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ON MAULDIN'S CLASSIFICATION OF REAL FUNCTIONS

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ABSTRACT. In this paper we investigate the Baire system generated by the family of all Darboux quasicontinuous, almost everywhere continuous functions, and prove that every function f of Mauldin's class $\alpha > 1$ is the limit of a sequence of Darboux functions f_n of Mauldin's class $\alpha_n < \alpha$, $n = 1, 2, \dots$.

Let us establish some terminology to be used later.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be quasicontinuous at a point $x \in \mathbb{R}$ if for all open neighbourhoods U of x and V of $f(x)$ there exists a nonempty open set $W \subset U \cap f^{-1}(V)$, ([5]).

Denote by $\mathcal{Q}$ the family of all quasicontinuous functions $f\colon \mathbb{R} \to \mathbb{R}$, by $\mathcal{A}$ the family of all almost everywhere continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ (with respect to the Lebesgue measure) and by $\mathcal{D}$ the family of all Darboux functions $f\colon \mathbb{R} \to \mathbb{R}$.

Given a fixed countable ordinal number $\alpha > 0$ and fixed family $\mathcal{K}$ of functions $f\colon \mathbb{R} \to \mathbb{R}$ we put

$$\mathcal{B}_0(\mathcal{K}) = \mathcal{K},$$

$$\mathcal{B}_\alpha(\mathcal{K}) = \Big\{ f\colon \mathbb{R} \to \mathbb{R} : \ f \text{ is the limit of the sequence of functions } f_n \in \bigcup_{\beta < \alpha} \mathcal{B}_\beta(\mathcal{K}),\ n = 1, 2, \dots \Big\}.$$

Let $\mathcal{P}$ denote the family of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the set $C(f)$ of its continuity points is dense.

In [3] it is proved that

$$\mathcal{B}_1(\mathcal{D} \cap \mathcal{Q}) = \mathcal{P}.$$

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In [6] Mauldin proved that for every countable ordinal number $\alpha > 0$,

$$\mathcal{B}_\alpha(\mathcal{A}) = \mathcal{M}_\alpha ,$$

where $f \in \mathcal{M}_\alpha$ if and only if there exists a function $g\colon \mathbb{R} \to \mathbb{R}$ of Baire class α and an F_σ set A of measure zero such that $\{x \in \mathbb{R} : f(x) \neq g(x)\} \subset A$.

In this paper I prove that $\mathcal{B}_1(\mathcal{D} \cap \mathcal{Q} \cap \mathcal{A}) = \mathcal{M}_1 \cap \mathcal{P}$,

$$B_1(\mathcal{M}_1 \cap \mathcal{P} \cap \mathcal{D}) = \mathcal{M}_2 \qquad \text{and} \qquad \mathcal{B}_1\Big(\mathcal{D} \cap \bigcup_{\beta<\alpha} \mathcal{M}_\beta\Big) = \mathcal{M}_\alpha .$$

THEOREM 1. *The following equality is true:*

$$\mathcal{B}_1(\mathcal{D} \cap \mathcal{Q} \cap \mathcal{A}) = \mathcal{M}_1 \cap \mathcal{P} .$$

Proof. Since $\mathcal{B}_1(\mathcal{D} \cap \mathcal{Q}) = \mathcal{P}$ and $\mathcal{B}_1(\mathcal{A}) = \mathcal{M}_1$ we have $\mathcal{B}_1(\mathcal{D} \cap \mathcal{Q} \cap \mathcal{A}) \subset \mathcal{M}_1 \cap \mathcal{P}$.

Let $f \in \mathcal{M}_1 \cap \mathcal{P}$. There exist a function $g\colon \mathbb{R} \to \mathbb{R}$ of Baire class 1 and an F_σ set B of measure zero such that $\{x \in \mathbb{R} : h(x) \neq g(x)\} \subset B$.

Put $h = f - g$. Evidently $h \in \mathcal{M}_1 \cap \mathcal{P}$ and

$$\{x \in \mathbb{R} : h(x) \neq 0\} \subset B .$$

Let

$$F_n = \{x \in \mathbb{R} : \operatorname{osc} h(x) \geq 2^{-n}\} , \qquad n = 1, 2, \dots . \tag{1}$$

Since all sets $B \cap F_1$ and $B \cap (F_n \setminus F_{n-1})$, $n = 1, 2, \dots$ are F_σ sets of measure zero, we can write

$$\begin{aligned} B \cap F_1 &= \bigcup_m F_{1,m} , \\ B \cap (F_n \setminus F_{n-1}) &= \bigcup_m F_{n,m} \qquad \text{for} \quad n = 2, 3, \dots , \end{aligned} \tag{2}$$

where all sets $F_{n,m}$ are closed and pairwise disjoint, $n, m = 1, 2, \dots$ ([8]).

For a fixed $k \geq 1$ there are pairwise disjoint closed intervals $I_{k,n,m,j} = [a_{k,n,m,j}, b_{k,n,m,j}]$ $(n + m \leq k + 1$, $F_{n,m} \neq \emptyset$ and $j = 1, 2, \dots)$, contained in $\mathbb{R} \setminus F_n \setminus \bigcup_{n+m\leq k+1} F_{n,m}$ such that:

(3) if $x \in I_{k,n,m,j}$ there is a point $y \in F_{n,m}$ such that $|x - y| < 1/k$;

(4) for each $x \in F_{n,m}$ and for each $r > 0$ there are indices j_1, j_2 such that $I_{k,n,m,j_1} \subset (x, x + r)$ and $I_{k,n,m,j_2} \subset (x - r, x)$;

(5) if there is the limit $\lim_{i\to\infty} x_i = x$, where $x_i \in I_{k,n,m,j(i)}$ $(j(i_1) > j(i_2)$ for $i_1 > i_2)$ then $x \in F_{n,m}$.

For each interval $I_{k,n,m,j}$ $(n+m \le k+1,\ j=1,2,\dots)$ there is a function $h_{k,n,m,j}\colon I_{k,n,m,j} \to \mathbb{R}$ such that:

(6) $h_{k,n,m,j}(a_{k,n,m,j}) = h_{k,n,m,j}(b_{k,n,m,j}) = 0$;

(7) $h_{k,1,m,j}(I_{k,1,m,j}) = \mathbb{R}$;

(8) $h_{k,n,m,j}(I_{k,n,m,j}) = [-2^{-n+2}, 2^{-n+2}]$ for $n > 1$;

(9) $h_{k,1,m,j}$ is continuous on the interval $(a_{k,1,m,j}, b_{k,1,m,j}]$ and for $n > 1$ a function $h_{k,n,m,j}$ is continuous on the interval $[a_{k,n,m,j}, b_{k,n,m,j}]$.

Let $h_k\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h_k(x) = \begin{cases} h_{k,n,m,j}(x) & \text{for } x \in I_{k,n,m,j}, \\ h(x) & \text{for } x \in F_{n,m}, \\ 0 & \text{otherwise} \end{cases}$$

if $n+m \le k+1$ and $j = 1,2,\dots$.

From (9), (6) and (5) it follows that h_k is continuous at all points of the set

$$G = \Big(\mathbb{R} \setminus \bigcup_{n+m \le k+1} F_{n,m}\Big) \setminus \bigcup_{m \le k} \bigcup_j \{a_{k,1,m,j}\}.$$

Since $\mathbb{R} \setminus G$ is of measure zero, h_k is almost everywhere continuous.

By (1), (2), (4), (7), (8) and (9), h_k is quasicontinuous and has the Darboux property.

The function g is the limit of a sequence of continuous functions g_k, $k = 1,2,\dots$. Let $f_k = g_k + h_k$ for $k = 1,2,\dots$.

The function f_k is quasicontinuous as the sum of the quasicontinuous function h_k and the continuous function g_k ([4]). The same f_k is almost everywhere continuous and continuous at each point of the set G.

Now we shall prove that every f_k $(k = 1,2,\dots)$ has the Darboux property. Assume the contrary that f_k does not have the Darboux property. There are real numbers a, b, c such that $a < b$, $c \in \big(\min(f_k(a), f_k(b)),\ \max(f_k(a), f_k(b))\big)$ and $c \notin f_k((a,b))$.

For definiteness assume that $f_k(a) < f_k(b)$. Let

$$d = \inf\{x \in (a,b] :\ f_k(x) > c\}.$$

Since g_k is continuous and $f_k = g_k + h_k$ is not continuous at the point d, h_k is not continuous at d. Consequently, $d \in \mathbb{R} \setminus G$.

If $f_k(d) < c$ and there are indices n, m, j such that $m \le k$ and $d = a_{k,1,m,j}$ then we may observe that the restricted function $h_k|I_{k,1,m,j}$ has the Darboux property and it is of Baire class 1. Consequently, $f_k|I_{k,1,m,j}$ has the Darboux property as the sum of continuous function $g_k|_{I_{k,1,m,j}}$ and the Darboux function $h_k|_{I_{k,1,m,j}}$ which is of Baire class 1 ([1]). If $f_k(d) < c$ and $d = \inf\{x \in (a,b] : f_k(x) > c\}$ then there is a point $z \in (a,b)$ such that $f_k(z) = c$. This contradicts the relation $c \notin f_k\big((a,b)\big)$.

If $f_k(d) < c$ and there is an index $m \leq k$ such that $d \in F_{1,m}$ then, by (4), there is an interval $I_{k,1,m,j} \subset (a,b)$. Since the restriction function $f_k|_{I_{k,1,m,j}}$ has the Darboux property, we have, by (7), $f_k\big((a,b)\big) = f_k(I_{k,1,m,j}) = \mathbb{R}$ and $c \in f_k\big((a,b)\big)$. This contradicts the relation $c \notin f_k((a,b))$.

If $f_k(d) < c$ and there are indices n, m such that $n > 1$, $n+m \leq k+1$ and $d \in F_{n,m}$, then $|h_k(d)| < 2^{-n+1}$. Since $f_k(d) = h_k(d) + g_k(d) < c$, it follows from the continuity of g_k at the point d and from (5) that there is an interval $I = [d,e]$ with $e \in (a,b) \setminus \bigcup_j I_{k,n,m,j}$ such that:

(10) $|g_k(x) - g_k(d)| < 2^{-n+1}$;
(11) $h_k(d) + g_k(x) < c$

for every $x \in (d,e)$.

From the definition of d there is a point $u \in (d,e)$ such that $f_k(u) > c$.

If there is an interval $I_{k,n,m,j}$ with $u \in I_{k,n,m,j}$ then from (8) and (11) there is a point $w \in I_{k,n,m,j}$ such that

$$f_k(w) = g_k(w) + h_k(w) < g_k(w) + h_k(d) < c.$$

Since $f_k|_{I_{k,n,m,j}}$ has the Darboux property,

$$c \in f_k(I_{k,n,m,j}) \subset f_k\big((a,b)\big),$$

which contradicts the relation $c \notin f_k\big((a,b)\big)$.

If $u \notin \bigcup_j I_{k,n,m,j}$ then $h_k(u) = 0$ or $u \in F_{n,m}$. Let $I_{k,n,m,j} \subset I$. Since $|h_k(u)| < 2^{-n+1}$, it follows from (8) and (10) that there is a point $v \in I_{k,n,m,j}$ such that:

$$\begin{aligned} f_k(v) &= h_k(v) + g_k(v) = 2^{-n+2} + g_k(v) \\ &> 2^{-n+2} + g_k(u) - 2^{-n+1} = 2^{-n+1} + g_k(u) \\ &> h_k(u) + g_k(u) > c. \end{aligned}$$

As above, it follows from (11) that there is a point $w \in I_{k,n,m,j}$ such that $f_k(w) < c$ and $c \in f_k\big((a,b)\big)$, which contradicts the relation $c \notin f_k\big((a,b)\big)$.

Similarly, we may consider the case, where $f_k(d) > c$.

So every function f_k $(k = 1,2,\dots)$ has the Darboux property.

Since $f_k = g_k + h_k$, $f = g + h$ and $g = \lim_{k\to\infty} g_k$, it is sufficient for the proof of the equality $f = \lim_{k\to\infty} f_k$ to prove that $h = \lim_{k\to\infty} h_k$.

If $x \in F_{n,m}$ then $h_k(x) = h(x)$ for $k > n+m$ and $h(x) = \lim_{k\to\infty} h_k(x)$.

Suppose that h is continuous at x. For fixed $\varepsilon > 0$ there is an index $k_0 > 1$ such that $2^{-k_0+2} < \varepsilon$. Since $x \notin F_{k_0}$, there is a positive number r such that

$$(x-r, x+r) \cap F_{k_0} = \emptyset .$$

Let $k_2 > k_0$ be an index such that $1/k_2 < r$. From (3), (8) and from the definition of h_k it follows, that for $k > k_2$, $|h_k(x)| \le 2^{-k_0+2} < \varepsilon$. So $\lim_{k\to\infty} h_k(x) = 0 = h(x)$.

Now, let $x \in F_n \setminus B$, for some index n. Since $F_n \subset F_k$ and every $I_{k,n,m,j} \subset \mathbb{R} \setminus F_k \subset \mathbb{R} \setminus F_n$ for $k > n$, it follows from (2) and from the definition of h_k that $h_k(x) = 0 = h(x)$ for $k > n$. So $\lim_{k\to\infty} h_k(x) = h(x)$. This completes the proof.

□

THEOREM 2. *The following equality is true:*

$$\mathcal{M}_2 = \mathcal{B}_1(\mathcal{M}_1 \cap \mathcal{P} \cap D).$$

P r o o f. Since $\mathcal{M}_2 = \mathcal{B}_1(\mathcal{M}_1)$, $\mathcal{M}_2 \supset \mathcal{B}_1(\mathcal{M}_1 \cap \mathcal{P} \cap \mathcal{D})$.

Now, let $f \in \mathcal{M}_2$. There exist a function g of Baire class 2 and an F_σ set B of measure zero such that:

$$\{x \in \mathbb{R} : \ f(x) \ne g(x)\} \subset B .$$

We can write $B = \bigcup_{n=1}^{\infty} B_n$, where all the sets B_n are closed and $B_n \subset B_{n+1}$ for $n = 1, 2, \dots$.

The function g is the limit of a sequence of functions g_n of Baire class 1. For $k = 1, 2, \dots$ let

$$h_k(x) = \begin{cases} g_k(x) & \text{for } x \in \mathbb{R} \setminus B, \\ f(x) & \text{for } x \in B_k . \end{cases}$$

Evidently, every function h_k ($k = 1, 2, \dots$) is pointwise discontinuous. For $k = 1, 2, \dots$ there is ([2]) an almost everywhere continuous function $t_k \colon \mathbb{R} \to \mathbb{R}$ of Baire class 1 such that:

- $\{x \in \mathbb{R} : \ t_k(x) \ne 0\}$ is F_σ set of measure zero;
- $\{x \in \mathbb{R} : \ t_k(x) \ne 0\} \cap B = \emptyset$;
- $\{x \in \mathbb{R} : \ t_{k_1}(x) \ne 0\} \cap \{x \in \mathbb{R} : \ t_{k_2}(x) \ne 0\} = \emptyset$ if $k_1 \ne k_2$ ($k_1, k_2 = 1, 2, \dots$);
- $h_k + t_k \in \mathcal{P} \cap \mathcal{D}$.

Let $f_k = h_k + t_k$, $k = 1,2,\dots$. Since

$$\{x \in \mathbb{R}:\ f_k(x) \neq g_k(x)\} \subset \{x \in \mathbb{R}:\ t_k(x) \neq 0\} \cup B_k,$$

we have $f_k \in \mathcal{M}_1$. So $f_k \in \mathcal{M}_1 \cap \mathcal{D} \cap \mathcal{P}$ for $k = 1,2,\dots$.

If $x \in B$ then there is an index n such that $x \in B_k$ for $k \geq n$ and consequently, $f_k(x) = f(x)$ for $k > n$. So $\lim_{k\to\infty} f_k(x) = f(x)$.

If $x \notin B$ then $h_k(x) = g_k(x)$ for $k = 1,2,\dots$. Since $\lim_{k\to\infty} g_k(x) = g(x)$ and $\lim_{k\to\infty} t_k(x) = 0$, we have

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} g_k(x) = g(x) = f(x).$$

This completes the proof. □

From Theorems 1 and 2 there follows:

COROLLARY 1. *For denumerable ordinal numbers $\alpha > 1$ the following equality is true:*

$$\mathcal{B}_\alpha(\mathcal{A}) = \mathcal{B}_\alpha(\mathcal{A} \cap \mathcal{Q} \cap \mathcal{D}).$$

THEOREM 3. *For every denumerable ordinal number $\alpha > 0$ the following equality is true:*

$$\mathcal{B}_1\Big(\mathcal{D} \cap \bigcup_{\beta<\alpha} \mathcal{M}_\beta\Big) = \mathcal{M}_\alpha.$$

Proof. For $\alpha = 2$ this theorem follows from Theorem 2. For $\alpha = 1$ the proof is the same as the proof of Theorem 2, where the g_k are continuous and consequently $h_k \in \mathcal{A}$. (Instead of [2] we need [7].)

Assume that $\alpha > 2$. The inclusion

$$\mathcal{B}_1\Big(\mathcal{D} \cap \bigcup_{\beta<\alpha} \mathcal{M}_\beta\Big) \subset \mathcal{M}_\alpha$$

is obvious. If $f \in \mathcal{M}_\alpha$ then there exist a function g of Baire class α and an F_σ set B of measure zero such that

$$\{x \in \mathbb{R}:\ f(x) \neq g(x)\} \subset B.$$

The function g is the limit of the sequence of functions g_n of Baire class β_n, where $\beta_n < \alpha$ $(n = 1,2,\dots)$ and $B = \bigcup_{n=1}^{\infty} B_n$ where $B_n \subset B_{n+1}$ and all the sets B_n are closed $(n = 1,2,\dots)$.

Let $C_{n,m} \subset \mathbb{R} \setminus B$ $(n,m = 1,2,\dots)$ be a family of pairwise disjoint perfect sets of measure zero such that for every open interval I and for every $n = 1,2,\dots$ there is m such that $C_{n,m} \subset I$. For all $n,m = 1,2,\dots$ let $h_{n,m}\colon C_{n,m} \to [-m,m]$ be a continuous function.

For $k = 1, 2, \ldots$ let us put

$$f_k(x) = \begin{cases} h_{k,m}(x) & \text{if } x \in C_{k,m}\,, \quad m = 1, 2, \ldots\,, \\ f(x) & \text{if } x \in B_k\,, \\ g_k(x) & \text{otherwise.} \end{cases}$$

Obviously, f_k has the Darboux property. Since

$$\{x \in \mathbb{R} : \ f_k(x) \neq g_k(x)\} \subset B_k \cup \bigcup_m C_{k,m}$$

and the set $B_k \cup \bigcup_m C_{k,m}$ is an F_σ set of measure zero, the function $f_k \in \mathcal{M}_{\beta_k}$, where $\beta_k < \alpha$.

The equality $f(x) = \lim_{k\to\infty} f_k(x)$ for every $x \in \mathbb{R}$, is obvious. □

Problem 1. *Is it true the following equality*

$$\mathcal{M}_\alpha \cap \mathcal{P} = \mathcal{B}_1\Big(\bigcup_{\beta<\alpha} \mathcal{M}_\beta \cap \mathcal{Q} \cap \mathcal{D}\Big) \quad \textit{for } \alpha > 1?$$

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