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ON THE EXTENSION OF POSITIVE OPERATORS

MARTA VONKOMEROVÁ

There are many papers devoted to measures and integrals with values in ordered spaces (e.g. [2], [4], [6], [11]). In some papers also group valued mappings are considered (e.g. [5], [7], [8], [9]). P. Volauf in [9] proved an extension theorem for lattice ordered group G-valued mappings where the measure extension theorem and the Daniell integral extension theorem are special cases. He assumed that G is complete. In this paper we assume that G is a σ-complete and strongly regular l-group. We use the construction from paper [3] by E. Futáš.

Let us introduce some notations first. If X is a lattice, then by $x \vee y$, $x \wedge y$ we shall denote lattice operations. The symbol $x_n \nearrow x$ $(x_n \searrow x)$ will be written if $x_n \leqq x_{n+1}$ $(x_n \geqq x_{n+1})$ for every n and $\bigvee_{n=1}^{\infty} x_n = x \left(\bigwedge_{n=1}^{\infty} x_n = x \right)$.

Definition 1. *An l-group H is strongly regular if there holds:*

if a, $a_k^i \in H$ for $i = 1, 2, \ldots$, $k = 1, 2, \ldots$ are such that $a_k^i \leqq a$ for $i = 1, 2, \ldots$, $k = 1, 2, \ldots$ and $a_k^i \searrow 0$ $(i \to \infty)$ for $k = 1, 2, \ldots$, then there exists a countable set $\mathscr{C} \subset N^N$ (N is the set of positive integers) such that if $b \in H$, $b \leqq \bigvee_{n=1}^{\infty} \sum_{k=1}^{n} a_k^{\Phi(k)}$ for every $\Phi \in \mathscr{C}$, then $b \leqq 0$.

As example of a strongly regular l-group let us take the set R of all real numbers. For all n and for each $\varepsilon > 0$ there exists i_n in this case such that $a_n^{i_n} \leqq \varepsilon$.

We form $\Phi_n \colon N \to N$ for $n = 1, 2, \ldots$. For this purpose let us put $\varepsilon = \frac{1}{2^k n}$ and we take $\Phi_n(k)$ such that

$$a_k^{\Phi_n(k)} \leqq \frac{1}{2^k n}$$

for $k = 1, 2, \ldots$. Then $\mathscr{C} = \{\Phi_n \colon n = 1, 2, \ldots\}$ is the countable set.

If $b \in R$ and if

$$b \leqq \bigvee_{m=1}^{\infty} \sum_{k=1}^{m} a_k^{\Phi_n(k)}$$

for $n = 1, 2, \ldots$, then

$$b \leqq \bigvee_{m=1}^{\infty} \sum_{k=1}^{m} \frac{1}{2^k n} = \sum_{k=1}^{\infty} \frac{1}{2^k n} = \frac{1}{n}$$

for $n 1, 2, \ldots$, hence $b \leqq 0$.

Similarly we can prove that the set R_n of all n-tuples of real numbers is a strongly regular l-group.

Now let us present further examples.

Example 1. Every regular K-space is a strongly regular l-group. A regular K-space (see 10 Th. VI.5.2) is a linear semiordered space which is relatively complete and such that every sequence of convergent sequences has a common regulator of convergence.

If $b_n \searrow 0$, then $u > 0$ is a regular of convergence of $\{b_n\}_{n=1}^{\infty}$ iff to any number $\varepsilon > 0$ there is n_0 such that $b_n < \varepsilon$ u for every $n \geqq n_0$. Further, every regular K-space is such that $\frac{1}{n} u \searrow 0$ for every $u \geqq 0$ (see 10 Th. IV. 1.5). Now let $a_k^i \searrow 0$ $(i \to \infty)$ for $k = 1, 2, \ldots$ and let u be the common regulator of convergence of all $\{a_k^i\}_{i=1}^{\infty}$, $k = 1, 2, \ldots$. Then for all k and every $\varepsilon > 0$ there exists $i_k \in N$ such that $a_k^i \leqq \varepsilon u$ for every $i \geqq i_k$. It suffices to choose $\Phi_n(k)$ such that

$$a_k^{\Phi_n(k)} \leqq \frac{1}{2^k \cdot n} u$$

for $k = 1, 2, \ldots$ and for $n = 1, 2, \ldots$ we obtain the countable subset of the set N^N.

Let

$$b \leqq \bigvee_{r=1}^{\infty} \sum_{k=1}^{r} a_k^{\Phi_n(k)}$$

for $n = 1, 2, \ldots$. Then

$$b \leqq \sum_{k=1}^{\infty} \frac{1}{2^k n} u = \frac{1}{n} u$$

for $n = 1, 2, \ldots$ and hence $b \leqq 0$.

Example 2. Let us have a set of all sequences of real numbers such that they are non zero only on the finite number of coordinates. Since $a_k^i \leqq a$ for every $i = 1, 2, \ldots, k = 1, 2, \ldots$ and the sequence $a = \{a_n\}_{n=1}^{\infty}$ is non zero only on the finite number of coordinates we have the case R_n.

Proposition. *Every σ-complete l-group is a commutative group.* (See Birkhoff G. [1])

Let G be a σ-complete and strongly regular l-group.

Let X be a conditionally σ-complete lattice. On X define further two operations $+$ and $-$. Suppose that on X the following relations hold:

1. If $x_n, y_n \in X$, $x_n \nearrow x$, $y_n \nearrow y$ ($x_n \searrow x$, $y_n \searrow y$), then $x_n \wedge y_n \nearrow x \wedge y$ ($x_n \vee y_n \searrow x \vee y$).
2. If $x, y \in X$, then $x+y=y+x$.
3. If $x, y, z \in X$, $x \leqq y$, then $x+z \leqq y+z$, $x-z \leqq y-z$, $z-x \geqq x-y$.
4. If $x_n, y_n \in X$, $x_n \nearrow x$, $y_n \nearrow y$ ($x_n \searrow x$, $y_n \searrow y$), then $x_n+y_n \nearrow x+y$ ($x_n+y_n \searrow x+y$).
5. If $x_n, y \in X$, $x_n \nearrow x$ ($x_n \searrow x$), then $x_n-y \nearrow x-y$, $y-x_n \searrow y-x$ ($x_n-y \searrow x-y$, $y-x_n \nearrow y-x$).
6. If $x, y \in X$, $x \geqq y$ then $x=y+(x-y)$.
7. There exists an element $0 \in X$ such that $x-x=0$ for every $x \in X$.
8. If $x, y, u, v \in X$, $x \geqq y$, $u \geqq v$, then $[(x+u)-(y+v)] \vee [(x-v)-(y-u)] \leqq (x-y)+(u-v)$.

Now let A be a sublattice of X closed under the operations $+$ and $-$. We also assume that to any $x \in X$ there are $a, b \in A$ such that $a \leqq x \leqq b$.

Further let J: $A \to G$ be an operator satisfying the following axioms:

(I) If $x, y \in A$, $x \geqq y$, then $J_0(x) \geqq J_0(y)$,

(II) if $x, y \in A$, then $J_0(x \vee y)+J_0(x \wedge y) = J_0(x)+J_0(y)$,

(III) if $x, y \in A$, $x \geqq y$, then $J_0(x)=J_0(y)+J_0(x-y)$,

(IV) if $x, y \in A$, then $J_0(x+y) \leqq J_0(x)+J_0(y)$,

(V) if $x_n \in A$, $x_n \searrow 0$, then $\bigwedge_{n=1}^{\infty} J_0(x_n)=0$.

From 5, (V) and (III) we get:

(V′)
$$\text{if } x_n \nearrow x,\ x_n, x \in A\ (n=1,2,\ldots), \text{ then } J_0(x)=\bigvee_{n=1}^{\infty} J_0(x_n),$$
$$\text{if } x_n \searrow x,\ x_n, x \in A\ (n=1,2,\ldots), \text{ then } J_0(x)=\bigwedge_{n=1}^{\infty} J_0(x_n).$$

Definition 2. *We put* $A_\sigma=\{x \in X\colon \exists x_n \in A,\ x_n \nearrow x\}$, $A_\delta = \{y \in X\colon \exists y_n \in A,\ y_n \searrow y\}$ *and we define* $J_1\colon A_\sigma \cup A_\delta \to G$ *by the formulas*

$$J_1(x)=\bigvee_{n=1}^{\infty} J_0(x_n), \text{ where } x_n \in A,\ x_n \nearrow x,$$

$$J_1(y)=\bigwedge_{n=1}^{\infty} J_0(y_n), \text{ where } y_n \in A,\ y_n \searrow y.$$

Further we put $A_{\sigma\delta} = \{x \in X: \exists x_n \in A_\sigma, x_n \searrow x\}$, $A_{\delta\sigma} = \{y \in X; \exists y_n \in A_\delta, y_n \nearrow y\}$ *and we define* J_2: $A_{\sigma\delta} \cup A_{\delta\sigma} \to G$ *by the formulas*

$$J_2(x) = \bigwedge_{n=1}^{\infty} J_1(x_n), \text{ where } x_n \in A_\sigma, x_n \searrow x,$$

$$J_2(y) = \bigvee_{n=1}^{\infty} J_1(y_n), \text{ where } y_n \in A_\delta, y_n \nearrow y.$$

Finally we put $S = \{x \in X: \exists y \in A_{\delta\sigma}, \exists z \in A_{\sigma\delta}, y \leqq x \leqq z$ *and* $J_2(y) = J_2(z)\}$ *and we define* J: $S \to G$ *such that*

$$J(x) = J_2(y) = J_2(z),$$

where y, z *are the elements from the definition of* S.

From 1, 4 ant the properties of the operations $\vee$, $\wedge$ we get that A_σ, A_δ, $A_{\delta\sigma}$, $A_{\sigma\delta}$ are lattices closed under the operation $+$.

We have to prove that the definitions of J_1, J_2 and J are correct.

Lemma 1. *If* $x_n \nearrow x$, $y_n \nearrow y$ $(x_n \searrow x, y_n \searrow y)$, x_n, $y_n \in A$ *for* $n = 1, 2, \ldots$, $x \leqq y$, *then*

$$\bigvee_{n=1}^{\infty} J_0(x_n) \leqq \bigvee_{n=1}^{\infty} J_0(y_n) \left(\bigwedge_{n=1}^{\infty} J_0(x_0) \leqq \bigwedge_{n=1}^{\infty} J_0(y_n)\right).$$

If $x_n \nearrow x$, $y_n \searrow x$, x_n, $y_n \in A$ *for* $n = 1, 2, \ldots$, *then*

$$\bigvee_{n=1}^{\infty} J_0(x_0) = \bigwedge_{n=1}^{\infty} J_0(y_n).$$

Proof. From 1, (I) and (V′) we have $x_m \wedge y_n \nearrow x_m \wedge y = x_m$, $J_0(x_m) = \bigvee_{n=1}^{\infty} J_0(x_m \wedge y_n) \leqq \bigvee_{n=1}^{\infty} J_0(y_n)$ for all m, hence

$$\bigvee_{m=1}^{\infty} J_0(x_m) \leqq \bigvee_{n=1}^{\infty} J_0(y_n).$$

If $y_n \geqq x \geqq x_n$, then by 4 and 5 there holds $y_n - x_n \searrow 0$ and from (V), (III) we get

$$\bigwedge_{n=1}^{\infty} J_0(y_n) = \bigvee_{m=1}^{\infty} J_0(x_m).$$

Lemma 2. *If* $u \in A_\sigma$, $v \in A_\delta$, *then* $u - v \in A_\sigma$, $v - u \in A_\delta$. *There further holds that if* $x \in A_\sigma$, $y \in A$, $x \geqq y$, *then* $J_1(x) = J_0(y) + J_1(x - y)$.

Proof. There exist u_n, $v_n \in A$, $u_n \nearrow u$, $v_n \searrow v$. Then $u_n - v_n \nearrow u - v$, $v_n - u_n \searrow v - u$ by 4 and 5, hence $u - v \in A_\sigma$, $v - u \in A_\delta$. If $x \in A_\sigma$, then thre exist $x_n \in A$, $x_n \nearrow x$. Further $x_n \vee y \nearrow x \vee y = x$, $x_n - y \nearrow x - y$ and from (III) we have

$$J_0(x_n \vee y) = J_0(y) + J_0(x_n \vee y - y)$$

for all n, hence

$$J_1(x) = J_0(y) + J_1(x - y).$$

Lemma 3. *If x, $x_n \in A_\sigma$ for $n = 1, 2, \ldots$, $x_n \nearrow x$, then $x \in A_\sigma$ and*

$$J_1(x) = \bigvee_{n=1}^{\infty} J_1(x_n).$$

Proof. For every $x_n \in A_\sigma$ there exist $x_n^i \in A$ such that $x_n^i \nearrow x_n$ $(i \to \infty)$. We put $y_n = \bigvee_{j=1}^{n} \bigvee_{i=1}^{n} x_j^i$. Then $y_n \in A$, $y_n \leqq y_{n+1}$, $y_n \leqq x_n$ for all n and $x = \bigvee_{n=1}^{\infty} y_n$.

Since $x_n \leqq x$, $x \in A_\sigma$, it follows from Lemma 1 that

$$J_1(x_n) \leqq J_1(x), \quad J_1(x_n) \geqq J_0(y_n) \text{ for all } n,$$

hence

$$\bigvee_{n=1}^{\infty} J_1(x_n) \leqq J_1(x), \quad J_1(x) = \bigvee_{n=1}^{\infty} J_0(y_n) \leqq \bigvee_{n=1}^{\infty} J_1(x_n)$$

and we have

$$J_1(x) = \bigvee_{n=1}^{\infty} J_1(x_n).$$

Lemma 4. *If x_n, $x \in A_\sigma$, $x_n \searrow x$, then $J_1(x) = \bigwedge_{n=1}^{\infty} J_1(x_n)$.*

Proof. There exist x_n, $x \in A_\sigma$, $x_n \searrow x$ and x_n^i, $x^n \in A$, $x_n^i \nearrow x_n$, $x^n \nearrow x$. Then $x_n - x^n \in A_\sigma$, $x_n - x^n \searrow 0$.

According to Lemma 2 for every n there holds

$$J_1(x_n) = J_0(x^n) + J_1(x_n - x^n),$$

hence

$$\bigwedge_{n=1}^{\infty} J_1(x_n) = \bigvee_{n=1}^{\infty} J_0(x^n) + \bigwedge_{n=1}^{\infty} J_1(x_n - x^n) = J_1(x) + \bigwedge_{n=1}^{\infty} J_1(x_n - x^n).$$

It suffices to prove that if $z_n \in A_\sigma$, $z_n \searrow 0$, then $\bigwedge_{n=1}^{\infty} J_1(z_n) = 0$.

Let $z_n \in A_\sigma$, $z_n \searrow 0$. For every z_n there exist $z_n^i \in A$, $z_n^i \nearrow z_n$.

Let $y_n^i = z_n^i \vee 0$. Then

$$y_n^i \nearrow z_n \vee 0 = z_n, \quad y_n^i \geqq 0, \quad J_0(y_n^i) \nearrow J_1(z_n) \quad (i \to \infty) \text{ for } n = 1, 2, \ldots$$

Let $b=\bigwedge_{n=1}^{\infty} J_1(z_n)$ and let Φ be an arbitrary element of the set N^N. Then

$$0 \leqq \bigwedge_{n=1}^{\infty} y_n^{\Phi(n)} \leqq \bigwedge_{n=1}^{\infty} z_n = 0, \text{ hence } \bigwedge_{n=1}^{\infty} y_n^{\Phi(n)} = 0.$$

There holds

$$J_0\Big(\bigwedge_{k=1}^{m} y_k^{\Phi(k)}\Big) \geqq \sum_{k=1}^{m-1} [J_0(y_k^{\Phi(k)}) - J_1(z_k)] + J_0(y_m^{\Phi(m)}).$$

Since

$$\bigwedge_{n=1}^{\infty} J_0\Big(\bigwedge_{k=1}^{n} y_k^{\Phi(k)}\Big) = J_0\Big(\bigwedge_{n=1}^{\infty} y_n^{\Phi(n)}\Big) = 0$$

we have

$$\bigwedge_{n=1}^{\infty} J_1(z_n) = \bigwedge_{n=1}^{\infty} J_1(z_n) - \bigwedge_{m=1}^{\infty} J_0\Big(\bigwedge_{k=1}^{m} y_k^{\Phi(k)}\Big) =$$

$$= \bigvee_{m=1}^{\infty} \Big[\bigwedge_{n=1}^{\infty} J_1(z_n) - J_0\Big(\bigwedge_{k=1}^{m} y_k^{\Phi(k)}\Big)\Big] \leqq \bigvee_{m=1}^{\infty} \Big[J_1(z_m) - J_0\Big(\bigwedge_{k=1}^{m} y_k^{\Phi(k)}\Big)\Big] \leqq$$

$$\leqq \bigvee_{m=1}^{\infty} \Big\{J_1(z_m) - \sum_{k=1}^{m-1} [J_0(y_k^{\Phi(k)}) - J_1(z_k)] - J_0(y_m^{\Phi(m)}) =$$

$$= \bigvee_{m=1}^{\infty} \sum_{k=1}^{m} [J_1(z_k) - J_0(y_k^{\Phi(k)})] = \bigvee_{m=1}^{\infty} \sum_{k=1}^{m} a_k^{\Phi(k)}.$$

With respect to the strong regularity of G we have

$$\bigwedge_{n=1}^{\infty} J_1(z_n) \leqq 0.$$

From the definition of J_1 we get $\bigwedge_{n=1}^{\infty} J_1(z_n) \geqq 0$.

Hence

$$\bigwedge_{n=1}^{\infty} J_1(z_n) = 0.$$

An analogous assertion to Lemma 3 and Lemma 4 holds also for the set A_δ.

We put

$$a_n^i = J_1(z_n) - J_0(y_n^i).$$

Evidently

$$a_n^i \searrow 0 \ (i \to \infty),\ a_n^i \leqq J_1(z_1) \text{ for } n = 1, 2, \ldots, i = 1, 2, \ldots$$

Lemma 5. *If $x \in A_\sigma$, $y \in A_\delta$, $x \geqq y$, then $J_1(x) = J_1(y) + J_1(x-y)$.*

Proof. There exist $x_n, y_n \in A$, $x_n \nearrow x$, $y_n \searrow y$. From Lemma 2 we have

$$J_1(x) = J_0(x_n \wedge y_m) + J_1(x - x_n \wedge y_m) \text{ for all } m, n.$$

For $m \to \infty$ we get $x_n \wedge y_m \searrow x_n \wedge y \in A_\delta$. Then according to 5 and Lemma 2 there holds $(x - x_n \wedge y_m) \nearrow (x - x_n \wedge y) \in A_\sigma$ and from Lemma 3 it follows that

$$J_1(x) = J_1(x_n \wedge y) + J_1(x - x_n \wedge y).$$

For $n \to \infty$ and from 1, 5 and Lemma 2 we have $x_n \wedge y \nearrow y$, $(x - x_n \wedge y) \searrow (x-y)$, $x - y \in A_\sigma$. Hence $J_1(x) = J_1(y) + J_1(x-y)$ by Lemma 4.

The following lemma shows that the definition of J_2 is correct.

Lemma 6. *If $x, y \in A_{\sigma\delta}$, $x \leqq y$, $x_n, y_n \in A_\sigma$, $x_n \searrow x$, $y_n \searrow y$, then $\bigwedge_{n=1}^{\infty} J_1(x_n) \leqq \bigwedge_{n=1}^{\infty} J_1(y_n)$. If $x, y \in A_{\delta\sigma}$, $x \leqq y$, $x_n, y_n \in A_\delta$, $x_n \nearrow x$, $y_n \nearrow y$, then $\bigvee_{n=1}^{\infty} J_1(x_n) \leqq \bigvee_{n=1}^{\infty} J_1(y_n)$.*

If further $x \in A_{\sigma\delta} \cap A_{\delta\sigma}$, $x_n \in A_\sigma$, $x_n \searrow x$, $y_n \in A_\delta$, $y_n \nearrow x$, then $\bigwedge_{n=1}^{\infty} J_1(x_n) = \bigvee_{n=1}^{\infty} J_1(y_n)$.

Proof. Analogous to that of Lemma 1. We shall use Lemma 1, Lemma 4 and Lemma 5.

Lemma 7. *If $x \in A_\sigma$, $y \in A_\delta$, $x \geqq y$, then $J_1(x) \geqq J_1(y)$. If further $u \in A_{\delta\sigma}$, $v \in A_{\sigma\delta}$, $u \leqq v$, then $J_2(u) \leqq J_2(v)$.*

Proof. According to Lemma 5 and Lemma 1 we have $J_1(x) = J_1(y) + J_1(x-y)$, $J_1(x-y) \geqq 0$. Hence $J_1(x) \geqq J_1(y)$. Let $u_n \in A_\delta$, $v_n \in A_\sigma$, $u_n \nearrow u$, $v_n \searrow v$; then $u_n \leqq u \leqq v \leqq v_m$, $J_1(u_n) \leqq J_1(v_m)$ for all n, m and hence

$$J_2(u) = \bigvee_{n=1}^{\infty} J_1(u_n) \leqq \bigwedge_{n=1}^{\infty} J_1(v_m) = J_2(v).$$

Lemma 8. *Let $x \in S$. We assume that $u, y \in A_{\delta\sigma}$, $v, z \in A_{\sigma\delta}$ are such that $u \leqq x \leqq v$, $y \leqq x \leqq z$ and $J_2(u) = J_2(v)$, $J_2(y) = J_2(z)$. Then $J_2(v) = J_2(z)$.*

Proof. Evidently $v \wedge z \in A_{\sigma\delta}$, $v \wedge z \geqq x$. According to Lemma 7 and Lemma 6 we have

$$J_2(u) \leqq J_2(v \wedge z) \leqq J_2(v) = J_2(u), \text{ hence } J_2(v \wedge z) = J_2(v),$$

$$J_2(y) \leqq J_2(v \wedge z) \leqq J_2(z) = J_2(y), \text{ hence } J_2(v \wedge z) = J_2(z),$$

and

$$J_2(z) = J_2(v).$$

The preceding lemma shows that the definition of J is correct.

Lemma 9. *IF $u, v \in A_\delta$, then $J_1(u) + J_1(v) = J_1(u \vee v) + J_1(u \wedge v)$ and if $x, y \in A_{\delta\sigma}$, then $J_2(x) + J_2(y) = J_2(x \vee y) + J_2(x \wedge y)$.*

Proof. From the definition of A_δ there exist $u_n, v_n \in A$, $u_n \searrow u$, $v_n \searrow v$. Then $u_n \wedge v_n \searrow u \wedge v$, $u_n \vee v_n \searrow u \vee v$ by 1 and from (II) we have $J_0(u_n) + J_0(v_n) = J_0(u_n \vee v_n) + J_0(u_n \wedge v_n)$ for every n. Hence

$$J_1(u) + J_1(v) = J_1(u \vee v) + J_1(u \wedge v).$$

If $x_n, y_n \in A_\delta$, $x_n \nearrow x$, $y_n \nearrow y$, then $x_n \vee y_n \nearrow x \vee y$, $x_n \wedge y_n \nearrow x \wedge y$ by 1. Applying the first assertion of this lemma and from the definition of J_2 we get the second assertion.

Lemma 10. *If $u, v \in A_\delta$, then $J_1(u+v) \leqq J_1(u) + J_1(v)$. If $x, y \in A_{\delta\sigma}$, then $J_2(x+y) \leqq J_2(x) + J_2(y)$.*

Proof. The first assertion follows from 4, (IV) and from the definitions of A_δ and J_1. If $x, y \in A_{\delta\sigma}$, then there exist $x_n, y_n \in A_\delta$, $x_n \nearrow x$, $y_n \nearrow y$. By 4, $x_n + y_n \nearrow x + y$. From the preceding there holds for every n

$$J_1(x_n + y_n) \leqq J_1(x_n) + J_1(y_n),$$

hence

$$J_2(x+y) \leqq J_2(x) + J_2(y).$$

Lemma 11. *If $x \in A_{\sigma\delta}$, $y \in A_{\delta\sigma}$, then $x - y \in A_{\sigma\delta}$. If $x \in A_{\sigma\delta}$, $y \in A_{\delta\sigma}$, $x \geqq y$, then $J_2(x) = J_2(y) + J_2(x-y)$.*

Proof. We have $x_n \in A_\sigma$, $y_n \in A_\delta$, $x_n \searrow x$, $y_n \nearrow y$ and $x_n \geqq y_n$, $x_n - y_n \in A_\sigma$. By 4 and 5 there holds $x_n - y_n \searrow x - y$. Hence $x - y \in A_{\sigma\delta}$. From Lemma 5 and the definition of J_2 we get

$$J_2(x) = J_2(y) + J_2(x-y).$$

Lemma 12. *If $x_n \in A_{\delta\sigma}$, $x_n \nearrow x$, then $x \in A_{\delta\sigma}$ and $J_2(x) = \bigvee_{n=1}^{\infty} J_2(x_n)$.*

Proof. We put $y_n = \bigvee_{j=1}^{n} \bigvee_{i=1}^{n} x_j^i$ where $x_j^i \in A_\delta$, $x_j^i \nearrow x_j (i \to \infty)$. Then $y_n \in A_\delta$, $y_n \leqq x_n$, $y_n \nearrow x$, hence $x \in A_{\delta\sigma}$.

From Lemma 6 we get

$$J_1(y_n) \leqq J_2(x_n) \leqq J_2(x) \text{ for every } n.$$

Since $J_2(x) = \bigvee_{n=1}^{\infty} J_1(y_n)$, we have

$$J_2(x) = \bigvee_{n=1}^{\infty} J_2(x_n).$$

Theorem. *The operator $J: S \to G$ is an extension of J_0 such that J satisfies the properties* (I)—(V). *If $L: S \to G$ is an extension of J_0 satisfying* (I)—(V), *then $L = J$. If $x \in X$ and there exist $y, z \in S$ such that $y \leqq x \leqq z$, $J(y) = J(z)$, then $x \in S$ and S is a conditionally σ-complete lattice.*

Proof. The properties (I)—(IV), the uniqueness of J and the completeness of S can be proved easily by aplling Lemmas 6, 7, 9, 10 and 11. The methods of this proof are analogous to the methods used in the proofs of Theorems 1, 2, 3, 4, 6 and 7 in paper [3].

We shall prove that:

If $x_n \in S$, $x_n \leqq x_{n+1}$ and there exist $a \in X$ such that $x_n \leqq a$ for all n, then $x = \bigvee_{n=1}^{\infty} x_n \in S$ and $J(x) = \bigvee_{n=1}^{\infty} J(x_n)$.

We may suppose that $a \in A$.

Let $r_n \in A_{\sigma\delta}$, $z_n \in A_{\delta\sigma}$, $z_n \leqq x_n \leqq r_n$, $J_2(z_n) = J_2(r_n)$, $z_n \leqq z_{n+1}$, $z_n \nearrow z$ for $n = 1, 2, \ldots$

From Lemma 12 it follows that

$$z \in A_{\delta\sigma}, J_2(z) = \bigvee_{n=1}^{\infty} J_2(z_n) = \bigvee_{n=1}^{\infty} J(x_n).$$

Evidently $z \leqq x$.

We put $y_n = r_n \wedge a$. Then $x_n \leqq y_n \leqq r_n$.

By Lemma 7 and Lemma 6

$$J_2(z_n) = J_2(y_n) = J(x_n) \text{ holds.}$$

Denote

$$y_n^i = r_n^i \wedge a \text{ and } a_n^i = J_1(y_n^i) - J_2(y_n) \text{ where } r_n^i \in A_\sigma,\ r_n^i \searrow r_n.$$

Then

$$a_n^i \leqq J_0(a) - J_2(z_1),\ a_n^i \searrow 0\ (i \to \infty) \text{ for } n = 1, 2, \ldots$$

By the strong regularity of G there exists the sequence $\Phi_1, \Phi_2, \ldots$ of elements from N^N such that if $b \leqq \bigvee_{m=1}^{\infty} \sum_{k=1}^{m} a_k^{\Phi_n(k)}$ for $n = 1, 2, \ldots$, then $B \leqq 0$. We put $u_n = \bigvee_{k=1}^{\infty} y_k^{\Phi_n(k)}$, $u = \bigwedge_{n=1}^{\infty} u_n$. Then $u \geqq \bigvee_{k=1}^{\infty} x_k$ and $u \in A_{\sigma\delta}$. Applying Lemma 6 and Lemma 3 we get

$$J_2(u) - J_2(z) \leqq J_1(u_n) - J_2(z) = J_1\Big(\bigvee_{r=1}^{\infty} \bigvee_{k=1}^{r} y_k^{\Phi_n(k)}\Big) -$$

$$- J_2(z) = \bigvee_{r=1}^{\infty} J_1\Big(\bigvee_{k=1}^{r} y_k^{\Phi_n(k)}\Big) - \bigvee_{r=1}^{\infty} J_2(z_r) \leqq$$

$$=\bigvee_{r=1}^{\infty}\left[J_1\left(\bigvee_{k=1}^{r} y_k^{\Phi_n(k)}\right)-J_2(z_r)\right]\leqq$$

$$=\bigvee_{r=1}^{\infty}\left[\sum_{k=1}^{r} J_1(y_k^{\Phi_n(k)})-\sum_{k=1}^{r-1} J_2(y_k)-J_2(y_r)\right]=\bigvee_{r=1}^{\infty}\sum_{k=1}^{r} a_k^{\Phi_n(k)}$$

for every n, and hence $J_2(u)\leqq J_2(z)$.

We have

$$J_1\left(\bigvee_{k=1}^{r} y_k^{\Phi_n(k)}\right)\leqq\sum_{k=1}^{r} J_1(y_k^{\Phi_n(k)})-\sum_{k=1}^{r-1} J_2(y_k).$$

By Lemma 7 we have

$$J_2(u)\geqq J_2(z).$$

From the preceding it follows that

$$x\in S \text{ and } J(x)=J_2(z)=\bigvee_{n=1}^{\infty} J(x_n).$$

An analogous assertion holds for $x_n\searrow x$.

Corollary 1. *Let $\mathscr{X}$ be the system of all subsets of a set M with the set-theoretical operations $\cap$, $\cup$ and $-$. Let $+$ be identical with $\cup$. Let $\mathscr{A}$ be a non-empty algebra of subsets of M on which we define a G-valued measure μ; i.e. μ: $\mathscr{A}\to G$ is the set function fulfilling the following conditions:*

(i) *$\mu(A)\geqq 0$ for every $A\in\mathscr{A}$ (0 is a zero element of G),*

(ii) *μ is finitely additive: i.e. if $A_i\in\mathscr{A}$, $i=1, 2, \ldots, n$, and $A_i\cap A_j=\emptyset$ for $i\neq j$, then*

$$\mu\left(\bigcup_{i=1}^{n} A_i\right)=\sum_{i=1}^{n}\mu(A_i),$$

(iii) *μ is continuous from above at $\emptyset$, i.e. if $A_i\in\mathscr{A}$, $i=1, 2, \ldots$, $A_i\searrow\emptyset$, then $\mu(A_i)\searrow 0$.*

Put $J_0=\mu$. Then the assumptions of the Theorem hold, and we have the measure extension theorem.

Proof. It is evident that $\mathscr{X}$ fulfils the relations 1—8. We need to prove that μ fulfils the axioms (I)—(V). If $A, B\in\mathscr{A}$, $A\subset B$, then $B=A\cup(B-A)$. By (ii) $\mu(B)=\mu(A)+\mu(B-A)$, and we have (III). Since $\mu(A)\geqq 0$ for every $A\in\mathscr{A}$ and by (III) there holds $\mu(B)\geqq\mu(A)$, which is (I). The axioms (II) and (IV) follow from the following

$$\begin{aligned}\mu(A\cup B)+\mu(A\cap B)&=\mu[(A-B)\cup(B-A)\cup(A\cap B)]+\mu(A\cap B)=\\&=\mu(A-B)+\mu(B-A)+\mu(A\cap B)+\mu(A\cap B)=\\&=\mu[(A-B)\cup(A\cap B)]+\mu[(B-A)\cup(A\cap B)]=\mu(A)+\mu(B).\end{aligned}$$

The property (V) holds by the definition of the G-valued measure.

Corollary 2. *Let X be system of all G-valued mappings defined on a set M with the operations $+$, $-$ defined as usually and the operations $\vee$, $\wedge$ where $u = x \vee y$ ($v = x \wedge y$) iff for all $t \in M$ we have $u(t) = x(t) \vee y(t)$ ($v(t) = x(t) \wedge y(t)$).*

Let A be such a sublattice of X that to any $f\colon X \to G$ there are $h, g \in A$ with $h \leqq f \leqq g$. Let $J_0\colon A \to G$ be a mapping satisfying the conditions:

(i) *$J_0(f+g) = J_0(f) + J_0(g)$ for all $f, g \in A$,*

(ii) *if $f, g \in A$, $f \leqq g$, then $J_0(f) \leqslant J_0(g)$,*

(iii) *if $f_n \in A$ ($n = 1, 2, \ldots$), $f_n \searrow 0$ (where 0 is the mapping which $0(t) = 0$ for all $t \in M$), then $J_0(f_n) \searrow 0$.*

Then the assumptions of the Theorem hold, and we have the Daniell integral extension theorem.

Proof. It is easy to show that every σ-complete l-group G fulfils the properties 1—8. Then the system X fulfils 1—8 too. We see that the conditions (I), (IV), (V) are evident. By (i) we have $J_0(f+g) = J_0(f \vee g + f \wedge g) = J_0(f \vee g) + J_0(f \wedge g)$ and $J_0(f) = J_0[g + (f-g)] = J_0(g) + J_0(f-g)$ for all $f, g \in A$, hence (II) and (III) hold too.

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О РАСШИРЕНИИ МОНОТОННЫХ ОПЕРАТОРОВ

Марта Вонкомерова

Резюме

Статья посвящена проблемам расширения операторов, определение которых дается на определенных подструктурах со значениями в частично упорядоченной группе G, обладающей свойством сильной регулярности. Специально получаются расширение меры, определенной на алгебре подмножеств данного множества, и расширение интеграла, определенного простыми функциями, причем значения меры и интеграла имеются в сильно регулярной, частично упорядоченной группе.