Mathematica Slovaca

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Mathematica Slovaca, Vol. 30 (1980), No. 2, 187--196

Persistent URL: http://dml.cz/dmlcz/131805

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ON A SUM OF OBSERVABLES IN A LOGIC

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A sum of two observables of a logic defined in a way differing from that of the mean values is studied and some properties are proved.

Introduction

In the classical probability theory the sum of observables is, doubtless, of great importance. Therefore there are made different attempts to introduce the sum into the theory of logic [2—6], as well as into the quantum measuring theory of noncompatible observables. We shall study the properties of the sum defined by (2.1).

1. Logic and observables

Let L be a σ-lattice with the first and the last elements 0 and 1, respectively, and an orthocomplementation $\perp : a \mapsto a^{\perp}$ which satisfies (i) $(a^{\perp})^{\perp} = a$ for all $a \in L$; (ii) if $a < b$, then $b^{\perp} < a^{\perp}$ for $a, b \in L$; (iii) $a \vee a^{\perp} = 1$ for all $a \in L$. We further assume that if $a < b$, then $b = a \vee (b \wedge a^{\perp})$. A poset L satisfying the above axioms will be called a logic.

We say that a, b are (i) orthogonal and we write $a \perp b$ if $a < b^{\perp}$; (ii) compatible and we write $a \leftrightarrow b$ if there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$.

An observable is a map x from $B(R_1)$ into L such that (i) $x(R_1) = 1$, $x(\emptyset) \cdot = 0$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$, $E, F \in B(R_1)$; (iii) $x\left(\bigcup_i E_i\right) = \bigvee_i x(E_i)$ if $E_i \cap E_j = \emptyset$, $i \neq j$, $\{E_i\} \subset B(R_1)$. If f is a Borel function on R_1 and x an observable, then $f \circ x$: $E \mapsto x(f^{-1}(E))$, $E \in B(R_1)$, is an observable. For an observable x we denote $\sigma(x) = \cap \{C \in B(R_1): x(C) = 1\}$ and we define $\|x\| = \sup \{|t| : t \in \sigma(x)\}$. We say that x is (i) bounded if $\|x\| < \infty$; (ii) bounded above (below) if there is a number $c \in R_1$ such that $\sigma(x) \subset (-\infty, c\rangle$ ($\sigma(x) \subset \langle c, \infty)$). Two observables x and y are compatible and we write $x \leftrightarrow y$ if $x(E) \leftrightarrow y(F)$ for every $E, F \in B(R_1)$.

The conventional measurable space $(\Omega, \mathscr{S})$ is a logic of compatible observables if we identify $x(E)=f^{-1}(E)$, $E \in B(R_1)$, where f is a $\mathscr{S}$ — measurable function. The logic $L(H)$, that is, the complete lattice of all closed subspaces of a Hilbert space H, is a very important example of a logic which has noncompatible observables and which is a model for quantum mechanics. In this logic the selfadjoint operators correspond to the observables [8].

Since the notion of observable is an analogy of a measurable function we will now investigate some properties of observables.

Theorem 1.1. *Let x be an observable of a logic L and $B_x(t) = x((-\infty, t))$, $t \in R_1$, then the system $\{B_x(t): t \in R_1\}$ has the following properties:*

(i) $B_x(s) < B_x(t)$ if $s < t$;

(ii) $\bigvee_t B_x(t) = 1, \quad \bigwedge_t B_x(t) = 0;$ (1.1)

(iii) $\bigvee_{t<s} B_x(t) = B_x(s)$.

Conversely, if a system $\{B(t): t \in R_1\}$ of the elements of a logic L fulfils (1.1), then there is a unique observable x such that $B_x(t) = B(t)$ for every $t \in R_1$.

Proof. Let x be an observable; then (i) is trivial. (ii): let $B_x(t) < a$ for every $t \in R_1$; then for every integer n we have $B_x(n) < a$. Hence $a > \bigvee_n B_x(n) = \bigvee_n x((-\infty, n)) = 1$. Similarly, $\bigwedge_t B_x(t) = 0$. (iii): let $a > B_x(t)$, $t < s$. If we choose $t_n \uparrow s$, then $a > \bigvee_n B_x(t_n) = B_x(s)$.

Let now on the logic L a system $\{B(t): t \in R_1\}$ satisfying (i)—(iii) be given. In the first place we show that there is a Boolean sub-σ-algebra of L generating by $\{B(t): t \in R_1\}$.

Let $r_1, r_2, \ldots$ be any distinct enumeration of the rational numbers in R_1. For every n let $\mathscr{A}_n$ be a Boolean subalgebra of L generated by $\{B(r_1), \ldots, B(r_n)\}$. This subalgebra surely exists, because if $(i_1, \ldots, i_n)$ is such an enumeration of $(1, \ldots, n)$ that $r_{i_1} < \ldots < r_{i_n}$, then the set of all finite lattice sums of orthogonal elements $\{B(r_{i_1}), B(r_{i_2}) \wedge B(r_{i_1})^\perp, \ldots, B(r_{i_n}) \wedge B(r_{i_{n-1}})^\perp, B(r_{i_n})^\perp\}$ is a Boolean subalgebra containing all $B(r_1), \ldots, B(r_n)$ and therefore it is $\mathscr{A}_n$. Let us put $\mathscr{A}_0 = \bigcup_n \mathscr{A}_n$; then $\mathscr{A}_0$ is a Boolean subalgebra of L, too.

By the Zorn lemma it is easy to see that there is a maximal Boolean subalgebra $\mathscr{M}$ of L containing $\mathscr{A}_0$. The $\mathscr{M}$ must be a Boolean sub-σ-algebra.

Let now $B(t)$ be an arbitrary element of $\{B(t): t \in R_1\}$. Since there is $r_{n_j} \uparrow t$, we have $B(t) = \bigvee_j B(r_{n_j}) \in \mathscr{M}$. We have shown that there is a Boolean sub-σ-algebra of

L generated by $\{B(t)\colon t \in R_1\}$ and let it be denoted by $\mathcal{A}$.

By the Loomis theorem there is a measurable space $(\Omega, \mathcal{S})$ and a homomorphism h from $\mathcal{S}$ onto $\mathcal{A}$. We claim to construct, by induction, the set s $A_1, A_2, \dots$ from $\mathcal{A}$ such that

(a) $h(A_i) = B(r_i)$;

(b) $A_i \subset A_j$ if $r_i < r_j$;

(c) $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

We note that if $A \subset B$, $A, B \in \mathcal{S}$ and if there is $c \in \mathcal{A}$ such that $h(A) < c < h(B)$, then there is $C \in \mathcal{S}$ such that $A \subset C \subset B$, $h(C) = c$. Indeed, since h maps $\mathcal{S}$ onto $\mathcal{A}$, there is $C_1 \in \mathcal{S}$ such that $h(C_1) = c$. If we define $C = (C_1 \cap B) \cup A$, then C has a given property.

Let A_1 be any set in $\mathcal{S}$ such that $h(A_1) = B(r_1)$. Suppose $A_1, \dots, A_n \in \mathcal{S}$ have been constructed so that (a) and (b) hold. We shall construct A_{n+1} as follows. Let $(i_1, \dots, i_n)$ be the permutation of $(1, \dots, n)$ such that $r_{i_1} < \dots < r_{i_n}$. Then only one condition holds $(*)$: (i) $r_{n+1} < r_{i_1}$; (ii) $r_{n+1} > r_{i_n}$; (iii) there is a unique $k = 1, \dots, n$ such that $r_{i_k} < r_{n+1} < r_{i_{k+1}}$; and by the above observation we can select A_{n+1} such that $h(A_{n+1}) = B(r_{n+1})$ and (i) $A_{n+1} \subset A_{i_1}$; (ii) $A_{n+1} \supset A_{i_n}$; (iii) $A_{i_k} \subset A_{n+1} \subset A_{i_{k+1}}$; according to $(*)$. Then the system $\{A_1, \dots, A_{n+1}\}$ fulfils (a) and (b). Thus, by induction, there follows that there is a sequence $\{A_j\}$ of sets in $\mathcal{S}$ with the properties (a) and (b). As

$$h\left(\bigcap_{j=1}^{\infty} A_j\right) = \bigwedge_{j=1}^{\infty} h(A_j) = \bigwedge_{j=1}^{\infty} B(r_j) = 0,$$

we may, replacing A_j by $A_j - \bigcap_i A_i$ if necessary, assume that $\bigcap_j A_j = \emptyset$.

We define an $\mathcal{S}$-measurable function f as follows:

$$f(\omega) = \begin{cases} 0 \text{ if } \omega \notin \bigcup_{j=1}^{\infty} A_j \\ \inf\{r_j\colon \omega \in A_j\} \text{ if } \omega \in \bigcup_{j=1}^{\infty} A_j. \end{cases}$$

A function f is everywhere well defined and it is finite. Moreover

$$f^{-1}((-\infty, r_k)) = \begin{cases} \bigcup_{r_j < r_k} A_j \text{ if } r_k \leq 0 \\ \bigcup_{r_j < r_k} A_j \cup \left(\Omega - \bigcup_i A_i\right) \text{ if } r_k > 0, \end{cases}$$

hence f is $\mathcal{S}$-measurable and $h(f^{-1}((-\infty, r_k))) = B(r_k)$. If we define an observable x by $x(E) = h(f^{-1}(E))$, $E \in B(R_1)$, then $x((-\infty, t)) = B(t)$ for every $t \in R_1$. Since $x_1((-\infty, t)) = x_2((-\infty, t))$ for every $t \in R_1$ implies $x_1 = x_2$, the uniqueness of x is shown and the proof is finished. Q.E.D.

Remark 1.2. (i) Theorem 1.1 holds if we consider a system $\{B(t)\colon t \in S\}$ satisfying (1.1), where S is a countable dense set in R_1.

(ii) If L is a non-lattice logic [7], then the assertions of Theorem 1.1 and the first part of Remark 1.2 remain valid, too.

Theorem 1.3. *For two observables x and y the following conditions are equivalent:*

(i) $x \leftrightarrow y$;

(ii) $B_x(t) \leftrightarrow B_y(s)$ *for every* $s, t \in R_1$;

(iii) $B_x(t) \leftrightarrow B_y(s)$ *for every* $s, t \in S$, *S is a countable dense set in* R_1.

Proof. The implication (i)$\Rightarrow$(ii)$\Rightarrow$(iii) is trivial. Let now (iii) hold. Let us denote for any $t \in S$

$$\mathscr{C}_t = \{E \in B(R_1)\colon x(E) \leftrightarrow B_y(t)\}.$$

If we take into account the assertion of Lemma 6.10 [8]: if $b \leftrightarrow a_n$, $n = 1, 2, \ldots,$ then $b \leftrightarrow a_n^{\perp}$, $n = 1, 2, \ldots$, $b \leftrightarrow \bigvee_n a_n$, $b \leftrightarrow \bigwedge_n a_n$; then $\mathscr{C}_t = B(R_1)$. Indeed, $\mathscr{C}_t$ contains the intervals $(-\infty, s)$ for every $s \in S$. Let $s \in R_1$; then there is $s_n \uparrow s$, $s_n \in S$. Hence $(-\infty, s) \in \mathscr{C}_t$ for every $s \in R_1$ and, consequently, $\mathscr{C}_t = B(R_1)$, $t \in S$. Similarly, $\mathscr{C}_t = B(R_1)$ for any $t \in R_1$. Analogically, $\mathscr{C} = \{F \in B(R_1)\colon x(E) \leftrightarrow y(F)$ for every $E \in B(R_1)\} = B(R_1)$. Therefore $x \leftrightarrow y$. Q.E.D.

2. The sum of two observables

If x and y are compatible observables, then, by [8, Theorem 6.9], there are an observable u and two Borel functions f, g such that $x = f \circ u$, $y = g \circ u$. Due to Theorem 6.17 [8] we may define the sum of x and y by $x + y = (f + g) \circ u$ idependently of the used f, g, u. Theorem 1.1 enables us to define the sum for noncompatible observables without using the mean values.

For two observables x, y we define the following system of the elements of a logic L:

$$B_{x \oplus y}(t) = \bigvee_{r \in Q} (B_x(r) \wedge B_y(t - r)), \quad t \in R_1, \tag{2.1}$$

where Q is the set of the rational numbers in R_1.

Lemma 2.1. *If $x \leftrightarrow y$, then a system $\{b_{x \oplus y}(t)\colon t \in R_1\}$ fulfils (1.1) of Theorem 1.1, and then an observable $x \oplus y$ coincides with the sum of compatible observables.*

Proof. There holds

$$B_{x \oplus y}(t) = \bigvee_{r \in Q} (x((-\infty, r)) \wedge y((-\infty, t - r))) =$$

$$\bigvee_{r\in Q}[u(f^{-1}((-\infty,r)))\wedge u(g^{-1}((-\infty,t-r)))]=u((f+g)^{-1}((-\infty,t)))=B_{(f+g)\circ u}(t).$$

Hence $B_x \oplus_y(t)$ fulfils (1.1) and $x\oplus y=(f+g)\circ u=x+y$. Q.E.D.

A logic L is σ-continuous if for $a_1<a_2<\dots$ and any a

$$a\wedge\left(\bigvee_n a_n\right)=\bigvee_n (a\wedge a_n)$$

holds. A logic L is said to satisfy the finite chain condition (f.c.c.) if $\{a_n\}\subset L$ with $a_1<a_2<\dots$ implies that there is N such that $a_n=a_N$ for $n>N$. It is easy to see that if L satisfies f.c.c., then it is σ-continuous.

Lemma 2.2. *Let L be a σ-continuous logic and S a countable dense set in R_1. Let us denote for the observables x, y $B^S_{x\oplus y}(t)=\bigvee_{s\in S}(B_x(s)\wedge B_y(t-s))$; then $B^S_{x\oplus y}(t)=B_{x\oplus y}(t)$ for every $t\in R_1$.*

Proof. We may show that if $t_n\uparrow t$, then $B^S_{x\oplus y}(t)=\bigvee_n B^S_{x\oplus y}(t_n)$. Indeed,

$$\bigvee_n B^S_{x\oplus y}(t_n)=\bigvee_n\bigvee_{s\in S}(B_x(s)\wedge B_y(t_n-s))=$$

$$=\bigvee_{s\in S}(B_x(s)\wedge\bigvee_n B_y(t_n-s))=\bigvee_{s\in S}(B_x(s)\wedge B_y(t-s)).$$

Let now n be any integer; then for each s there is $r=r(s)\in Q$ such that we have $s<r<s+n^{-1}$. Therefore $B_x(s)\wedge B_y(t-n^{-1}-s)<B_x(r)\wedge B_y(t-r)$ and

$$B^S_{x\oplus y}(t-n^{-1})<B_{x\oplus y}(t)$$

$$B^S_{x\oplus y}(t)=\bigvee_n B^S_{x\oplus y}(t-n^{-1})<B_{x\oplus y}(t).$$

Similarly we show that $B_{x\oplus y}(t)<B^S_{x\oplus y}(t)$. Q.E.D.

Theorem 2.3. *Let L be a σ-continuous logic and x, y be observables. Then for $\{B_{x\oplus y}(t)\colon t\in R_1\}$ we have*

(i) $B_{x\oplus y}(s)<B_{x\oplus y}(t)$, $s<t$ *(on any logic, too);*

(ii) $\bigvee_t B_{x\oplus y}(t)=1$ *(if x, y are bounded above, then* (ii) *holds on any logic);*

(iii) $\bigwedge_t B_{x\oplus y}(t)=0$ *(on any logic) if x, y are bounded below;*

(iv) $\bigvee_{t<s} B_{x\oplus y}(t)=B_{x\oplus y}(s)$;

(v) $B_{x\oplus y}(t)=B_{y\oplus x}(t)$ *for every $t\in R_1$.*

Proof. Because of (i) of (1.1) the (i) is evident. (ii)

$$\bigvee_t B_{x\oplus y}(t)=\bigvee_t\bigvee_{r\in Q}(B_x(r)\wedge B_y(t-r))=$$

$$= \bigvee_{r\in Q} \bigvee_{t} (B_x(r) \wedge B_y(t-r)) > \bigvee_{r\in Q} \bigvee_{n=1}^{\infty} (B_x(r) \wedge B_y(n-r)) =$$

$$= \bigvee_{r\in Q} (B_x(r) \wedge \bigvee_{n=1}^{\infty} B_y(n-r)) = \bigvee_{r\in Q} (B_x(r) \wedge 1) = 1,$$

by the σ-continuity of L. Similarly for (iv).

If x, y are bounded above, then there is $c \in R_1$ such that $\sigma(x)$, $\sigma(y) \subset (-\infty, c\rangle$. Then for any $\varepsilon > 0$ $B_x(c+\varepsilon) = 1 = B_y(c+\varepsilon)$. Hence $B_{x\oplus y}(2c+2\varepsilon) = 1$.

(iii) There is $c \in R_1$ such that $\sigma(x)$, $\sigma(y) \subset \langle c, \infty)$. Then $B_{x\oplus y}(2c) = 0$.

(v) Let $t \in R_1$; then the set $S_t = \{s = t - r : r \in Q\}$ is countable dense in R_1 and, by Lemma 2.2, we have

$$B_{x\oplus y}(t) = \bigvee_{r\in Q} (B_x(r) \wedge B_y(t-r) = \bigvee_{s\in S_t} (B_y(s) \wedge B_x(t-s)) = B_{y\oplus x}(t).$$

Q.E.D.

Lemma 2.4. *Let x, y be two observables bounded below on a σ-logic L. Then*

$$\|x \oplus y\| \leq \|x\| + \|y\|. \tag{2.2}$$

Proof. If x or y is unbounded, then (2.2) holds. Therefore let x, y be bounded observables. Let us denote

$$a_1 = \inf \sigma(x), \quad b_1 = \sup \sigma(x)$$
$$a_2 = \inf \sigma(y), \quad b_2 = \sup \sigma(y).$$

Then $B_{x\oplus y}(a_1 + a_2) = 0$ and $B_{x\oplus y}(b_1 + b_2 + \varepsilon) = 1$ for every $\varepsilon > 0$. We prove only $B_{x\oplus y}(b_1 + b_2 + \varepsilon) = 1$. If we choose a rational number r such that $b_1 + \varepsilon/4 < r <$- $b_1 + \varepsilon/2$, then $-r > -b_1 - \varepsilon/2$ and

$$B_x(r) > B_x(b_1 + \varepsilon/4) = 1,$$
$$B_y(b_1 + b_2 + \varepsilon - r) > B_y(b_1 + b_2 + \varepsilon - b_1 - \varepsilon/2) = B_y(b_2 + \varepsilon/2) = 1.$$

We have proved that $\sigma(x \oplus y) \subset \langle a_1 + a_2, b_1 + b_2 \rangle$. If $a = \inf \sigma(x \oplus y)$, $b = \sup \sigma(x \oplus y)$, then $a_1 + a_2 \leq a \leq b \leq b_1 + b_2$. We calculate $\|x \oplus y\| = \max\{|a|, |b|\} \leq \max\{|a_1 + a_2|, |b_1 + b_2|\} \leq \max\{|a_1|, |b_1|\} + \max\{|a_2|, |b_2|\} = \|x\| + \|y\|$.

Q.E.D.

We denote by o such an observable that $o(\{0\}) = 1$. For $\alpha \in R_1$ and x we denote by αx such an observable that $(\alpha x)(E) = x(\{t: \alpha(t) \in E\})$, where $\alpha(t) \equiv \alpha t$, $t \in R_1$ and finally, for x, y we denote $x \ominus y = x \oplus (-y)$.

Theorem 2.5. *Let $O_B(L)$ be the set of all bounded observables on a σ-continuous logic L. Then $O_B(L)$ is a normed space with respect to the norm $\|x\| = \sup\{|t| : t \in \sigma(x)\}$ and the following properties hold*

(i) $\|x\| \geq 0$, $x \in O_B(L)$, $\|x\| = 0$ *iff* $x = o$;

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in R_1$, $x \in O_B(L)$;

(iii) $\|x \oplus y\| \leq \|x\| + \|y\|$, x, $y \in O_B(L)$;

(iv) $x \oplus y = y \oplus x$, $x, y \in O_B(L)$;
(v) $x \oplus o = x$; $x \in O_{B(L)}$;
(vi) $x \ominus x = o$, $x \in O_B(L)$;
(vii) $(\alpha+\beta)x = \alpha x \oplus \beta x$, $\alpha, \beta \in R_1$, $x \in O_B(L)$;
(viii) $\alpha(x \oplus y) = \alpha x \oplus \alpha y$, $\alpha \geqslant 0$, $x, y \in O_B(L)$.

Proof. The properties (i)—(ii) follow from [3, Theorem 4.2], (iii) follows from Lemma 2.4, (iv) from Lemma 2.3; (v)—(vii) are the corollaries of the calculus for compatible observables; (viii) follows from the definition of the sum and from Lemma 2.2. Q.E.D.

For a given element $a \in L$ we define a question observable q_a by $q_a(\{0\}) = a^\perp$, $q_a(\{1\}) = a$, and an observable x is a question observable iff $\sigma(x) \subset \{0, 1\}$ [4].

Remark 2.6. The sum defined by (2.1) is not associative in general.

Indeed, let $a, b, c \in L$; then

$$B_{(q_a \oplus q_b) \oplus q_c}(t) = \begin{cases} 0 & t \leqslant 0 \\ (a \vee b \vee c)^\perp & 0 < t \leqslant 1 \\ (a \vee b)^\perp \vee ((a \wedge b)^\perp \wedge c^\perp) & 1 < t \leqslant 2 \\ (a \wedge b \wedge c)^\perp & 2 < t \leqslant 3 \\ 1 & 3 < t \end{cases}$$

$$B_{q_a \oplus (q_b \oplus q_c)}(t) = \begin{cases} 0 & t \leqslant 0 \\ (a \vee b \vee c)^\perp & 0 < t \leqslant 1 \\ (b \vee c)^\perp \vee ((b \wedge c)^\perp \wedge a^\perp) & 1 < t \leqslant 2 \\ (a \wedge b \wedge c)^\perp & 2 < t \leqslant 3 \\ 1 & 3 < t. \end{cases}$$

If now $L = L(R_2)$ and a, b, c are three mutually distinct noncompatible subspaces, then

$$B_{(q_a \oplus q_b) \oplus q_c}(2) = c^\perp, \quad B_{q_a \oplus (q_b \oplus q_c)}(2) = a^\perp. \qquad \text{Q.E.D.}$$

Lemma 2.7. *If for $x_1, \ldots, x_n$ we define, by the recurrence formula, $x_1 \oplus \ldots \oplus x_n = (x_1 \oplus \ldots \oplus x_{n-1}) \oplus x_n$, $n = 1, 2, \ldots$, then*

(i) $q_a \oplus q_b(\{i\}) = \begin{cases} (a \vee b)^\perp & \text{if } i = 0 \\ (a \vee b) \wedge (a \wedge b)^\perp & \text{if } i = 1 \\ a \wedge b & \text{if } i = 2; \end{cases}$

(ii) $q_{a_1} \oplus \ldots \oplus q_{a_n}(\{0\}) = (a_1 \vee \ldots \vee a_n)^\perp$;

(iii) $q_{a_1} \oplus \ldots \oplus q_{a_n}(\{n\}) = a_1 \wedge \ldots \wedge a_n$;

(iv) $\sigma(q_{a_1} \oplus \ldots \oplus q_{a_n}) \subset \{0, 1, n\}$.

Proof. The property (i) follows from the definition of the sum, and (ii)—(iv) may be proved by induction. Q.E.D.

3. Comparison with the sum defined by mean values

Gudder in [4] studied the sum of bounded observables defined by mean values. Let m be a state, that is, a map from L into $\langle 0, 1\rangle$ such that (i) $m(1)=1$; (ii) $m\left(\bigvee_i a_i\right)=\sum_i m(a_i)$, if $a_i \perp a_j$, $i \neq j$, then the mean value of an observable in m is $m(x)=\int t \, dm_x(t)$ if the integral on the right-hand side exists and is finite, where m_x is a measure on $B(R_1)$: $m_x(E)=m(x(E))$, $E \in B(R_1)$. If there is a quite full system M of states [4] such that for any two bounded observables x, y there is a unique observable z such that

$$m(z)=m(x)+m(y), \quad \text{for every} \quad m \in M, \tag{3.1}$$

then z is called the sum of x, y and it is written $z=x+y$.

It is easy to see that this sum is associative and it coincides with the sum of compatible observables.

Example 3.1. Let $L=L(R_2)$ and let x, y, z correspond to

$$\mathbf{M}_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \mathbf{M}_2=\begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}, \quad \mathbf{M}=\mathbf{M}_1+\mathbf{M}_2=\begin{pmatrix}3/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}.$$

Then the sum of x, y defined (i) by (3.1) is z; (ii) by (2.1) is q_1. The logic $L(R_2)$ is isomorphic to a logic L of subsets of the set $\Omega=\langle 0, \pi/2\rangle$, that is, with the logic $L=\{\emptyset, \Omega, \{\pi/2, \varphi\}, \{\pi/2, \varphi\}^c, 0 \leq \varphi < \pi/2\}$. Let f, g, h correspond to x, y, z in this isomorphism, where

$$f(\omega)=\begin{cases}0 & \text{if } \omega \in \{\pi/2, 0\}\\ 0 & \text{if } \omega \notin \{\pi/2, 0\};\end{cases} \quad g(\omega)=\begin{cases}1 & \text{if } \omega \in \{\pi/2, \pi/4\}\\ 0 & \text{if } \omega \notin \{\pi/2, \pi/4\};\end{cases}$$

$$h(\omega)=\begin{cases}(2-\sqrt{2})/2 & \text{if } \omega \notin \{\pi/2, \operatorname{arctg}(1+\sqrt{2})\}\\ (2+\sqrt{2})/2 & \text{if } \omega \in \{\pi/2, \operatorname{arctg}(1+\sqrt{2})\}\end{cases}$$

Now, if we define the sum of measurable functions f, g:

(i) by points, that is, $(f+g)(\omega)=f(\omega)+g(\omega) \Rightarrow f+g$ is no observable;

(ii) by (3.1), then $f+g=h$;

(iii) by (2.1), then $f+g=1$.

This example refers to the splitting of the notion of the sum in a transition from a measurable space into logics. Moreover, in [1] it is shown that although $(f+g)(\omega)=f(\omega)+g(\omega)$ is a measurable function, the additivity of the mean value does not hold in general. (f, g in [1] are unbounded observables.)

Lemma 3.2. *The following propositions are equivalent*

(i) $q_a \oplus q_b$ *is a question observable;*

(ii) $q_a \oplus q_b = q_{a \vee b}$;

(iii) $a \wedge b = 0$.

S. P. Gudder in [4] showed that $a \perp b$ iff $q_a + q_b = q_{a \vee b}$. This property does not hold for the sum defined by (2.1).

Corollary 3.2.1. *If there holds* $a \perp b$ *iff* $q_a \oplus q_b = q_{a \vee b}$, *then* L *is a Boolean* σ*-algebra.*

Proof. If $q_a \oplus q_b = q_{a \vee b}$, then by (ii) of Lemma 3.2 there follows that $a \perp b$ iff $a \wedge b = 0$. By Zierler [9, Lemma 1.5] there implies that L is a Boolean σ-algebra.

Q.E.D.

Lemma 3.3. *There holds*

$$B_{q_a \ominus q_b}(t) = \begin{cases} 0 & t \leq -1 \\ a^\perp \wedge b & -1 < t \leq 0 \\ a^\perp \cup b & 0 < t \leq 1 \\ 1 & 1 < t \end{cases}$$

Moreover, the following propositions are equivalent

(i) $q_a \ominus q_b$ *is a question observable;*

(ii) $q_a \ominus q_b = q_{a \wedge b^\perp}$

(iii) $a^\perp \wedge b = 0$.

We see that the sum of two observables $x \oplus y$ has not the same properties as the sum defined by (3.1) and the investigation of the sum defined by (2.1) may be made mainly for compatible observables.

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Received July 24, 1978

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О СУММЕ НАБЛЮДАЕМЫХ В ЛОГИКЕ

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Резюме

Сумма двух наблюдаемых в логике определяется отличным способом от определения суммы посредством средниих значении. Некоторые свойства этой суммы доказаны