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HYPERINVARIANT SUBSPACE LATTICE OF ISOMETRIES

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1. Introduction

Let $\mathfrak{H}$ be a complex separable Hilbert space and let $B(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$. A subspace (i.e. a closed linear manifold) $\mathfrak{L} \subset \mathfrak{H}$ is called invariant for $T \in B(\mathfrak{H})$ if $T\mathfrak{L} \subset \mathfrak{L}$. $\mathfrak{L}$ is hyperinvariant for T if it is invariant under each $A \in B(\mathfrak{H})$ that commutes with T. We denote Lat(T) and Hyplat(T) the set of all invariant and hyperinvariant subspaces of T, respectively.

If $\{\mathfrak{L}_\gamma, \gamma \in \Gamma\}$ is a family of hyperinvariant subspaces of T, then both the intersection $\bigcap_{\gamma \in \Gamma} \mathfrak{L}\gamma$ and the closed linear span $\bigvee_{\gamma \in \Gamma} \mathfrak{L}\gamma$ are from Hyplat(T), i.e. Hyplat(T) is a complete lattice.

Let $T \in B(\mathfrak{H})$. We denote by $\{T\}'$, $\{T\}''$ the commutant and the double commutant of T, respectively:

$$\{T\}' = \{S \in B(\mathfrak{H}) \colon ST = TS\}, \{T\}'' = \bigcap_{S \in \{T\}'} \{S\}'.$$

If $S \in \{T\}''$, then both

$\operatorname{Ker} S = \{h \in \mathfrak{H} \colon Sh = 0\}$ and $\overline{\operatorname{Ran}} S = \overline{SH}$

belong to Hyplat(T) (see [9]).

In [7], [8], [9] it was proved that some completely non-unitary contractions (among them all c.n.u. weak contractions) have the following property:

Definition 1.1. *An operator $T \in B(\mathfrak{H})$ is said to have the property* (L) *if* Hyplat(T) *is the smallest complete lattice which contains all subspaces of the forms* Ker S *and* $\overline{\operatorname{Ran}}\, S$ *for* $S \in \{T\}''$.

It was not known whether there exists an operator not having the property (L). The purpose of this paper is to show that an isometry need not have the property (L). We shall even obtain a characterization of the isometries having the property (L).

Let V be an arbitrary isometry on the space $\mathfrak{H}$ (i.e. $\|Vh\| = \|h\|$ for all $h \in \mathfrak{H}$). According to the Wold decomposition ([5], Theorem I.1.1) $\mathfrak{H}$ then decomposes into an orthogonal sum $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ such that $\mathfrak{H}_0$ and $\mathfrak{H}_1$ belong to Lat(T), $V|\mathfrak{H}_0$ is unitary and $V|\mathfrak{H}_1$ is a unilateral shift. Moreover

$$\mathfrak{H}_0 = \bigcap_{n=0}^{\infty} V^n\mathfrak{H} \text{ and so } \mathfrak{H}_0 \in \mathrm{Hyplat}(V). \tag{1.1}$$

It is easy to show that every unitary operator has the property (L) [9]. We shall show that the unilateral shift has the property (L) as well.

2. Unilateral shifts.

An isomtery $V \in B(\mathfrak{H})$ is called a unilateral shift if there exists a subspace $\mathfrak{L} \subset \mathfrak{H}$ such that $V^n\mathfrak{L}$ is orthogonal to $V^m\mathfrak{L}$ for all pairs of non-negative integers $n \neq m$ and $\bigoplus_{n=0}^{\infty} V^n\mathfrak{L} = \mathfrak{H}$. In what follows we shall use a functional model for unilateral shifts on Hardy spaces.

Let H^p, $1 < p \leqq \infty$, be the Hardy spaces of analytic functions in the unit disc D. It is well known that we may consider H^p as a subspace of the space L^p on the unit circle of those $f \in L^p$ which have the Fourier coefficients with negative indices zero. For more detail see Chapter III.1 of [5].

Let $\varphi \in L^\infty$, we denote $M(\varphi)$ the operator of multiplication by φ on L^2. If $\varphi \in H^\infty$, then $M(\varphi)H^2 \subset H^2$ and we denote $T(\varphi) = M(\varphi)|H^2$ the analytic Toeplitz operator with symbol φ.

If $\chi(e^{it}) = e^{it}$, then $S = T(\chi)$ is the unilateral shift of multiplicity 1. Then $\{S\}' = \{S\}'' = \{T(\varphi) : \varphi \in H^\infty\}$ ([6], Chapter 3). We call an inner function every $u \in H^\infty$ such that $|u(e^{it})| = 1$ almost everywhere with respect to the Lebesgue measure on the unit circle (a.e.). As was shown in [6], p. 42 $\mathrm{Lat}(S) = \{T(\varphi)H^2 : \varphi$ is an inner function$\}$. Obviously, Hyplat(S) Lat(S) and S has the property (L).

Every unilateral shift of multiplicity $n (1 \leqq n \leqq \infty)$ S_n is unitary equivalent to the orthogonal sum of n copies of S. S_n is defined on the space $H_n^2 = \bigoplus_{i=1}^{n} H_i$, $H_i = H^2$ for

$$1 \leqq i < n + 1 .$$

With respect to this orthogonal sum every operator in the commutant of S_n is an operator of multiplication by an $n \times n$ matrix over H^∞. The following lemma describes the double commutant of the unilateral shift of arbitrary multiplicity.

Lemma 2.1. $\{S_n\}'' = \{\varphi(S_n) : \varphi \in H^\infty\}$.

Proof. For $A\in\{S_n\}'$ denote A_{ij} the (i,j)th entry of the corresponding matrix. Let $A\in\{S_n\}''$ and $i\neq j$. If X is an operator with $X_{ij}=1$ and $X_{k1}=0$ for all $(k,1)\neq(i,j)$, then $X\in\{S_n\}'$ and we have

$$A_{ji}=(AX)_{jj}=(XA)_{jj}=0$$

$$A_{ii}=(AX)_{ij}=(XA)_{ij}=A_{jj}.$$

This means that $A=\varphi(S_n)$, where $\varphi=A_{ii}(1\leqq i<n+1)$.

By Chapter V.3.4 of [5] or by [3] every $\mathfrak{L}\in\mathrm{Hyplat}(S_n)$ is of the form $\mathfrak{L}=\varphi H^2_n$, where φ is an inner function. This means that S_n has the property (L).

3. The operator $U\oplus S$ does not have the propety (L)

Let $V\in B(\mathfrak{H})$ be an arbitrary isometry. By the Wold decomposition $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_1$, where $\mathfrak{H}_0$ and $\mathfrak{H}_1$ reduce V, $\mathfrak{H}_0\in\mathrm{Hyplat}(V)$ and $V_0=V|\mathfrak{H}_0$ is unitary, $V_1=V|H_1$ is a unilateral shift. We have shown that both V_0 and V_1 have the property (L). Now we shall show that V need not have (L).

Example 3.1. *The operator $V=U\oplus S$ on the space $L^2\oplus H^2$, where U is the bilateral shift*: $(Uf)(e^{it})=e^{it}f(e^{it}), f\in L^2$ and *S is the unilateral shift, has not the property* (L).

Proof. Denote by J: $H^2\to L^2$ the natural imbedding of H^2 into L^2, i.e. $Ju=u$ for $u\in H^2$. Then the operator

$$\begin{pmatrix}0 & J\\ 0 & 0\end{pmatrix}$$

commutes with V. $\{S\}'=\{T(\varphi)\colon\ \varphi\in H^\infty\}=\{S\}''$ and $\{U\}'=\{M(f)\colon\ f\in L^\infty\}=$ $=\{U\}''.\{V\}''\subset\{U\}''\oplus\{S\}''$ (Lemma 1.1 of [1]). Every $T\in\{V\}''$ is of the form

$T=\begin{pmatrix}M(f) & 0\\ 0 & T(\varphi)\end{pmatrix}$, $f\in L^\infty$, $\varphi\in H^\infty$. T commutes with $\begin{pmatrix}0 & J\\ 0 & 0\end{pmatrix}$, it follows that $JT(\varphi)=M(f)\mathrm{J}$, i.e. $M(f)|H^2=T(\varphi)$ and so $f=\varphi$. We conclude

$\{V\}''=\{\varphi(V)\colon\varphi\in H^\infty\}$. Let $\varphi=\varphi_i\varphi_e$ be the inner-outer factorization of $\varphi\in H^\infty$. By the theorem of Beurling [5, Proposition III.1.2] we have that if φ is not identically zero, then

$$\overline{\mathrm{Ran}}\ \varphi(V)=\overline{\varphi L^2}\oplus\overline{\varphi H^2}=L^2+\varphi_i H^2$$

and Ker $\varphi(V)=(0)$.

It follows that the smallest complete lattice containing all $\overline{\mathrm{Ran}}\,A$ and $\mathrm{Ker}\,A$ for $A\in\{V\}''$ is formed by the zero subspace and $L^2\oplus mH^2$ for all inner functions m. But Hyplat(V) contains also the subspace $\mathfrak{L}=L^2(0,\pi)\oplus(0)$, where $L^2(0,\pi)=$

$= \{u \in L^2 : u(e^{it}) = 0$ for $t \in (\pi, 2\pi)\}$. To show this we observe that every $A \in \{V\}'$ has the form

$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, where $A_1 \in \{U\}'$, $A_4 \in \{S\}'$, $A_2 S = U A_2$ and $A_3 = 0$ because $L^2 \oplus \oplus (0) = \bigcap_{n=0}^{\infty} V^n(L^2 \oplus H^2) \in \mathrm{Hyplat}(V)$. This means that $A_1 = M(u)$ for a function $u \in L^\infty$ and $A\mathfrak{L} \subset \mathfrak{L}$.

We conclude this section by some remarks

1) Example 3.1 shows that $T_1 \oplus T_2$ need not have (L) if both T_1 and T_2 have (L).

2) We have proved that all completely non-unitary weak contractions have the property (L). The operator $U \oplus S$ is not a weak contraction because its spectrum is the whole unit disc. In a subsequent paper we shall prove that every weak contraction has the property (L).

3) The operator $U \oplus S$ is subnormal. So our example shows that not every subnormal operator has the property (L). It shows also that if T has the property (L) and $\mathfrak{L} \in \mathrm{Lat}(T)$, than $T|\mathfrak{L}$ need not have the property (L).

4) The example 3.1 is a special case of the result of the following chapter where we shall give a characterization of the isometries having the property (L).

4. General case

Let $V \in B(\mathfrak{H})$ be an isometry. Similarly as in [2] we consider the unique decomposition $\mathfrak{H} = \mathfrak{H}_{0S} \oplus \mathfrak{H}_{0A} \oplus H_1$ such that $V_{0S} = V|\mathfrak{H}_{0S}$ is a singular unitary operator (i.e. its spectral measure is singular with respect to the Lebesgue measure), $V_{0A} = V|\mathfrak{H}_{0A}$ is an absolutely continuous unitary operator and $V_1 = V|\mathfrak{H}_1$ is a unilateral shift. R. G. DOUGLAS [2] showed that

$$\mathrm{Hyplat}(V) = \mathrm{Hyplat}(V_{0S}) \oplus \mathrm{Hyplat}(V_{0A} \oplus V_1). \tag{4.1}$$

According to [1] this is equivalent to both following relations:

$$\{V\}' = \{V_{0S}\}' \oplus \{V_{0A} \oplus V_1\}', \tag{4.2}$$

$$\{V\}'' = \{V_{0S}\}'' \oplus \{V_{0A} \oplus V_1\}''. \tag{4.3}$$

If $A \in \{V_{0S}\}''$, $B \in \{V_{0A} \oplus V_1\}''$, then the operators $A \oplus 0$, $A \oplus I$, $0 \oplus B$, $I \oplus B$ belong to $\{V\}''$ and

$$\left.\begin{aligned} \mathrm{Ker}\, A \oplus (0) &= \mathrm{Ker}(A \oplus I), \\ \overline{\mathrm{Ran}}\, A \oplus (0) &= \overline{\mathrm{Ran}}(A \oplus 0), \\ (0) \oplus \mathrm{Ker}\, B &= \mathrm{Ker}(I \oplus B), \\ (0) \oplus \overline{\mathrm{Ran}}\, B &= \overline{\mathrm{Ran}}(0 \oplus B). \end{aligned}\right\} \tag{4.4}$$

V_{0S} has the property (L) because every unitary operator has (L). Relations (4.1) — (4.4) show that V has the property (L). if and only if $V_{0A} \oplus V_1$ has the property (L).

According to the theory of spectral representations of normal operators [4] we may suppose that V_{0A} is the operator of multiplication by e^{it} on the space

$$L^2(E_1) \oplus L^2(E_2) \oplus \dots, \tag{4.5}$$

where $E_1 \supset E_2 \supset \dots$ are measurable subsets of the unit circle and the measure considered is always the normalized Lebesgue measure. If χ_{E_n} is the characteristic function of E_n, then $L^2(E_n) = \chi_{E_n} L^2$. Denote by $M(E_n)$ the restriction of the bilateral shift U to $L^2(E_n)$. From [6, Theorem 1.20] it follows similarly as for the whole shift that

$$\{M(E_n)\}' = \{M(E_n)\}'' = \{M(f) : f \in L^\infty(E_n)\}.$$

Lemma 4.1. *Let $E_1 \supset E_2 \supset \dots$ be measurable subsets of the unit circle and let $V_{0A} = M(E_1) \oplus M(E_2) \oplus \dots$*

Then $\{V_{0A}\}'' = \{M(f) \oplus M(f) \oplus \dots : f \in L^\infty(E_1)\}$

Proof. $\{V_{0A}\}'' \subset \{M(E_1)\}'' \oplus \{M(E_2)\}'' \oplus \dots$ because for each $i = 1, 2, \dots$ the operator A_i given by the matrix (corresponding to the decomposition (4.5)) with $A_{ii} = I$ and all the other entries zero commutes with V_{0A} (See [1]). If $B \in \{V_{0A}\}''$, then its matrix representation is a diagonal matrix:

$$B_{jj} = M(f_j),\ f_j \in L^\infty,\ j = 1, 2, \dots$$
$$B_{kj} = 0 \text{ if } k \neq j.$$

For $k > j$ let P_{kj} be the orthogonal projection of $L^2(E_k)$ into $L^2(E_j)$, i.e. the operator $M(\chi_{E_j}) | L^2(E_k)$.

Let X be the operator with the matrix:

$X_{kj} = P_{kj}$ and all the other entries zero. $X \in \{V_{0A}\}'$, therefore $XB = BX$. It follows that $f_j = \chi_{E_j} f_k$ (for each $k > j$). This means that B is the operator of multiplication by f_1 and the proof is finished.

Lemma 4.2. *If $\mathfrak{H} = L^2(E) \oplus H^2$, where E is a measurable subset of the unit circle and $V = M(E) \oplus S$, then*

$$\{V\}'' = \{\varphi(V) : \varphi \in H^\infty\}.$$

Proof. The proof is essentially the same as that in Example 3.1 for $U \oplus S$. Instead of the operator $Ju = u$ $(u \in H^2)$ we have to use the operator $J_E u = \chi_E u$ $(u \in H^2)$. Then the operator

$$\begin{pmatrix} 0 & J_E \\ 0 & 0 \end{pmatrix}$$

commutes with V. Every operator from $\{V\}''$ is of the from $f(M(E)) \oplus \varphi(S)$, $f \in L^\infty$, $\varphi \in H^\infty$. It follows that

$$J_E \varphi(S) = f(M(E)) J_E .$$

If we apply this equation to the constant function 1 (which is from H^2), we obtain:

$$\chi_E \varphi = \chi_E f, \text{ i.e. } f(M(E)) \oplus \varphi(S) = \varphi(V) .$$

Theorem 4.3. *Let $E_0 \supset E_1 \supset E_2 \supset \ldots$ be measurable subsets of the unit circle. Let us suppose that at least E_0 is of the positive Lebesgue measure and $1 \leqq n \leqq \infty$. If $\mathfrak{H} = \bigoplus_{i=0}^{\infty} L^2(E_i) \oplus H_n^2$ and $V = \bigoplus_{i=0}^{\infty} M(E_i) \oplus S_n$, then $\{V\}'' = \{\varphi(V) : \varphi \in H^\infty\}$.*

Proof. According to [1, Lemma 1.1]

$$\{V\}'' \subset \left\{ \bigoplus_{i=0}^{\infty} M(E_i) \right\}'' \oplus \{S_n\}'' .$$

Applying the lemmas 2.1 and 4.1 we obtain that every $A \in \{V\}''$ is of the form $A = (M(f) \oplus M(f) \oplus \ldots) \oplus \varphi(S_n)$, $f \in L^\infty$, $\varphi \in H^\infty$.

Let us consider the following decomposition of H:

$$\mathfrak{H} = L^2(E_0) \oplus H^2 \oplus H_{n-1}^2 \oplus \bigoplus_{i=1}^{\infty} L^2(E_i) .$$

Because $A | L^2(E_0) \oplus H^2 = M(f) \oplus M(\varphi)$ belongs to $\{M(E_0) \oplus S\}''$ lemma 4.2 asserts that $f(z) = \varphi(z)$ for $z \in E_0$, i.e. $A = \varphi(V)$.

Theorem 4.4. *Let V be an arbitrary isometry on a separable Hilbert space $\mathfrak{H}$. Then V has the property (L) if and only if either V is unitary or the absolutely continuous unitary part of V is zero.*

Proof. As mentioned at the beginning of this section we may assume that the singular unitary part of V is zero. If V is unitary or if V has no unitary part (i.e. V is the unilateral shift), then V has the property (L). If V is as in Theorem 4.3 with at least E_0 of the positive Lebesgue measure, then $\{V\}'' = \{\varphi(V) : \varphi \in H^\infty\}$. For each $\varphi \in H^\infty$, $\varphi \not\equiv 0$ $\operatorname{Ker} \varphi(V) = (0)$ and $\overline{\operatorname{Ran} \varphi(V)} = \bigoplus_{j=0}^{\infty} L^2(E_j) \oplus \varphi_i H_n^2$, where φ_i is the inner factor of φ. (See the proof of Example 3.1). If $F_0 \subset E_0$ is a meassurable subset of E_0 such that both F_0 and its complement in E_0 have the positive Lebesgue measure, then the subspace $\chi_{F_0}\left(\bigoplus_{j=0}^{\infty} L^2(E_j) \right) \oplus (0)$ is hyperinvariant for V, but it is not contained in the smallest complete lattice containing all $\operatorname{Ker} \varphi(V$: and $\operatorname{Ran} \varphi(V)$ for $\varphi \in H^\infty$.

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РЕШЕТКА ГИПЕРИНВАРИАНТНЫХ ПОДПРОСТРАНСТВ ИЗОМЕТРИЙ

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Резюме

В статье изучаются условия, при которых изометрия V в гильбертовом пространстве обладает следующим свойством:

(L) Решетка подпространств, гиперинвариактных для V порождена подпространствами, являющимися нуль-пространством или замыканием области значения оператора А из второго коммутанта V.

Доказывается, что V обладает свойством (L) тогда и только тогда, когда либо V — унитарный оператор, либо его абсолютно непрерывная унитарная часть нулевая.