in $\mathbf{L}^+$ such that

$$g_n \leqslant g \text{ and } p(I_1)(f_n - g_n) < \varepsilon$$

for every $n \in \mathbf{N}$. Since I is saturable, the sequence $\{I(f_n)\}_{n \in \mathbf{N}}$ which is Cauchy in the quasi-complete space X, is convergent there. Theorem 3.3 now applies.

Let $\mathbf{A}$ be a family of subsets of Ω. The space of all $\mathbf{A}$-simple functions on Ω is denoted by sim $(\mathbf{A})$. For every set A in $\mathbf{A}$, let

$$A \cap \mathbf{A} = \{B \in \mathbf{A} : B \subset A\}$$

For a Daniell integral I: $\mathbf{L} \to X$ let

$$\mathbf{R}(I) = \{A \subset \Omega : A \in \mathbf{K}_1(I)\} \tag{13}$$

Lemma 5.3. *If I: $\mathbf{L} \to X$ is a saturable Daniell integral, then*

(i) *$\mathbf{R}(I)$ is a ring,*

(ii) *if the constant function 1 belongs to $\mathbf{L}$, then $\mathbf{R}(I)$ is a σ algebra;*

(iii) *if $\mathbf{L} \wedge 1$ is included in $\mathbf{L}$, then every function in $\mathbf{L}^+$ is the pointwise limit of some non-negative, increasing sequence in sim $(\mathbf{R}(I))$, where*

$$\mathbf{L} \wedge 1 = \{f \wedge 1 : f \in \mathbf{L}\}.$$

Proof. Statement (i) holds since $\mathbf{K}_1$ is a Riesz space.

Statement (ii) follows from Lemma 5.2.

To prove (iii), let $f \in \mathbf{L}^+$. For a positive number α let $A = f^{-1}((\alpha, \infty))$. Given every $n \in \mathbf{N}$, let

$$f_n = (n(f - f \wedge \alpha)) \wedge 1;$$

then $f_n \in \mathbf{L}^+$ and $f_n \leqslant f - \alpha$. Since the sequence $\{f_n\}_{n \in \mathbf{N}}$ is increasing and pointwise convergent to A, Lemma 5.2 ensures that $A \in \mathbf{K}_1$. Hence if $b \in \mathbf{R}$ and $b > \alpha$, then $\{\omega \in \Omega : \alpha < f(\omega) \leqslant b\}$ belongs to $\mathbf{R}(I)$. Given $n \in \mathbf{N}$, let

$$A_k^n = \{\omega \in \Omega : (k-1)2^{-n} < f(\omega) \leqslant k2^{-n}\}$$

for every $k \in \mathbf{N}$, $2 \leqslant k < 2^n$, and let

$$g_n = \sum_{k=1}^{2^n} (k-1)2^{-n} A_k^n .$$

Then the sequence $\{g_n\}_{n \in \mathbf{N}}$ in sim $(\mathbf{R}(I))$ is increasing and pointwise convergent to f.

We say that an additive set function on a ring of sets, with values in a locally convex space, is a vector measure.

Lemma 5.4. *If the constant function* 1 *belongs to* $\boldsymbol{L}$ *and if* $I: \boldsymbol{L} \to X$ *is a saturable Daniell integral, then the set function* $\mu(I): \boldsymbol{R}(I) \to X$ *defined by*

$$\mu(I)(A) = I_1(A), \quad A \in \boldsymbol{R}(I) \tag{14}$$

is a σ*-additive vector measure.*

Proof follows from Theorem 4.5 and Lemma 5.3.

The following lemma is due to [11: Theorem on Extension].

Lemma 5.5. *Let* $\boldsymbol{R}$ *be a ring of subsets of* Ω. *For a scalarly* σ*-additive vector measure* $\mu: \boldsymbol{R} \to X$, *the following statements are equivalent:*

(i) μ *extensible to an* X*-valued vector measure* $\hat{\mu}$ *on the* δ*-ring* $\delta(\boldsymbol{R})$ *generated by* $\boldsymbol{R}$;

(ii) *for every* $A \in \boldsymbol{R}$, *the set* $\mu(A \cap \boldsymbol{R})$ *is relatively weakly compact in* X.

If (i) *or* (ii) *holds, then* $\hat{\mu}(B \cap \delta(\boldsymbol{R}))$ *is a relatively weakly compact set in* X *for every* $B \in \delta(\boldsymbol{R})$.

Given a subset V of X, its balanced convex hull is denoted by bco V. A characterization of saturability is given in the following

Theorem 5.6. *A Stone integral* $I: \boldsymbol{L} \to X$ *is saturable if and only if, for every function* $f \in \boldsymbol{L}$, *the set* $I(\boldsymbol{L}, f)$ *is relatively weakly compact in* X.

Proof. The 'if' part is obvious (cf. [8: Théorème 4.4]).

Suppose now that I is a saturable integral. Fix a function $f \in \boldsymbol{L}^+$. The Riesz space

$$\boldsymbol{M}(f) = \{g/f: g \in \boldsymbol{L}, |g| \leq af \text{ for some } \alpha \in \mathbf{R}\}$$

contains the constant function 1. Let us define the saturable Daniell integral $J: \boldsymbol{M}(f) \to X$ by

$$J(g/f) = I(g)$$

for every $g/f \in \boldsymbol{M}(f)$. Then, without loss of generality we can assume that $f = 1$ since $J(\boldsymbol{M}(f), 1) = I(\boldsymbol{L}, f)$.

By Lemmas 5.3 and 5.4, the set function $\mu(I)$ from the σ-algebra $\boldsymbol{R}(I)$ into X is a σ-additive vector measure. Thus it follows from Lemma 5.5 that the set $\mu(I)(\boldsymbol{R}(I))$ is relatively weakly compact. Let $g \in \boldsymbol{L}$ such that $0 \leq g \leq 1$. By Lemma 5.3, there exist non-negative functions $g_n \in \text{sim}\,(\boldsymbol{R}(I))$, $n \in \mathbf{N}$, such that the sequence $\{I_1(g_n)\}_{n \in \mathbf{N}}$ is convergent to $I_1(g)$ in X. Since every $I_1(g_n)$ belongs to bco $\mu(I)(\boldsymbol{R}(I))$ by Abel's summation, the set $I(\boldsymbol{L}^+, f)$ is included in the closure of bco $\mu(I)(\boldsymbol{R}(I))$. From the Krein theorem (cf. [13: 24.4 (4′)]) it follows that $I(\boldsymbol{L}, 1)$ is a relatively weakly compact set.

Lemma 5.7 ([8: Lemme 1.2]). *Let W_r, $n \in \mathbf{N}$, be non empty sub ets of X. If the series $\sum_{n \ 1}^{\infty} W_n$ is convergent in X, then o is the rie $\sum_{n \ 1}^{\infty} W_n$*

Lemma 5.8. *Let Λ be an index s t. If the s ries $\sum_{\Lambda} W_\lambda$ of compact subsets of X is convergent, then $\sum_{\lambda \in \Lambda} W_\lambda$ is a compact ub et f X.*

Proof. We use the map Φ from the Cartes'an product W of sets W_λ, $\lambda \in \Lambda$, into X defined by $\Phi((x_\lambda)) \quad \sum_{\lambda \in \Lambda} x$ for every $(x_\lambda) \in W$. Since Φ is continuous, the image $\Phi(W) = \sum_{\lambda \in \Lambda} W_\lambda$ is compact

The following lemma is an application of [7: 17 12].

Lemma 5.9. *A complete subset A of X is weakly compact if and only if $\pi \ (A)$ is weakly compact in X for every $p \in P(X)$.*

In the case of Banach space valued Daniell integrals, the following theorem has been proved by Kluvánek [8: Théorème 4.1].

Theorem 5.10. *A Daniell integral $I: \mathbf{L} \to X$ is saturable if and only if its Stone extension is.*

Proof. If the Stone extension I_1 is saturable, then it is clear that I is saturable

Conversely, suppose that I is aturable. Fir t as ume that is a Banach space with norm $\| \ \|$. For brevity, let $\sigma \quad \sigma_{\| \|}$ Fix a function $f \in$ Choose functions $f_n \in \mathbf{L}^+$, $n \in \mathbf{N}$, such that (12) holds and

$$\sum_{n \ 1}^{\infty} \sigma(f_n) < \sigma(f) \quad 1 .$$

If g_i is a function in $\mathbf{K}_1$ such that $|g \ | < f_i$ for every $i \in \mathbf{N}$, then

$$\left\| I_1 \left(\sum_{i \ n}^{m} g_i \right) \right\| < \sigma\left(\sum_{n}^{m} g_i \right) \leq \sum_{i \ n}^{m} \sigma(f_i)$$

whenever $m \in \mathbf{N}$ and $n \in \mathbf{N}$, $m > n$. Hence the eries

$$\sum_{n \ 1}^{\infty} I_1(\mathbf{K}_1, f \ ) \tag{15}$$

is convergent in X. Given $n \in \mathbf{N}$, the set $I_1(\mathbf{K}_1, f_n)$, which is included in the closure of $I(\mathbf{L}, f_n)$, is relatively weakly compact in X by Theorem 5.6. By Lemmas 5.7 and 5.8, the set (15) is relatively weakly compact. To prove that the set $I_1(\mathbf{K}_1, f)$ is included in the set (15), take a function $h \in \mathbf{K}_1^+$ for which $h < f$. For every $n \in \mathbf{N}$ let

$$h_n \quad \left(\sum_{i \ 1}^{n} f_i \right) \wedge h \quad \left(\sum_{i \ 1}^{n \ 1} f_i \right) \wedge h ,$$

where $\sum_{i=1}^{0} f_i = 0$. Then $h_n \in \mathbf{K}_1^+$, $h_n \leqslant f_n$ for every $n \in \mathbf{N}$, and $\sum_{n=1}^{\infty} h_n = h$. Consequently, it follows that $I_1(h) = \sum_{n=1}^{\infty} I_1(h_n)$ lies in the set (15).

Now let X be an arbitrary quasi-complete space. The above argument asserts that since $\pi_p \circ I_1 = (\pi_p \circ I)_1$ on $\mathbf{K}(I)$, the set $\pi_p(I_1(\mathbf{K}_1, f))$ is relatively weakly compact in X_p for every $p \in P(X)$. Then, by Lemma 5.9, the set $I_1(\mathbf{K}_1, f)$ is relatively weakly compact in X. Thus the Stone extension I_1 is saturable by Theorem 5.6.

Corollary 5.11. *A Daniell integral $I: \mathbf{L} \to X$ is saturable if and only if the sequence $\{I(f_n)\}_{n \in \mathbf{N}}$ is summable in X for every sequence $\{f_n\}_{n \in \mathbf{N}}$ of functions in $\mathbf{L}$ such that there exists a function $f \in \mathbf{L}$ which satisfies that*

$$0 \leqslant \sum_{n=1}^{\infty} f_n(\omega) \leqslant f(\omega)$$

for a.a. $\omega \in \Omega$.

Now we give the Lebesgue convergence theorem with respect to a Daniell integral.

Theorem 5.12. *A Daniell integral $I: \mathbf{L} \to X$ is saturable if and only if it satisfies the following condition:*

(LC) *If $\{f_n\}_{n \in \mathbf{N}}$ is a sequence of functions in $\mathbf{K}_1(I)$ which converges to a function $F \in \mathbf{R}^{\Omega}$ a.e. in Ω, and if there is a function $g \in \mathbf{K}(I)$ such that $|f_n| \leqslant g$ a.e. for every $n \in \mathbf{N}$, then f belongs to $\mathbf{K}_1(I)$ and the sequence $\{f_n\}_{n \in \mathbf{N}}$ is convergent to f in $\mathbf{K}_1(I)$.*

Proof. It suffices the prove the 'only if' part. By Proposition 2.9, we may assume that $f(\omega) = \lim f_n(\omega)$ for every $\omega \in \Omega$ and that $|f_n(\omega)| \leqslant g(\omega)$ for every $\omega \in \Omega$ and every $n \in \mathbf{N}$.

First suppose that the sequence $\{f_n\}_{n \in \mathbf{N}}$ is increasing. Then the condition (LC) holds by Theorem 3.3.

Similarly the condition (LC) holds even if the sequence $\{f_n\}_{n \in \mathbf{N}}$ is decreasing. In the general case, let

$$g_n(\omega) = \inf \{f_i(\omega) : i \in \mathbf{N}, i \geqslant n\} \quad \text{and} \quad h_n(\omega) = \sup \{f_i(\omega) : i \in \mathbf{N}, i \geqslant n\}$$

for every $\omega \in \Omega$ and every $n \in \mathbf{N}$. From the above arguments we have $g_n, h_n \in \mathbf{K}_1$ for every $n \in \mathbf{N}$. Since $|g_n| \leqslant g$ for every $n \in \mathbf{N}$ and since the sequence $\{g_n\}_{n \in \mathbf{N}}$ is increasing and pointwise convergent to f, the first step assures us that $f \in \mathbf{K}_1$. Further, since the sequence $\{h_n - g_n\}_{n \in \mathbf{N}}$ is decreasing and pointwise convergent to 0, the sequence $\{p(I_1)(h_n - g_n)\}_{n \in \mathbf{N}}$ is convergent to 0 for every $p \in P(X)$. Consequently, $f_n \to f$ as $n \to \infty$ in $\mathbf{K}_1$ since $|f - f_n| \leqslant h_n - g_n$ for every $n \in \mathbf{N}$. Thus the condition (LC) holds.

Corollary 5.13. *If* $I: \boldsymbol{L} \to X$ *is a saturable Daniell integral, then* $\boldsymbol{K}_1(I)$ *is a Dedekind* σ*-complete Riesz space.*

A subset of a locally convex space Y is called quasi-closed if it contains all limit points of its bounded subsets. For a subset M of Y, the intersection of all quasi-closed subsets of Y which include M is said to be the quasi-closure of M (cf. [13: 23.1]).

Corollary 5.14. *If* $I: \boldsymbol{L} \to X$ *is a saturable Daniell integral, then* $\boldsymbol{K}_1(I)$ *is equal to the quasi-closure of* $\boldsymbol{L}$ *in* $\boldsymbol{F}(I)$.

Proof. Fix a non-negative function $f \in \boldsymbol{K}_1$. Take a sequence $\{g_n\}_{n \in \mathbf{N}}$ and a net $\{f_\gamma\}_{\gamma \in \Gamma}$ from $\boldsymbol{L}^+$ as in the proof of Lemma 4.4. Then, since

$$f = \lim_{n \to \infty} \lim_{\gamma \in \Gamma} f_\gamma \wedge g_n$$

in $\boldsymbol{K}_1$, the function f belongs to the quasi-closure of $\boldsymbol{L}$ in $\boldsymbol{F}$.

6. Direct sum

Let Λ be an index set. Let $\{\Omega_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint sets and Ω its union. For every $\lambda \in \Lambda$, let $\boldsymbol{L}_\lambda$ be a Riesz subspace of $\mathbf{R}^{\Omega_\lambda}$; then we may regard $\boldsymbol{L}_\lambda$ as a Riesz subspace of $\mathbf{R}^\Omega$.

Let X be a quasi-complete locally convex Hausdorff space. For every $\lambda \in \Lambda$ let $I_\lambda: \boldsymbol{L}_\lambda \to X$ be a Stone integral. We denote by $\boldsymbol{L}$ the Riesz subspace of $\mathbf{R}^\Omega$ which consists of the functions $f \in \mathbf{R}^\Omega$ such that $f\Omega_\lambda$ belongs to $\boldsymbol{L}_\lambda$ for every $\lambda \in \Lambda$ and the series

$$\sum_{\lambda \in \Lambda} I_\lambda(\boldsymbol{L}_\lambda, f\Omega_\lambda) \tag{16}$$

is convergent in X.

Define the map $I: \boldsymbol{L} \to X$ by

$$I(f) = \sum_{\lambda \in \Lambda} I_\lambda(f\Omega_\lambda)$$

for every $f \in \boldsymbol{L}$. This map I is called the direct sum of the family $\{I_\lambda\}_{\lambda \in \Lambda}$ of Stone integrals.

Proposition 6.1. *If* $I_\lambda: \boldsymbol{L}_\lambda \to X$, $\lambda \in \Lambda$, *are Daniell integrals, then their direct sum* $I: \boldsymbol{L} \to X$ *is a Daniell integral.*

Proof. We take functions $F \in \boldsymbol{L}^+$ and $f_n \in \boldsymbol{L}^+$, $n \in \mathbf{N}$, such that (10) holds for every $\omega \in \Omega$. The Orlicz—Pettis theorem implies that

$$I(f) = \sum_{\lambda \in \Lambda} I_\lambda\Big(\sum_{n=1}^{\infty} (f_n \Omega_\lambda)\Big) = \sum_{n=1}^{\infty} \sum_{\lambda \in \Lambda} I_\lambda(f_n \Omega_\lambda) = \sum_{n=1}^{\infty} I(f_n).$$

Consequently, I is a Daniell integral.

From Proposition 1.5 we have

Corollary 6.2. *The direct sum of a family of Stone integrals is a Stone integral.*

The family of all finite subsets of Λ is a directed set with respect to inclusion and denoted by $\boldsymbol{D}(\Lambda)$.

Proposition 6.3. *Let $I: \mathbf{L} \to X$ be the direct sum of Stone integrals $I_\lambda: \mathbf{L}_\lambda \to X$, $\lambda \in \Lambda$. Then, for every function $f \in \mathbf{L}$, the net*

$$\left\{ \sum_{\lambda \in \Lambda} (f\Omega_\lambda) \right\}_{\Delta \in \boldsymbol{D}(\Lambda)} \tag{17}$$

is convergent to f in $\mathbf{L}$; that is,

$$f = \sum_{\lambda \in \Lambda} (f\Omega_\lambda) \tag{18}$$

in the mean convergence topology on $\mathbf{L}$.

Proof. Let $p \in P(X)$. By applying Theorem 3.3, we can easily prove that the set

$$\Xi = \{\lambda \in \Lambda : p(I)(f\Omega_\lambda) > 0\}$$

is countable, that the function $\sum_{\lambda \in \Xi} (f\Omega_\lambda)$ defined pointwise belongs to $\boldsymbol{K}_1(I)$ and that, for every $\varepsilon > 0$, there exists a $\Delta \in \boldsymbol{D}(\Lambda)$ such that

$$p(I_1)\left(\sum_{\lambda \in \Xi \setminus \Delta} f\Omega_\lambda \right) < \varepsilon .$$

Thus, if g if a function in $\mathbf{L}$ such that $|g| \leq]hf - \sum_{\lambda \in \Delta} f\Omega_\lambda|$, then $p(I(g)) < \varepsilon$. Thus (18) holds in $\mathbf{L}$.

Proposition 6.4. *Let $I: \mathbf{L} \to X$ be the direct sum of Stone integrals $I_\lambda: \mathbf{L}_\lambda \to X$, $\lambda \in \Lambda$. Then $\mathbf{L}$ is quasi-complete if and only if $\mathbf{L}_\lambda$ is quasicomplete for every $\lambda \in \Lambda$.*

In particular, if X is complete, then $\mathbf{L}$ is complete if and only if $\mathbf{L}_\lambda$ is complete for every $\lambda \in \Lambda$.

Proof. Suppose that $\mathbf{L}$ is quasi-complete. Then, for every $\lambda \in \Lambda$, the space $\mathbf{L}_\lambda$ is also quasi-complete since $\mathbf{L}_\lambda / \mathbf{L}_\lambda \cap \boldsymbol{N}(I_\lambda)$ can be regarded as a closed subspace of the quasi-complete Hausdorff space

$$\mathbf{L} / \mathbf{L} \cap \boldsymbol{N}(I). \tag{19}$$

Suppose now that $\mathbf{L}_\lambda$ is quasi-complete for every $\lambda \in \Lambda$. To prove that $\mathbf{L}$ is quasi-complete, take a bounded Cauchy net $\{f^{(\gamma)}\}_{\gamma \in \Gamma}$ in $\mathbf{L}$. Without loss of generality, we may assume that $f^{(\gamma)} \geq 0$ for every $\gamma \in \Gamma$. For every $p \in P(X)$, there exists a real number M_p such that

$$p(I)(f^{(\gamma)}) \leqslant M_p, \ \gamma \in \Gamma. \tag{20}$$

Since $\{f^{(\gamma)}\Omega_\lambda\}_{\gamma \in \Gamma}$ is a bounded Cauchy net, it converges to a function f_λ in $\boldsymbol{L}_\lambda$ for every $\lambda \in$. Let

$$f(\omega) = \sum_{\lambda \in \Lambda} f_\lambda(\omega)$$

for every $\omega \in \Omega$. We shall show that $f \in \boldsymbol{L}$. For every $\lambda \in \Lambda$ let g_λ be a function in $\boldsymbol{L}_\lambda^+$ such that $g_\lambda \leqslant f_\lambda$. For every $\lambda \in \Lambda$ and every $\gamma \in \Gamma$ let $g_\lambda^{(\gamma)} = g_\lambda \wedge f^{(\gamma)}$. Then the net $\{g_\lambda^{(\gamma)}\}_{\gamma \in \Gamma}$ is convergent to g_λ in $\boldsymbol{L}_\lambda$ for every $\lambda \in \Lambda$. For every $\gamma \in \Gamma$ define the function $g^{(\gamma)} \in \boldsymbol{L}$ by

$$g^{(\gamma)}(\omega) = \sum_{\lambda \in \Lambda} g_\lambda^{(\gamma)}(\omega), \quad \omega \in \Omega.$$

Clearly $\{g^{(\gamma)}\}_{\gamma \in \Gamma}$ is a Cauchy net in $\boldsymbol{L}$; therefore, given $p \in P(X)$ and $\varepsilon > 0$, there exists a $\delta \in \Gamma$ such that

$$p\Big(\sum_{\lambda \in \Delta} I_\lambda(g_\lambda^{(\delta)}) - \sum_{\lambda \in \Delta} I_\lambda(g_\lambda)\Big) < \varepsilon$$

for every $\Delta \in \boldsymbol{D}(\Lambda)$. On the other hand, since the series $\sum_{\lambda \in \Lambda} I_\lambda(g_\lambda^{(\delta)})$ is convergent in X, there exists a $\Xi \in \boldsymbol{D}(\Lambda)$ such that if Δ, $\Theta \in \boldsymbol{D}(\Lambda)$ and Δ, $\Theta \supset \Xi$, then

$$p\Big(\sum_{\lambda \in \Delta} I_\lambda(g_\lambda^{\delta)}) - \sum_{\lambda \in \Theta} I_\lambda(g_\lambda^{(\delta)})\Big) < \varepsilon;$$

therefore

$$p\Big(\sum_{\lambda \in \Delta} I_\lambda(g_\lambda) - \sum_{\lambda \in \Theta} I_\lambda(g_\lambda)\Big) < 3\varepsilon.$$

Hence the net

$$\Big\{\sum_{\lambda \in \Delta} I_\lambda(g_\lambda)\Big\}_{\Delta \in \boldsymbol{D}(\Lambda)} \tag{21}$$

is Cauchy in X. Moreover, the net (21) is bounded. Indeed, (20) implies that

$$p\Big(\sum_{\lambda \in \Delta} I_\lambda(g_\lambda)\Big) \leqslant p(I)\Big(\sum_{\lambda \in \Delta} f_\lambda\Big) = \lim_{\gamma \in \Gamma} p(I)\Big(\sum_{\lambda \in \Delta} (f^{\gamma)}\Omega_\lambda)\Big) \leqslant M_p.$$

Thus the net (21) is convergent in X. In other words, the series (16) is convergent; therefore we have $f \in \boldsymbol{L}$.

We now claim that the net $\{f^{(\gamma)}\}_{\gamma \in \Gamma}$ is convergent to f in $\boldsymbol{L}$. Given $p \in P(X)$ and $\varepsilon > 0$, there exists an $\alpha \in \Gamma$ such that

$$p(I)(f^{(\gamma)} - f^{(\delta)}) < \varepsilon$$

for all $\gamma \in \Gamma$ and $\delta \in \Gamma$ such that $\gamma, \delta \geqq \alpha$. Fix a $\gamma \in \Gamma$ such that $\gamma \geqq \alpha$. Let h be a function in $\boldsymbol{L}$ such that $|h| \leqslant |f^{(\gamma)} - f|$. Then

$$p(I)\Big(\sum_{\lambda \in \Delta}(h\Omega_\lambda)\Big) \geqslant p(I)\Big(\sum_{\lambda \in \Delta}(f^{(\gamma)}\Omega_\lambda - f_\lambda)\Big) =$$
$$= \lim_{\delta \in \Gamma} p(I)\Big(\sum_{\lambda \in \Delta}(f^{(\gamma)} - f^{(\delta)})\Omega_\lambda\Big) \leqslant \varepsilon .$$

By Proposition 6.3, we have $p(I)(h) \leqslant \varepsilon$. That is, $p(I)(f^{(\gamma)} - f) \leqslant \varepsilon$.

If X is complete, then the completeness of $\boldsymbol{L}$ can be proved similarly.

From Theorem 5.6 and Lemma 5.8 we have

Proposition 6.5. *Let I be the direct sum of the family $\{I_\lambda\}_{\lambda \in \Lambda}$ of Stone integrals. Then I is saturable if and only if I_λ is saturable for every $\lambda \in \Lambda$.*

7. Vector measure

Let Ω be a set and let X be a quasi-complete locally convex Hausdorff space.

Let $\boldsymbol{R}$ be a ring of subsets of Ω and let $\mu: \boldsymbol{R} \to X$ be a vector measure. We say that μ is locally bounded if, for every $A \in \boldsymbol{R}$, the set $\mu(A \cap \boldsymbol{R})$ is bounded in X.

Let I_μ: sim $(\boldsymbol{R}) \to X$ be the linear map which extends μ.

Proposition 7.1. *The map I_μ is a Stone integral if and only if μ is a locally bounded, scalarly σ-additive vector measure.*

Proof. Suppose that I_μ is a Stone integral. Then, by Proposition 1.5 the measure μ is locally bounded and scalarly σ-additive.

Conversely suppose that μ is locally bounded and scalarly σ-additive. Fix any functional $x' \in X'$. To prove that $x' \circ I_\mu$ is a Daniell integral, take a sequence $\{f_n\}_{n \in \mathbf{N}}$ of non-negative fun tions in sim $(\boldsymbol{R})$, which is decreasing and pointwise convergent to 0. There exists a set $A \in \boldsymbol{R}$ such that

$$S(f_1) \subset A. \tag{22}$$

Since $x' \circ \mu$ can be extended to a σ-additive measure on the σ-algebra $A \cap \delta(\boldsymbol{R})$ by Lemma 5.5, the sequence $\{x' \circ I_\mu(f_n)\}_{n \in \mathbf{N}}$ is convergent to 0. Thus I_μ is a Stone integral.

The proof of the following proposition is straightforward.

Proposition 7.2. *If the map I_μ is a Daniell integral, then μ is a σ-additive vector measure.*

Problem. If μ is a σ-additive vector measure, is I_μ a Daniell integral?

The following example shows that not every Stone integral is a Daniell integral.

Example 7.3. For every $n \in \mathbf{N}$ let e_n be the unit vector with the n-th co-ordinate one. Let $\boldsymbol{R}$ be the ring of all finite subsets of $\mathbf{N}$ and their complements. Define the vector measure $\mu: \boldsymbol{R} \to c_0$ by letting

$$\mu(A)) = \sum_{n \in A} (e_n - e_{n-1})$$

for every finite set $A \in \boldsymbol{R}$ and $\mu(B) = -\mu(\mathbf{N} \setminus B)$ for every cofinite set $B \in \boldsymbol{R}$, where $e_0 = 0$. Then μ is locally bounded and scalarly σ-additive. But μ is not σ-additive. Hence I_μ: sim $(\boldsymbol{R}) \to c_0$ is not a Daniell integral but a Stone integral.

Proposition 7.4. *The map I_μ is a saturable Daniell integral if and only if μ can be extended to a σ-additive measure on the δ-ring $\delta(\boldsymbol{R})$.*

Proof. If I_μ is a saturable Daniell integral, then Lemma 5.5 and Theorem 5.6 ensure that μ can be extended to a σ-additive measure on $\delta(\boldsymbol{R})$.

Suppose now that μ is extensible to a σ-additive vector measure on $\delta(\boldsymbol{R})$, which we denote also by μ. Let $\{f_n\}_{n \in \mathbf{N}}$ be a sequence of non-negative functions in sim $(\boldsymbol{R})$ which is decreasing and pointwise convergent to 0. Choose a set $A \in \boldsymbol{R}$ which satisfies (22). Since μ is σ-additive on the σ-algebra $A \cap \delta(\boldsymbol{R})$, the Lebesgue convergence theorem for a vector measure (cf. [12: Theorem II.4.2]) implies that the sequence $\{I_\mu(f_n)\}_{n \in \mathbf{N}}$ is convergent to 0. In other words, I_μ is a Daniell integral. Given a function $f \in$ sim $(\boldsymbol{R})$, it follows from Abel's summation that the set

$$I_\mu \,(\text{sim}\,(\boldsymbol{R}), f) \tag{23}$$

is a linear combination of sets of the form:

$$\text{bco}\, \mu(B \cap \boldsymbol{R}), \quad B \in \boldsymbol{R}.$$

Thus, by Lemma 5.5 and the Krein theorem, the set (23) is relatively weakly compact.

Let $\boldsymbol{M}(\boldsymbol{R})$ be the linear space of the functions $f \in \mathbf{R}^\Omega$ such that there exist non-negative functions f_n, $g_n \in$ sim $(\boldsymbol{R})$, $n \in \mathbf{N}$, for which the sequence $\{f_n\}_{n \in \mathbf{N}}$ is increasing and pointwise convergent to f^+ and the sequence $\{g_n\}_{n \in \mathbf{N}}$ to f^-.

Suppose first that $\boldsymbol{R}$ is a σ-algebra. Recall that a function $f \in \boldsymbol{M}(\boldsymbol{R})$ is said to be μ-integrable if f is $\langle x', \mu \rangle$-integrable for every $x' \in X'$ and if, given $A \in \boldsymbol{R}$, there exists an $x_A \in X$ such that

$$\langle x', x_A \rangle = \int_A f \, \mathrm{d}\langle x', \mu \rangle, \quad x' \in X'.$$

We denote by $\boldsymbol{E}(\mu)$ the space of all μ-integrable functions. For every $f \in \boldsymbol{E}(\mu)$, let

$$p(\mu)(f) = \left\{ \int_\Omega |f| \, \mathrm{d}\,|\langle x', \mu \rangle| : x' \in U_p' \right\},$$

where $|\langle x', \mu \rangle|$ is the total variation of $\langle x', \mu \rangle$.

Proposition 7.5. *Let μ be a vector measure defined on a σ-algebra $\boldsymbol{R}$ of subsets of Ω. Then*

$$\boldsymbol{E}(\mu)=\boldsymbol{M}(\boldsymbol{R})\cap\boldsymbol{K}_1(I_\mu) \tag{24}$$

and $p(\mu)=p((I_\mu)_1)$ on $\boldsymbol{E}(\mu)$ for every $p\in P(X)$.

Proof follows from Theorem 3.3 and [12: Theorem II.4.1].

Now we wish to extend the definition of μ-integrability to the case where $\boldsymbol{R}$ is a δ-ring. Proposition 7.5 shows us how to proceed: in this more general situation, the space $\boldsymbol{E}(\mu)$ of μ-integrable functions is defined by (24) and endowed with the topology induced from $\boldsymbol{K}_1(I_\mu)$. For simplicity, we write

$$p(\mu)(f)=p((I_\mu)_1)(f)$$

for every $p\in P(X)$ and every $f\in\boldsymbol{E}(\mu)$.

The following example shows that the equality $\boldsymbol{E}(\mu)=\boldsymbol{K}_1(I_\mu)$ does not always hold.

Example 7.6. (cf. [12: Example IV.6.1]). Let $\Omega=[0, 1]$ and $\boldsymbol{R}$ σ-algebra of all Borel subsets of Ω. Let $X=\mathbf{R}^\Omega$ equipped with the product topology. Define the vector measure $\mu\colon\boldsymbol{R}\to X$ by

$$\mu(A)=A \tag{25}$$

for every $A\in\boldsymbol{R}$. Then $\boldsymbol{K}_1(I_\mu)=\mathbf{R}^\Omega$, and $(I_\mu)_1$ is equal to the identity map of $\mathbf{R}^\Omega$. On the other hand, $\boldsymbol{E}(\mu)=\boldsymbol{M}(\boldsymbol{R})$.

Next we start from a saturable Daniell integral I from a Riesz subspace $\boldsymbol{L}$ of $\mathbf{R}^\Omega$ into X.

The following lemma is a direct consequence of Theorem 5.12 (cf. Lemmas 5.3 and 5.4).

Lemma 7.7. (i) *The family defined by* (13) *is a δ-ring.*

(ii) *The set function $\mu(I)\colon\boldsymbol{R}(I)\to X$ given by* (14) *is a σ-additive vector measure.*

(iii) *if I satisfies the following inclusion:*

$$\boldsymbol{K}_1(I)\wedge 1\subset\boldsymbol{K}_1(I), \tag{26}$$

then

$$\boldsymbol{K}_1(I)\subset\boldsymbol{M}(\boldsymbol{R}(I)). \tag{27}$$

Theorem 7.8. *If I satisfies* (26), *then its Stone extension $I_1:\boldsymbol{K}_1(I)\to X$ coincides with the Stone extension $(I_{\mu(I)})_1:\boldsymbol{K}_1(I_{\mu(I)})\to X$ of the Daniell integral $I_{\mu(I)}$:* sim $(\boldsymbol{R}(I))\to X$. *In particular, $\boldsymbol{K}_1(I)=\boldsymbol{E}(\mu(I))$.*

Proof. For simplicity, let

$$J=I_{\mu(I)},\ \mu=\mu(I)\ \text{ and }\ \boldsymbol{R}=\boldsymbol{R}(I).$$

By Theorem 5.12 and Lemma 7.7 (iii) we have

$$p(I_1)(f) = p(I_1 \text{ (sim } \boldsymbol{R}), f)) = p(J)(f)$$

for every $p \in P(X)$ and every $f \in \text{sim}(\boldsymbol{R})$. From Theorem 3.3 and Lemma 7.7 (iii) it follows that $\boldsymbol{K}_1(I)$ is included in $\boldsymbol{K}_1(J)$, that $J_1 = I_1$ on $\boldsymbol{K}_1(I)$ and that $p(J_1) = p(I_1)$ on $\boldsymbol{K}_1(I)$. Hence Lemma 2.7 ensures that $\boldsymbol{F}(I) = \boldsymbol{F}(J)$. Thus $\boldsymbol{K}_1(I) = \boldsymbol{K}_1(J)$.

Corollary 7.9. *If I satisfies the inclusion* (26), *then* $\mu(I_{\mu(I)}) = \mu(I)$ *and* $\boldsymbol{R}(I_{\mu(I)}) = \boldsymbol{R}(I)$.

8. Closed Daniell integral

Let Ω be a set and $\boldsymbol{L}$ a Riesz subspace of $\mathbf{R}^{\Omega}$. Let X be a quasi-complete locally convex Hausdorff space.

A Stone integral $I: \boldsymbol{L} \to X$ is called closed (resp. quasi-closed) if $\boldsymbol{L}$ is a complete (resp. quasi-complete) locally convex space with respect to the mean convergence topology. A Stone integral is said to be closable (resp. quasi-closable) if its Stone extension is closed (resp. quasi-closed).

By Proposition 2.5, every metrizable space-volued Stone integral is closed. Example 2.12 or 3.5 shows that not every saturable Daniell integral is closable; Example 2.14 shows that not every closed Daniell integral is saturable.

Let $I: \boldsymbol{L} \to X$ be a Stone integral. Let $L(I)$ denote the quotient space (19), and $K_1(I)$ the quotient space $\boldsymbol{K}_1(I)/\boldsymbol{N}(I)$. The seminorms on $L(I)$ (resp. $\boldsymbol{K}_1(I)$) derived from $p(I)$ (resp. $p(I_1)$), $p \in P(X)$, are also denoted by $p(I)$ (resp. $p(I_1)$). Let us take a function f from $\boldsymbol{L}$ (resp. $\boldsymbol{K}_1$); the element of $L(I)$ (resp. $\boldsymbol{K}_1(I)$) which contains f will be written $[f]$. The Stone integral I (resp. I_1) induces the linear map $[I]$ (resp. $[I_1]$) from $L(I)$ resp. $K_1(I)$ into X.

Proposition 8.1. *Let* $I: \boldsymbol{L} \to X$ *be either a closed Stone integral or a saturable, quasi-closed Daniell integral. Then*

$$\boldsymbol{L} + \boldsymbol{N}(I) = \boldsymbol{K}_1(I);$$

that is, $L(I)$ *is identical with* $K_1(I)$.

Proof. If I is closed, then the statement is obvious. If I is saturable and quasi-closed, then the statement follows from Corollary 5.14.

Let $\boldsymbol{R}$ be a δ-ring of subsets of Ω. A σ-additive vector measure $\mu: \boldsymbol{R} \to X$ is called closed (resp. quasi-closed) if $\boldsymbol{R}$ (resp. every bounded closed subset of $\boldsymbol{R}$) is complete with respect to the uniformity induced from $\boldsymbol{K}_1(I_\mu)$, where $\boldsymbol{R}$ is considered to be a subset of $\boldsymbol{K}_1(I_\mu)$. Note that μ is not always closed even when I_μ is closable (see Example 7.6). If $\boldsymbol{R}$ is a σ-algebra, then μ is closed if and only if μ is quasi-closed. But not every quasi-closed vector measure is closed as the following example shows.

Example 8.2. Let Ω be an uncountable set and $\boldsymbol{R}$ the δ-ring of all finite subsets of Ω. Let X be the Hilbert space $l_2(\Omega)$ equipped with the weak topology. The vector measure from $\boldsymbol{R}$ into X defined by (25) is not closed but quasi-closed.

Proposition 8.3. *Let $I: \boldsymbol{L} \to X$ be a closable (resp. quasi-closable), saturable Daniell integral which satisfies* (26). *Then the σ-additive vector measure $\mu(I): \boldsymbol{R}(I) \to X$ given by* (13) *and* (14) *is closed (resp. quasi-closed).*

Proof follows from the fact that the quotient space $\boldsymbol{R}(I)/\boldsymbol{R}(I) \cap \boldsymbol{N}(I)$ is a closed subset of $\boldsymbol{K}_1(I)$.

Let $E(\mu)$ denote the quotient space $\boldsymbol{E}(\mu)/\boldsymbol{E}(\mu) \cap \boldsymbol{N}(I_\mu)$. The equivalence class of a function $f \in \boldsymbol{E}(\mu)$ is denoted by $[f]$.

Proposition 8.4. *If $\boldsymbol{R}$ is a σ-algebra and if $\mu: \boldsymbol{R} \to X$ is a closed vector measure, then*

(i) *$\boldsymbol{E}(\mu)$ is complete;*

(ii) *$\boldsymbol{E}(\mu) + \boldsymbol{N}(I_\mu) = \boldsymbol{K}_1(I_\mu)$, that is, $E(\mu) = K_1(I_\mu)$;*

(iii) *I_μ is a closable Daniell integral.*

Proof. Let $j: X \to \hat{X}$ be the natural injection. By [12: Theorem IV.4.1], the space $\boldsymbol{E}(j \circ \mu)$ is complete. Lemma 4.4 implies that $\boldsymbol{E}(\mu) = \boldsymbol{E}(j \square \mu)$. Thus Statements (i) to (iii) follow.

Lemma 8.5. *Let $\boldsymbol{R}$ be a δ-ring and let $\mu: \boldsymbol{R} \to X$ be a quasi-closed vector measure. If $\{f_\gamma\}_{\gamma \in \Gamma}$ is a Cuachy net in $\boldsymbol{E}(\mu)$ and if there are pairwise disjoint sets A_n, $n \in \mathbf{N}$, in $\boldsymbol{R}$ such that*

$$\bigcup_{\gamma \in \Gamma} S(f_\gamma) \subset \bigcup_{n=1}^{\infty} A_n,$$

then the net $\{f_\gamma\}_{\gamma \in \Gamma}$ is convergent in $\boldsymbol{E}(\mu)$.

Proof. Given $n \in \mathbf{N}$, by Proposition 8.4 there exists a function $f_n \in \boldsymbol{E}(\mu)$ such that $S(f_n) \subset A_n$ and the net $\{f_\gamma A_n\}_{\gamma \in \Gamma}$ is convergent to f_n in $\boldsymbol{E}(\mu)$. Since the sequence $\left\{\sum_{i=1}^{n} f_i\right\}_{n \in \mathbf{N}}$ is Cauchy in $\boldsymbol{E}(\mu)$, the series

$$\sum_{n=1}^{\infty} (I_\mu)_1(\boldsymbol{K}_1(I_\mu), f_n)$$

is convergent in X. Hence the function f defined pointwise by (10) belongs to $\boldsymbol{E}(\mu)$ and (11) holds in $\boldsymbol{E}(\mu)$. Applying Theorem 5.12, we can prove that the net $\{f_\gamma\}_{\gamma \in \Gamma}$ is convergent to f in $\boldsymbol{E}(\mu)$.

Lemma 8.6. *Let μ and $\boldsymbol{R}$ be as in Lemma 8.5. If $f_\gamma \in \boldsymbol{E}(\mu)$, $\gamma \in \Gamma$, are functions such that the net $\{[f_\gamma]\}_{\gamma \in \Gamma}$ is increasing and bounded above in the Riesz space*

$E(\mu)$, *then there exists a function* $g \in \boldsymbol{E}(\mu)$ *such that* $\{[f_\gamma]\}_{\gamma \quad \Gamma}$ *is convergent to* $[g]$ *in* $E(\mu)$ *and*

$$[g] = \sup \{[f_\gamma] : \gamma \in \Gamma\} \tag{28}$$

in $E(\mu)$.

Proof. We may assume that $f_\gamma \geqslant 0$ for every $\gamma \in \Gamma$. There exists a function $f > 0$ in $\boldsymbol{E}(\mu)$ such that $[f_\gamma] < [f]$ for every $\gamma \in \Gamma$. Since $f \in \boldsymbol{M}(\boldsymbol{R})$, there exists an increasing sequence $\{A_n\}_{n \in \mathbf{N}}$ in $\boldsymbol{R}$ such that

$$S(f) \subset \bigcup_{n \quad 1}^{\infty} A_n \tag{29}$$

Fix a positive number ε and a seminorm $p \quad P(X)$ By Theorem 12 there exists n $N \in \mathbf{N}$ such that

$$p(\mu)(f - fA_N) \quad \varepsilon$$

Let ν denote the restriction of μ to A_N. There exi ts a non negative measure λ defined on the σ algebra $A \quad \cap \boldsymbol{R}$ such that $\lambda_p(A) \to 0$, $A \in A_N \cap \boldsymbol{R}$ if and only if $p(\nu)(A) \to 0$ (cf. [12: Theorem II.1.1]). Thus the net $\{f \, A_N\} \quad _{\in \Gamma}$ is Cauchy with respect to λ_p, so that it is also Cauchy with respect to $p(\nu)$ (cf. [12: Lemma III.2.1]). Hence we can choo e an $\alpha \in \Gamma$ such that if γ, $\delta \in \Gamma$ nd γ, $\delta > \alpha$, then

$$p(\nu)(f_\gamma A_N - f_\delta A_N) < \varepsilon .$$

Consequently

$$p(\mu)(f_\gamma - f_\delta) < p(\nu)(f_\gamma A_N - f \; A_N) + 2p(\mu)(f - fA_N) < 3\varepsilon .$$

Thus $\{f_\gamma\}_{\gamma \in \Gamma}$ is a Cauchy net in $\boldsymbol{E}(\mu)$. See that, by Lemma 8.5, there exists a function $g \in \boldsymbol{E}(\mu)$ to wh ch the net $\{f_\gamma\}_{\gamma \in \Gamma}$ is convergent in $\boldsymbol{E}(\mu)$. Furthermore, (28) holds in $E(\mu)$.

Propos'tion 8.7. *Let* μ *be a quasi-clo ed,* σ *add tive vector measure on a* δ *ring* $\boldsymbol{R}$ *of subsets of* Ω, *with values in* X. *Then*

(i) $\boldsymbol{E}(\mu)$ *is sequentially complete;*

(ii) $\boldsymbol{E}(\mu) + \boldsymbol{N}(I_\mu) - \boldsymbol{K}_1(I_\mu)$, *th t is,* $E(\mu) - K_1(I_\mu)$;

(iii) $E(\mu)$ *is a Dedekind complete Riesz space.*

Proof. Statement (i) is a direct con equence of Lemma 8.5.

To show (ii), let $f \in \boldsymbol{K}_1$. Then there exists pairwise disjoint sets $A_n \in \boldsymbol{R}$, $n \in \mathbf{N}$, such that (29) holds. Given $n \in \mathbf{N}$, by Propo ition 8.4 there exist function $g_n \in \boldsymbol{E}(\mu)$ and $h_n \in \boldsymbol{N}(I_\mu)$ such that

$$S(g_n) \subset A_n \quad \text{and} \quad fA_n - g_n + h_n .$$

Let

$$g(\omega)=\sum_{n=1}^{\infty} g_n(\omega) \text{ and } h(\omega)=\sum_{n=1}^{\infty} h_n(\omega)$$

for every $\omega \in \Omega$. Then we obtain $g \in \boldsymbol{E}(\mu)$ by Theorem 5.12 and $h \in \boldsymbol{N}(I_\mu)$ by Proposition 2.9. Thus

$$f=g+h \in \boldsymbol{E}(\mu)+\boldsymbol{N}(I_\mu)$$

Statement (iii) follows from Lemma 8.6.

Lemma 8.8. *Let $\boldsymbol{R}$ be a δ-ring (resp. a σ-algebra) and let $\mu: \boldsymbol{R} \to X$ be a σ-additive vector measure. Then there exist a set $\tilde{\Omega}$, a δ-ring (resp. a σ-algebra) $\boldsymbol{Q}$ of subsets of $\tilde{\Omega}$, an injective ring homomorphism $\alpha: \boldsymbol{R} \to \boldsymbol{Q}$ and a quasi-closed (resp. closed) vector measure $\tilde{\mu}: \boldsymbol{Q} \to X$ such that*

(i) *$\alpha(\boldsymbol{R})$ is a dense subset of $\boldsymbol{Q}$;*

(ii) *$\tilde{\mu}(\alpha(A))=\mu(A)$ for every $A \in \boldsymbol{R}$.*

(iii) *$p(\tilde{\mu})(\alpha(A))=p(v)(A)$ for every $p \in P(X)$ and every $A \in \boldsymbol{R}$.*

In particular, if $\boldsymbol{R}$ separates points of Ω, then

(iii) *$\Omega \subset \tilde{\Omega}$; and*

(iv) *$\boldsymbol{R} \subset \boldsymbol{Q} \cap \Omega$.*

Proof. A sketch of the proof has been given in [11: Theorem on Closure]. If $\boldsymbol{R}$ is a δ-ring we just apply [13: 23.1.(4)] to make the vector measure $\tilde{\mu}$ map $\boldsymbol{Q}$ into X.

Lemma 8.9. *Under the same notation as in Lemma 8.8 there exists a Riesz homomorphism $\psi: \boldsymbol{E}(\mu) \to E(\tilde{\mu})$ such that*

(i) *$\psi(\boldsymbol{E}(\mu))$ is a dense subspace of $E(\tilde{\mu})$;*

(ii) *$[(I_\mu)_1](\psi(f))=(I_\mu)_1(f)$ for every $f \in \boldsymbol{E}(\mu)$;*

(iii) *$p(\tilde{\mu})(\psi(f))=p(\mu)(f)$ for every $p \in P(X)$ and every $f \in \boldsymbol{E}(\mu)$.*

Proof. For simplicity, let $I=I_\mu$ and $J=I_{\tilde{\mu}}$. There exists a unique Riesz homomorphism φ: sim $(\boldsymbol{R}) \to$ sim $(\boldsymbol{Q})$ which extends α. Fix a function $f \in$ sim $(\boldsymbol{R})$ and a seminorm $p \in P(X)$. Since $I=J \circ \varphi$ on sim $(\boldsymbol{R})$, we have

$$p(I)(f) \leqslant p(J)(\varphi(f)).$$

Given $\varepsilon>0$, there exists a function $h \in$ sin $(\boldsymbol{Q})$ with $|h| \leqslant |\varphi(f)|$ such that

$$p(J)(\varphi(f)) \leqslant p(J(h))+\varepsilon.$$

There exist pairwise disjoint sets $A_i \in \boldsymbol{Q}$ and real numbers a_i, $i=1, 2, \ldots, n$, $n \in \mathbf{N}$, such that

$$h=\sum_{i=1}^{n} a_i A_i.$$

For every $i \in \mathbf{N}$ such that $1 \leq i \leq n$, there exists a set $B_i \in \boldsymbol{R}$ for which

$$p(J)(A_i \ominus \alpha(B_i)) < \varepsilon \Big/ \Big(\sum_{i=1}^{n} |a_i| + 1\Big).$$

Let

$$g = \Big(\sum_{i=1}^{n} a_i B_i\Big) \wedge f.$$

Since $p(J)(h - \varphi(g)) < \varepsilon$, we have

$$p(J)(\varphi(f)) - 2\varepsilon \leq p(I(g)) \leq p(I)(f).$$

Thus

$$p(J)(\varphi(f)) = p(I)(f).$$

Let $\psi = \pi \circ \varphi$, where $\pi: \boldsymbol{K}_1(J) \to (J)$ is the natural projection. We shall show that ψ has a unique extension to $\boldsymbol{E}(\mu)$. Given a non-negative function f in $\boldsymbol{E}(\mu)$, we choose an increasing sequence $\{f_n\}_{n \in \mathbf{N}}$ in sim $(\boldsymbol{R})$ which is pointwise convergent to f. Since the sequence $\{f_n\}_{n \in \mathbf{N}}$ is convergent to f in $\boldsymbol{E}(\mu)$ by Theorem 5.12, the sequence $\{\pi \circ \varphi(f_n)\}_{n \in \mathbf{N}}$ is Cauchy in the sequentially complete space $E(\tilde{\mu})$, and so there exists a unique limit of $\{\pi \circ \varphi(f_n)\}_{n \in \mathbf{N}}$ is $E(\tilde{\mu})$, independent of the sequence $\{f_n\}_{n \in \mathbf{N}}$. This unique limit is denoted by $\psi(f)$. Now Statements (i) to (iii) hold.

Theorem 8.10. *Let $I: \boldsymbol{L} \to X$ be a saturable Daniell integral which satisfies* (26). *Then there exist a set $\tilde{\Omega}$, a Riesz subspace $\tilde{\boldsymbol{L}}$ of $\mathbf{R}^{\tilde{\Omega}}$, a saturable Daniell integral $\tilde{I}: \tilde{\boldsymbol{L}} \to X$ and a Riesz homomorphism $\Psi: \boldsymbol{L} \to \tilde{L}(\tilde{I})$ such that*

(i) *$\tilde{\boldsymbol{L}}$ is a sequentially complete space;*

(ii) *$\tilde{L}(\tilde{I})$ is a Dedikind complete Riesz space;*

(iii) *if $\{[g_\gamma]\}_{\gamma \in \Gamma}$ is a decreasing net in $\tilde{L}(\tilde{I})$ such that* $\inf \{[g_\gamma] : \gamma \in \Gamma\} = 0$ *in $\tilde{L}(\tilde{I})$, then it is convergent to 0 in $\tilde{L}(\tilde{I})$;*

(iv) *$\Psi(\boldsymbol{L})$ is a dense subspace of $\tilde{L}(\tilde{I})$;*

(v) *$[\tilde{I}] \circ \Psi = I$ on $\boldsymbol{L}$;*

(vi) *$p(\tilde{I}) \circ \Psi = p(I)$ on $\boldsymbol{L}$ for every $p \in P(X)$;*

(vii) *Ψ induces a topological Riesz isomorphism $\hat{\Psi}$ from $L(I)$ onto a dense subspace of $\tilde{L}(\tilde{I})$ such that $\hat{\Psi}([f]) = \Psi(f)$ for every $f \in \boldsymbol{L}$.*

In particular, if $\boldsymbol{K}_1(I)$ separates points of Ω, then $\tilde{\Omega}$ can be chosen such that

(viii) $\Omega \subset \tilde{\Omega}$.

Proof. For brevity, let $\mu = \mu(I)$. Theorem 7.8 implies that $\boldsymbol{K}_1(I) = \boldsymbol{E}(\mu)$ and $I_1 = (I_\mu)_1$. We take a set $\tilde{\Omega}$, a δ-ring $\boldsymbol{Q}$ and a vector measure $\tilde{\mu}: \boldsymbol{Q} \to X$ such that (i) to (iii) of Lemma 8.8 hold. Then there exists a Riesz homomorphism $\psi: \boldsymbol{E}(\mu) \to E(\tilde{\mu})$ such that (i) to (iii) of Lemma 8.9 hold. Let $\iota: \boldsymbol{L} \to \boldsymbol{K}_1(I)$ be the natural injection and let

$$\Psi = \psi \circ \iota, \ \tilde{\boldsymbol{L}} = \tilde{\boldsymbol{E}}(\tilde{\mu}) \text{ and } \tilde{I} = (I_{\tilde{\mu}})_{\iota}.$$

Then Statements (i) to (vii) hold.

If $\boldsymbol{K}_{\iota}(I)$ separates points of Ω, then the δ-ring $\boldsymbol{R}(I)$ separates points of Ω, and so (viii) follows from Lemma 8.8 (iv).

Lemma 8.11. *Under the same notation as in Theorem* 8.10 *there exist an index set Λ and a system $\{e_\lambda\}_{\lambda \in \Lambda}$ of non-negative functions in $\tilde{\boldsymbol{L}}$ such that*

(i) $[e_\lambda] \wedge [e_{\lambda'}] = 0$ if $\lambda, \lambda' \in \Lambda$, $\lambda \neq \lambda'$;

(ii) *if f is a function in $\tilde{\boldsymbol{L}}^+$ such that $[e_\lambda] \wedge [f] = 0$ for every $\lambda \in \Lambda$, then $[f] = 0$;*

(iii) *if f is a function in $\tilde{\boldsymbol{L}}^+$, then*

$$[f] = \sup \left\{ \sum_{\lambda \in \Delta} [f\Omega_\lambda] : \Delta \in \boldsymbol{D}(\Lambda) \right\}$$

in $\tilde{L}(\tilde{I})$, where $\Omega_\lambda = S(e_\lambda)$;

(iv) *if $f \in \tilde{\boldsymbol{L}}$ is a function, then the net* (17) *is convergent to f in $\tilde{\boldsymbol{L}}$.*

Proof. By Zorn's lemma we can choose an index set Λ and a system $\{e_\lambda\}_{\lambda \in \Lambda}$ in $\tilde{\boldsymbol{L}}^+$ such that (i) and (ii) hold. Statements (iii) and (iv) follow from Theorem 8.10 (ii), (iii).

Lemma 8.12. *Let $I: \boldsymbol{L} \to X$ be a saturable Daniell integral. If $\boldsymbol{L}$ contains a function e such that $e(\omega) > 0$ for every $\omega \in \Omega$, then there exist a Riesz subspace $\boldsymbol{M}$ of $\mathbf{R}^\Omega$, a Riesz isomorphism φ from $\boldsymbol{L}$ onto $\boldsymbol{M}$ and a Daniell integral $J: \boldsymbol{M} \to X$ such that*

(i) *$\boldsymbol{M}$ contains the constant function* 1;

(ii) *$J(\varphi(f)) = I(f)$ for every $f \in \boldsymbol{L}$;*

(iii) *$p(J)(\varphi(f)) = p(I)(f)$ for every $p \in P(X)$ and every $f \in \boldsymbol{L}$.*

Proof. Define the linear map φ from $\boldsymbol{L}$ onto the Riesz subspace $\boldsymbol{M} = \{f/e : f \in \boldsymbol{L}\}$ of $\mathbf{R}^\Omega$ by $\varphi(f) = f/e$ for every $f \in \boldsymbol{L}$. Let $J: \boldsymbol{M} \to X$ be defined by $J(\varphi(f)) = I(f)$ for every $f \in \boldsymbol{L}$. Then the statements follow.

We are now ready to show that every saturable Daniell integral can be extended to a quasi-closed Daniell integral.

Theorem 8.13. *Let $I: \boldsymbol{L} \to X$ be a saturable Daniell integral which satisfies* (26). *Then there exist a set $\hat{\Omega}$, a Riesz subspace $\hat{\boldsymbol{L}}$ of $\mathbf{R}^{\hat{\Omega}}$, a saturable, quasi-closed Daniell integral $\hat{I}: \hat{\boldsymbol{L}} \to X$ and a Riesz homomorphism $\Phi: \boldsymbol{L} \to L_1(I) = \hat{\boldsymbol{L}}/\hat{\boldsymbol{L}} \cap \boldsymbol{N}(\hat{I})$ such that*

(i) *$\Phi(\boldsymbol{L})$ is a dense subspace of $L_1(I)$;*

(ii) *$[\hat{I}](\Phi(f)) = I(f)$ for every $f \in \boldsymbol{L}$;*

(iii) *$p(\hat{I})(\Phi(f)) = p(I)(f)$ for every $p \in P(X)$ and every $f \in \boldsymbol{L}$;*

(iv) *Φ induces a topological Riesz isomorphism $\hat{\Phi}$ from $L(I)$ onto a dense subspace of $L_1(I)$ such that $\hat{\Phi}([f]) = \Phi(f)$ for every $f \in \boldsymbol{L}$.*

In particular, if X is complete, then $\hat{I}$ is closed.

Proof. (I) Take a saturable Daniell integral $\hat{I}: \tilde{\boldsymbol{L}} \to X$ as in Theorem 8.10. There exist an index set Λ and a system $\{g_\lambda\}_{\lambda \in \Lambda}$ in $\tilde{\boldsymbol{L}}^+$ such that (i) to (iv) of Lemma 8.11 hold. Given $\lambda \in \Lambda$, let $\boldsymbol{L}_\lambda$ denote the Riesz subspace $\{f\Omega_\lambda : f \in \tilde{\boldsymbol{L}}\}$ of $\mathbf{R}^{\Omega}\lambda$. The restriction I_λ of $\tilde{I}$ to $\boldsymbol{L}_\lambda$ is a saturable Daniell integral.

(II) Fix an arbitrary $\lambda \in \Lambda$. There exist a Riesz subspace $\boldsymbol{M}_\lambda$ of $\mathbf{R}^{\Omega}\lambda$, a Riesz isomorphism $\varphi_\lambda : \boldsymbol{L}_\lambda \to \boldsymbol{M}_\lambda$ and a Daniell integral $J_\lambda : \boldsymbol{M}_\lambda \to X$ such that (i) to (iii) of Lemma 8.12 are valid. Let $\mu(\lambda) = \mu(J_\lambda)$. Theorem 7.8 implies that

$$(J_\lambda)_1 = (I_{\mu(\lambda)})^1 \text{ and } \boldsymbol{K}_1(J_\lambda) = \boldsymbol{E}(\mu(\lambda)).$$

Let $j_\lambda : \boldsymbol{M}_\lambda \to \boldsymbol{E}(\mu(\lambda))$ be the natural injection. By Proposition 8.4 and Lemmas 8.8, 8.9, there exist a set $\hat{\Omega}_\lambda$, a Riesz subspace $\boldsymbol{H}_\lambda$ of $\mathbf{R}^{\hat{\Omega}_\lambda}$, a saturable and closed Daniell integral $\hat{J}_\lambda : \boldsymbol{H}_\lambda \to X$ and a Riesz homomorphism ψ_λ which maps $\boldsymbol{E}(\mu(\lambda))$ onto a dense subspace of $H_\lambda(\hat{J}_\lambda)$ such that $[\hat{J}_\lambda] \circ \psi_\lambda = (J_\lambda)_1$ and $p(\hat{J}_\lambda) \circ \psi_\lambda = p(\mu(\lambda))$ for every $p \in P(X)$.

(III) Let $\hat{\Omega}$ be the disjoint union of the family $\{\hat{\Omega}_\lambda : \lambda \in \Lambda\}$ of sets. Let $J: \boldsymbol{H} \to X$ be the direct sum of the family $\{\hat{J}_\lambda : \lambda \in \Lambda\}$ of Daniell integrals. Then J is a quasi-closed, saturable Daniell integral by Propositions 6.4 and 6.5.

(IV) For every $\lambda \in \Lambda$, let $\Phi_\lambda = \psi_\lambda \circ j_\lambda \circ \varphi_\lambda$. Fix a seminorm $p \in P(X)$ and a set $\Delta \in \boldsymbol{D}(\Lambda)$. We claim that

$$p(J)\Big(\sum_{\lambda \in \Delta} \Phi_\lambda(f_\lambda)\Big) = p(\tilde{I})\Big(\sum_{\lambda \in \Delta} f_\lambda\Big)$$

for every $f_\lambda \in \boldsymbol{L}_\lambda$, $\lambda \in \Delta$. Indeed, this follows from the fact that $\Phi_\lambda(\boldsymbol{L}_\lambda)$ is a dense subspace of $H_\lambda(\hat{J}_\lambda)$ for every $\lambda \in \Delta$.

(V) Let $f \in \tilde{\boldsymbol{L}}$. For every $\lambda \in \Lambda$, there exists a function $g_\lambda \in \boldsymbol{H}_\lambda$ such that $[g_\lambda] = \Phi_\lambda(f\Omega_\lambda)$. Let

$$g(\omega) = \sum_{\lambda \in \Lambda} g_\lambda(\omega)$$

for every $\omega \in \hat{\Omega}$. To show that the function g belongs to $\boldsymbol{H}$, we may assume that $f \geqslant 0$ and $g_\lambda \geqslant 0$, $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, we take any function $h_\lambda \in \boldsymbol{H}_\lambda^+$ such that $h_\lambda \leqslant g_\lambda$. Fix a seminorm $p \in P(X)$. Since the net (17) is convergent to f in $\tilde{\boldsymbol{L}}$, Proposition 2.9 and Theorem 5.12 ensure that the set

$$\Xi = \{\lambda \in \Lambda : p(\tilde{I})(f\Omega_\lambda) > 0\},$$

is countable, and so we can write $\Xi = \{\lambda_n : n \in \mathbf{N}\}$. Note that $p(I)(h_\lambda) = 0$ whenever $\lambda \in \Lambda \setminus \Xi$. For every $n \in \mathbf{N}$, let

$$\boldsymbol{L}_n = \boldsymbol{L}_{\lambda_n}, \quad \Phi_n = \Phi_{\lambda_n} \text{ and } h_n = h_{\lambda_n}.$$

Given $\varepsilon > 0$ and $n \in \mathbf{N}$, there exists a function $k_n \in \boldsymbol{L}_n^+$ such that $k_n \leqslant f_n$ and

$$p(J)([h_n] - \Phi_n(k_n)) < \varepsilon 2^{-n}.$$

Thus if m and n are natural numbers such that $m \geqslant n$, then it follows from (IV) that

$$p(J)\Big(\sum_{i=n}^{m} h_i\Big) \leqslant p(J)\Big(\sum_{i=n}^{m}([h_i] - \Phi_i(k_i))\Big) + p(\tilde{I})\Big(\sum_{i=n}^{m} k_i\Big) \leqslant$$

$$\leqslant \varepsilon + p(\tilde{I})\Big(\sum_{i=n}^{m} k_i\Big).$$

Since by Theorem 5.12 the sequence $\Big\{\sum_{i=1}^{n} k_i\Big\}_{n \in \mathbf{N}}$ is Cauchy in $\tilde{\boldsymbol{L}}$, the net

$$\Big\{\sum_{\lambda \in \Delta} J(h_\lambda)\Big\}_{\Delta \in \boldsymbol{D}(\Lambda)}$$

is Cauchy in X. Since this net is bounded in X, it is convergent in X. Hence the function g belongs to $\boldsymbol{H}$. Furthermore, from (IV), Proposition 6.3 and Lemma 8.11 (iv), it follows that

$$p(J)(g) = p(\tilde{I})(f) \tag{30}$$

for every $p \in P(X)$.

(VI) Take a function $f \in \tilde{\boldsymbol{L}}$. For every $\lambda \in \Lambda$, let g_λ be as in (V) and let $h_\lambda \in \boldsymbol{H}_\lambda$ be a function such that $[g_\lambda] = [h_\lambda]$ in $H(J)$. It follows from Proposition 6.3 and (V) that

$$p(J)\Big(\sum_{\lambda \in \Lambda}(g_\lambda - h_\lambda)\Big) = 0$$

for every $p \in P(X)$. This enables us to define the Riesz homomorphism Π: $\tilde{\boldsymbol{L}} \to H(J)$ by

$$\Pi(f) = \Big[\sum_{\lambda \in \Lambda} g_\lambda\Big], \quad f \in \tilde{\boldsymbol{L}}.$$

Then it follows from (30) that

$$p(J)(\Pi(f)) = p(\tilde{I})(f)$$

for every $p \in P(X)$. Moreover, Proposition 6.3 and Lemma 8.11 (iv) ensure that $[J] \circ \Pi = \tilde{I}$ on $\tilde{\boldsymbol{L}}$.

(VII) Let $\Phi = \Pi \circ \Psi: \boldsymbol{L} \to H(J)$. Then Φ is a Riesz homomorphism. Let π: $\boldsymbol{H} \to H(J)$ be the natural projection, and let $\hat{\boldsymbol{L}}$ be the quasi-closure of $\pi^{-1}(\Phi(\boldsymbol{L}))$ in $\boldsymbol{H}$. The restriction $\hat{I}$ of J to $\hat{\boldsymbol{L}}$ is a saturable Daniell integral. We can identify $L_1(I) = \hat{\boldsymbol{L}}/\hat{\boldsymbol{L}} \cap \boldsymbol{N}(\hat{I})$ with the quasi-closure of $\Phi(\boldsymbol{L})$ in $H(J)$. Hence we may regard Φ as a map into $L_1(I)$. Statements (i) and (ii) are now clear.

(VIII) Given a seminorm $p \in P(X)$, we claim that the equality

$$p(\hat{I}) = p(J) \tag{31}$$

holds on $\hat{\boldsymbol{L}}$. Since $p(J) \geqslant p(\hat{I})$ on $\hat{\boldsymbol{L}}$, the seminorm $p(\hat{I})$ is continuous on $\hat{\boldsymbol{L}}$ with respect to the topology induced from $\boldsymbol{H}$. Hence, it suffices to prove that $p(\hat{I}) \geqslant p(J)$

on $\pi^{-1}(\Phi(\boldsymbol{L}))$. For each function $g \in \pi^{-1}(\Phi(\boldsymbol{L}))$ there exists a function $f \in \boldsymbol{L}$ such that $[g] = \Phi(f)$. Then Theorem 8.10 (vi) and (V) imply that

$$p(J)(g) = p(I)(f).$$

Thus $p(J)(g) \leq p(\hat{I})(g)$. That is, (31) holds on $\hat{\boldsymbol{L}}$.

(IX) By (VIII) the topology of convergence in mean on $\hat{\boldsymbol{L}}$ is identical to the topology induced from $\boldsymbol{H}$. Since $\hat{\boldsymbol{L}}$ is a quasi-closed subspace of the quasicomplete space $\boldsymbol{H}$, it is quasi-complete. In other words, $\hat{I}$ is quasi-closed. Moreover, from Theorem 5.6, the integral $\hat{I}$ is saturable. Statement (iii) now follows from (V) and (VIII).

(X) If X is complete, then $\boldsymbol{H}$ is complete by Proposition 6.4. Thus $\hat{I}$ is closed.

The Riesz space constructed in Theorem 8.13 will be called the L_1-extension of $\boldsymbol{L}$ with respect to I.

Proposition 8.14. *If I: $\boldsymbol{L} \to X$ is a saturable, quasi-closable Daniell integral such that* (26) *holds, then $K_1(I)$ coincides with $L_1(I)$.*

Proof. Since $K_1(I)$ is quasi-complete, it is topologically Riesz isomorphic to $L_1(I)$.

Anknowledgement. The author expresses his gratitude to Professor I. Kluvánek for his invaluable assistance and encouragement, and wishes to thank Dr. P. G. Dodds for indicating an error in the original version.

REFERENCES

[1] ALIPRANTIS, C. D. and BURKINSHAW, O.: Locally Solid Riesz Spaces. Academic Press, New York, 1978.

[2] BAUER, H.: Sur l'équivalence des théorie de l'intégration selon N. Bourbaki et selon M. H. Stone. Bull. Soc. Math. France, 85, 1957, 51—75.

[3] BOURBAKI, N.: Intégration, Chapitre 6. Hermann, Paris, 1959.

[4] BOURBAKI, N.: Topologie Générale, Chapitres 1, 2, Quatrième edition. Hermann, Paris, 1965.

[5] DANIELL, P. J.: A general form of integral. Ann, of Math. (2) 19, 1917—18, 279—294.

[6] DODDS, P. G.: σ-weakly compact mappings of Riesz spaces. Trans. Amer. Math. Soc., 214, 1975, 389—402.

[7] KELLEY, J. L. and NAMIOKA, I.: Linear Topological Spaces. Van Nostrand, New York, 1963.

[8] KLUVÁNEK, I.: Intégrale vectorielle de Daniell. Mat. fyz. časop. SAV, 15, 1965, 146—161.

[9] KLUVÁNEK, I.: Completion of vector measure spaces. Rev. Roum. Math. Pures et Appl. 12, 1967, 1483—1488.

[10] KLUVÁNEK, I.: The range of a vector-valued measure. Math. Systems Theory 7, 1973, 44—54.

[11] KLUVÁNEK, I.: The extension and closure of vector measure. In: Vector and Operator Valued Measures and Applications, Academic Press, New York, 1973, 175—190.

[12] I. KLUVÁNEK, I.—KNOWLES, G.: Vector Measures and Control Systems. North Holland, Amsterdam, 1976.

[13] KÖTHE, G.: Topological Vector Spaces I. Springer-Verlag, Berlin, 1969.

[14] MAŘÍK, J.: Predstavlenie funkcionala v vide integrala. Czechoslovak Math. J., 5, 1955, 467—487.
[15] RIESZ, F.—SZ.-NAGY, B.: Leçons d'Analyse Fonctionelle. Académiai Kiadó, Budapest, 1952.
[12] STONE, M. H.: Notes on integration I, II. Proc. Nat. Acad. Sci. U.S.A., 34, 1948, 336—342 and 447—455.
[17] THOMAS, E.: L'intégration par rapport à une mesure de Radon vectorielle. Ann. Inst. Fourier, Grenoble, 20 — 2, 1970, 55—191.

Received May 1981

School of Mathematical Sciences,
The Flinders University of
South Australia,
BEDFORD PARK, S.A., 5042
AUSTRALIA

ВЕКТОРНЫЕ ИНТЕГРАЛЫ ДАНИЭЛЯ

Susumu Okada

Резюме

В работе строится теория интегралов Даниэля на абстрактном множестве со значениями в локально выпуклых пространствах. Главное внимание уделено следующим вопросам: теоремам Беппо Леви и Лебега и полноте пространства L_1. Для получения теоремы Беппо Леви следует расширить интеграл по схеме Стоуна. Теорема Лебега имеет место тогда и только тогда, когда интеграл Даниэля отображает упорядоченные промежутки в слабо компактные множества. Пространство L_1, полученное нами из стоуновского расширения, не всегда является квази полным. Поэтому в § 8 мы строим другое расширение, обеспечивающее квази полноту соответствующего пространства L_1. Здесь предполагается выполнение известного min $(f, 1)$ — условия Стоуна. Прямые суммы интегралов Даниэля рассматриваем в § 6, а связь между интегралом Даниэля и векторными мерамы в § 7.