Mathematica Slovaca

G. D. Lugovaya; A. N. Sherstnev

On the extension problem for unbounded measures on projections

Mathematica Slovaca, Vol. 50 (2000), No. 4, 473--481

Persistent URL: http://dml.cz/dmlcz/131771

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 2000

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Math. Slovaca, **50** (2000), No. 4, 473–481

Mathematica Slovaca
©2000
Mathematical Institute
Slovak Academy of Sciences

ON THE EXTENSION PROBLEM FOR UNBOUNDED MEASURES ON PROJECTIONS

G. D. LUGOVAYA — A. N. SHERSTNEV

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We construct an unbounded semifinite finitely additive measure on projections in a von Neumann algebra with no direct summands of type I_2 not extending to a weight. Thus we give a negative solution to the problem set in [SHERSTNEV, A. N.: *On certain problems in the theory of unbounded measures on projections*, Atti Sem. Mat. Fis. Univ. Modena **42** (1994), 357–366]

1. Introduction

The famous problem of linearity in non-commutative measure theory deals with extending orthoadditive measures on projections in a von Neumann algebra to linear functionals. The problem has been set up and solved for factors and unitarily-invariant measures by F. Murray and J. von Neumann in their classical works on rings of operators. The full extension program for unitary-invariant measures to integrals has been carried out by I. Segal [1]. For general orthoadditive bounded measures the above problem has been solved exhaustively due to the efforts of several mathematicians during 1957–1983 ([2]–[5]). The problem was positively solved for all types of von Neumann algebras with no direct summand of type I_2. In 1984, the authors [6] obtained the positive solution to the problem for unbounded σ-orthoadditive measures on the algebra of all bounded linear operators on a Hilbert space, and then this result was extended to the case of semifinite von Neumann algebras by M. Matveichuk [7]. The aim of this paper is to present a negative solution to the problem for the case of unbounded finitely additive measures.

In the next section, the main result of the paper is stated. In Section 3, we introduce the notion of an orthoideal of orthoprojections in a von Neumann alge-

2000 Mathematics Subject Classification: Primary 46L10, 46L51.
Key words: von Neumann algebra, measure on projections, orthoideal.

This research was supported by the RFFI (Russia), Grant No. 98-01-00103 and URFI, Grant No. 1617.

bra. This notion plays a key role in constructing the measure. The construction is described in Section 4. In Section 5, all necessary proofs are adduced. The results of this paper were announced in [8], [9].

2. The main result

Let M be a von Neumann algebra. We shall denote by M^{pr} and M^+ the classes of all orthoprojections and positive operators (respectively) in M. We denote by $B(H)$ the algebra of all bounded linear operators on a Hilbert space H. For $x \in B(H)$ we denote by $\mathcal{R}(x)$ the range of x. Let I $(= I_H)$ be the identity operator on H.

1. DEFINITION. A mapping $m\colon M^{\mathrm{pr}} \to [0, +\infty]$ is said to be a *finitely additive measure* if

$$m(p+q) = m(p) + m(q) \qquad (p, q \in M^{\mathrm{pr}}\,, \quad pq = 0)\,.$$

A measure m is said to be *unbounded* if $m(I) = +\infty$.
We call m *semifinite* if there exists a net $(p_\alpha)_{\alpha \in A} \subset M^{\mathrm{pr}}$ with $p_\alpha \nearrow I$ and $m(p_\alpha) < +\infty$ for every $\alpha \in A$.

Nonsemifinite measures are rather difficult to view even if $\dim H < +\infty$, and we shall not examine them here. Recall that a *weight* on a von Neumann algebra M is a map $\varphi\colon M^+ \to [0, +\infty]$ such that

$$\varphi(x+y) = \varphi(x) + \varphi(y)\,, \quad \varphi(\lambda x) = \lambda \varphi(x) \qquad (x, y \in M^+\,, \ \ \lambda \geq 0\,. \ \ 0 \cdot \infty \equiv 0)\,.$$

When $\varphi(I) < +\infty$, a weight φ extends to a positive linear functional $\varphi \in M^*$ by linearity (we use the same letter φ for this extension). The fundamental result of Christensen-Yeadon [3], [4] lies in the fact that every bounded finitely additive measure on the orthoprojections in a von Neumann algebra with no direct summand of type I_2 extends to a positive linear functional. The main result of this paper is:

2. THEOREM. *There exists a von Neumann algebra M with no direct summands of type I_2 and an unbounded semifinite finitely additive measure m: $M^{\mathrm{pr}} \to [0, +\infty]$ not extending to a weight.*

Thus we give the negative answer to a question raised by one of the authors in 1992 ([10; Problem 2.7]).

The proof of the theorem consists in constructing an appropriate von Neumann algebra M and a two-valued measure $m\colon M^{\mathrm{pr}} \to \{0, +\infty\}$ with the required properties.

3. Orthoideals of projections

3. Definition. A set $J \subset M^{\mathrm{pr}}$ is said to be an *orthoideal of projections* in a von Neumann algebra M if the following properties are satisfied:

(i) $p, q \in J$, $pq = 0$ implies $p + q \in J$;

(ii) $p \leq q$ ($p \in M^{\mathrm{pr}}$, $q \in J$) implies $p \in J$.

4. If S is a subset of M^{pr}, then among all orthoideals containing S, there exists the smallest. This is the intersection of all orthoideals containing S. It is natural to call it the orthoideal generated by S. We shall denote the orthoideal by $J(S)$. This orthoideal can be described simply when $S = \{p, q\}$ is a pair of projections in M^{pr}. Consider the following increasing sequence of sets $(J_n) \subset M^{\mathrm{pr}}$, which is defined by induction:

$$J_0 = \{p, q, 0\},$$
$$J_n = \{r \in M^{\mathrm{pr}} : \ (\exists s, t \in J_{n-1})(st = 0 \ \& \ r \leq s + t)\} \qquad (n \in \mathbb{N}).$$

Then $J(p, q) = \bigcup_{n=0}^{\infty} J_n$.

Denote by M^{fpr} the set of all finite-dimensional orthoprojections in M.

5. Proposition. *Let $p, q \in M^{\mathrm{pr}}$ and $S = \{p, q\} \cup M^{\mathrm{fpr}}$. Then*

$$J(S) = \{r + k : \ r \in J(p, q), \ k \in M^{\mathrm{fpr}}, \ rk = 0\}.$$

Proof. It is obvious that projections of the form $r + k$, where $r \in J(p, q)$, $k \in M^{\mathrm{fpr}}$ and $rk = 0$ are contained in the orthoideal generated by S. Thus it is sufficient to verify that $\{r + k : \ r \in J(p, q), \ k \in M^{\mathrm{fpr}}, \ rk = 0\}$ is itself an orthoideal.

The condition (i) in Definition 3 is obviously satisfied. Now, let $r \in J(p, q)$, $k \in M^{\mathrm{fpr}}$, $rk = 0$, and $s \in M^{\mathrm{pr}}$ be such that $s \leq r + k$. We show that s has the form $s = r_1 + k_1$ ($r_1 \in J(p, q)$, $k_1 \in M^{\mathrm{fpr}}$). Put $r_1 = s \wedge r$. Then $r_1 \in J(p, q)$ because $r_1 \leq r$. It remains to be proved that $k_1 \equiv s - r_1 \in M^{\mathrm{fpr}}$. We may assume (and we do this) that $k_1 \wedge k = 0$. Suppose on the contrary $\dim k_1 = +\infty$ and let $\{e_1, e_2, \dots\}$ be an orthonormal sequence in $\mathcal{R}(k_1)$. Consider the decompositions

$$e_j = f_j + g_j \quad (j \in \mathbb{N}), \qquad \text{where} \quad f_j = re_j, \ \ g_j = ke_j.$$

Note that $f_j, g_j \neq \theta$ ($j \in \mathbb{N}$). (Indeed, if $f_j = \theta$, then $e_j \in \mathcal{R}(k_1 \wedge k) = \{\theta\}$; if $g_j = \theta$, then $e_j \in \mathcal{R}(k_1 \wedge r) = \{\theta\}$). Let $n = \dim k$. Then there are numbers $\lambda_1, \dots, \lambda_{n+1}$ not all equal to zero with $\sum_{j=1}^{n+1} \lambda_j g_j = \theta$. Hence

$$\sum_{j-1}^{n+1} \lambda_j e_j = \sum_{j=1}^{n+1} \lambda_j f_j + \sum_{j=1}^{n+1} \lambda_j g_j = \sum_{j=1}^{n+1} \lambda_j f_j \in \mathcal{R}(k_1 \wedge r) = \{\theta\}.$$

This contradicts the linear independence of the system $\{e_1, e_2, \dots\}$. □

6. Remark. If J ($\subset M^{\mathrm{pr}}$) is an orthoideal, then a two-valued finitely additive measure on M^{pr} is naturally associated with J by

$$m_J(r) \equiv \begin{cases} 0 & \text{if } r \in J, \\ +\infty & \text{if } r \in M^{\mathrm{pr}} \setminus J. \end{cases}$$

4. The construction

7. Let K be an infinite-dimensional separable Hilbert space and $T \geq 0$ a compact operator acting on K with $\operatorname{Ker} T = \{\theta\}$ (θ is the null vector in K). We will denote the projection-valued measure on the Borel algebra $\mathfrak{B}(\mathbb{R})$ constructed by the resolution of the identity for T by $e(\cdot) = e^T(\cdot)$. Let $\mathcal{N} = \{T\}''$ be the commutative von Neumann algebra in K generated by T. Put $H = K \oplus K \oplus K$ and introduce the von Neumann algebra $M \equiv \{[u_{ij}] : u_{ij} \in \mathcal{N}\}$ acting on H (here, $[u_{ij}]$ are 3×3-matrices). This algebra has no direct summands of type I_2. For the construction of the required orthoideal we fix two orthoprojections,

$$p = \begin{bmatrix} \Delta_T & \Delta_T T & 0 \\ T\Delta_T & T\Delta_T T & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad q = \begin{bmatrix} \Delta_{2T} & 2\Delta_{2T} T & 0 \\ 2T\Delta_{2T} & 4T\Delta_{2T} T & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(for $V \in B(K)$ we set $\Delta_V \equiv (I_K + V^*V)^{-1}$). Consider the orthoideal $J \equiv \{r + k : r \in J(p,q),\ k \in M^{\mathrm{fpr}},\ rk = 0\}$ introduced in Proposition 5. The measure m_J canonically associated with the orthoideal due to Remark 6 is semifinite. (In fact, let $\omega_n = (\frac{1}{n}, +\infty)$. By the spectral theorem for the compact operator T, $e(\omega_n) \in \mathcal{N}^{\mathrm{fpr}}$ and $e(\omega_n) \nearrow I_K$ (since $\operatorname{Ker} T = \{\theta\}$). Therefore, $[e(\omega_n)\delta_{ij}] \nearrow I_H$ and $m_J([e(\omega_n)\delta_{ij}]) = 0$ ($n \in \mathbb{N}$).)

8. The proof of Theorem 2. We will show that the measure m_J constructed in Paragraph 7 does not extend to a weight. We should prove two key lemmas (this will be done in Section 5):

9. Lemma. *Let E be a Hilbert space and let r, s, t be orthoprojections in E. Then the following conditions are equivalent.*

(i) $\mathfrak{R}(r) \subset \mathfrak{R}(s) + \mathfrak{R}(t)$;

(ii) *there exists $C > 0$ such that $r \leq C(s+t)$.*

10. Lemma. *The orthoprojection $r = \begin{bmatrix} I_K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ belongs to $M^{\mathrm{pr}} \setminus J$.*

Now, we complete the proof of Theorem 2. Suppose that $\varphi\colon M^+ \to [0,+\infty]$ is a weight such that $\varphi|_{M^{\mathrm{pr}}} = m_J$. Note that the orthoprojections r, p, q satisfy the condition (i) in Lemma 9:

$$\begin{aligned} r\{f,g,h\} &= \{f,\theta,\theta\} = \{2f,2Tf,\theta\} + \{-f,-2Tf,\theta\} \\ &= p\{2f,2Tf,\theta\} + q\{-f,-2Tf,\theta\} \qquad (f,g,h \in K). \end{aligned}$$

Hence there exists $C > 0$ such that $r \le C(p+q)$. Since $p, q \in J$ and $r \in M^{\mathrm{pr}} \setminus J$ (see Lemma 10), we obtain the contradiction

$$+\infty = m(r) = \varphi(r) \le C\varphi(p+q) = C\big(\varphi(p) + \varphi(q)\big) = C\big(m(p) + m(q)\big) = 0,$$

which proves the theorem.

5. Proofs

In this section, we give the proofs of Lemmas 9 and 10.

11. Lemma. *Let E be a Hilbert space and $x_j \in B(E)^+$ $(1 \le j \le n)$. Then*

$$\mathcal{R}\big((x_1 + \cdots + x_n)^{1/2}\big) = \mathcal{R}\big(x_1^{1/2}\big) + \cdots + \mathcal{R}\big(x_n^{1/2}\big).$$

(*Here, the right-hand side is regarded as the algebraic sum of linear subspaces.*)

Proof. Put $a = x_1 + \cdots + x_n$. Then $x_j \le a$ and [9; I, §1, Lemma 2] give

$$x_j^{1/2} = d_j a^{1/2} = a^{1/2} d_j^* \qquad (1 \le j \le n),$$

where d_j are operators in $B(E)$. Hence $\mathcal{R}\big(x_1^{1/2}\big) + \cdots + \mathcal{R}\big(x_n^{1/2}\big) \subset \mathcal{R}\big(a^{1/2}\big)$.

Conversely, observe that $b \equiv \sum d_j^* d_j$ is the range projection of a. In fact, we have $a = a^{1/2} b a^{1/2}$ and $d_j h = \theta$ $(1 \le j \le n)$, and for all $f, g \in E$, $h \in \operatorname{Ker} a$,

$$\begin{aligned} \big\langle \big(\textstyle\sum d_j^* d_j\big) a^{1/2} f, a^{1/2} g + h \big\rangle &= \big\langle \big(\textstyle\sum d_j^* d_j\big) a^{1/2} f, a^{1/2} g \big\rangle \\ &= \langle a^{1/2} b a^{1/2} f, g \rangle = \langle a f, g \rangle = \langle a^{1/2} f, a^{1/2} g \rangle \\ &= \langle a^{1/2} f, a^{1/2} g + h \rangle. \end{aligned}$$

For every vector of the form $a^{1/2} f$ we now have

$$\begin{aligned} a^{1/2} f &= a^{1/2} b f \\ &= \sum a^{1/2} d_j^* d_j f \\ &= \sum x_j^{1/2} (d_j f) \in \mathcal{R}\big(x_1^{1/2}\big) + \cdots + \mathcal{R}\big(x_n^{1/2}\big), \end{aligned}$$

and the lemma follows. □

12. The proof of Lemma 9. Let us first assume that the condition (ii) is satisfied. From the inequality $r \le C(s+t)$ it follows that there exists an operator $x \in B(E)$, $\|x\| \le \sqrt{C}$ such that $r = x(s+t)^{1/2} = (s+t)^{1/2}x^*$. Therefore

$$\mathcal{R}(r) = \mathcal{R}\big((s+t)^{1/2}x^*\big) \subset \mathcal{R}\big((s+t)^{1/2}\big) = \mathcal{R}(s) + \mathcal{R}(t)$$

by Lemma 11.

Conversely, let the condition (i) holds and $F = \overline{\mathcal{R}(s) + \mathcal{R}(t)}$. The operator $(s+t)^{1/2}|_F$ is bounded and injective. Hence there exists a closed inverse operator $(s+t)^{-1/2}\colon \mathcal{R}(s) + \mathcal{R}(t) \to F$. Since $\mathcal{R}(r) \subset \mathcal{R}(s) + \mathcal{R}(t)$, the closed operator $A \equiv (s+t)^{-1/2}r$ is defined everywhere on E. Therefore it is bounded. Putting $C = \|A\|^2$ we have

$$r = (s+t)^{1/2}A = (s+t)^{1/2}AA^*(s+t)^{1/2} \le C(s+t),$$

and the Lemma follows.

13. The proof of Lemma 10. We start with the observation that it is sufficient to prove the "reduced" version of this lemma. To be precise, let $\pi = \begin{bmatrix} I_K & 0 & 0 \\ 0 & I_K & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $(\in M^{\mathrm{pr}})$. Consider the reduced von Neumann algebra $M_\pi \equiv \left\{\pi x|_{\mathcal{R}(\pi)} : x \in M\right\}$ acting on the Hilbert space $H_\pi = K \oplus K$,

$$p_\pi = \begin{bmatrix} \Delta_T & \Delta_T T \\ T\Delta_T & T\Delta_T T \end{bmatrix} = \pi p|_{H_\pi}, \qquad q_\pi = \begin{bmatrix} \Delta_{2T} & 2\Delta_{2T} T \\ 2T\Delta_{2T} & 4T\Delta_{2T} T \end{bmatrix} = \pi q|_{H_\pi}.$$

It should be noted that $p_\pi = P_{\Gamma(T)}$ (respectively $q_\pi = P_{\Gamma(2T)}$) is the orthoprojection onto the graph of T (respectively $2T$). Let J_π be the orthoideal generated by $S = \{p_\pi, q_\pi\} \cup M_\pi^{\mathrm{fpr}}$. Our next goal is to prove the following lemma.

14. Lemma. *The orthoprojection* $r_\pi = \begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix}$ *belongs to* $M_\pi^{\mathrm{pr}} \setminus J_\pi$.

We will slightly simplify the problem by restricting ourselves to consideration of the ideal $J(p_\pi, q_\pi)$ generated by the two-element set $\{p_\pi, q_\pi\}$ (see Paragraph 4). Namely, we will prove:

15. Proposition. *If* $r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \in J(p_\pi, q_\pi)$, *then* $\operatorname{Ker}(I_K - r_{11}) = \{\theta\}$.

Now, since every orthoprojection $r' \in J_\pi$ is of the form $r' = r \dotplus k$, $r \in J(p_\pi, q_\pi)$, $\dim k < +\infty$, it follows from Proposition 15 that $\dim\operatorname{Ker}(I_K - r'_{11}) = \dim\operatorname{Ker}(I_K - r_{11} - k_{11}) < +\infty$, whereas

$$\dim\operatorname{Ker}\big(I_K - (r_\pi)_{11}\big) = \dim\operatorname{Ker} 0 = +\infty,$$

which proves Lemma 14.

We begin with some general facts concerning the matrix structure of orthoprojections onto the graphs of operators.

16. Proposition. *Let T be a closed densely defined operator acting on a Hilbert space K, $E = K \oplus K$. Then $\Gamma(T)^{\perp} = \{\{-T^*f, f\} : f \in D(T^*)\}$, and the orthoprojection in E onto the graph $\Gamma(T)$ has the form*

$$P_{\Gamma(T)} = \begin{bmatrix} \Delta & (T\Delta)^* \\ T\Delta & T(T\Delta)^* \end{bmatrix}, \qquad \textit{where} \quad \Delta = (I + T^*T)^{-1}.$$

Moreover, if r is a subprojection of $P_{\Gamma(T)}$, then r can be represented in the form $r = \begin{bmatrix} \alpha & (T\alpha)^* \\ T\alpha & T(T\alpha)^* \end{bmatrix}$, where the operator α satisfies $0 \le \alpha = \alpha^2 + (T\alpha)^* T\alpha$.

P r o o f. The operator matrix of an orthoprojection in $B(E)$ has the form $\begin{bmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{bmatrix}$. In addition

$$\alpha^2 + \gamma\gamma^* = \alpha, \qquad \beta^2 + \gamma^*\gamma = \beta, \qquad \alpha\gamma + \gamma\beta = \gamma.$$

If that is the matrix of $P_{\Gamma(T)}$, then

$$\begin{bmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{bmatrix} \begin{bmatrix} f \\ \theta \end{bmatrix} \in P_{\Gamma(T)} \ (f \in K) \implies \gamma^* = T\alpha, \ \gamma = (T\alpha)^* = \overline{\alpha T^*},$$
$$\begin{bmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{bmatrix} \begin{bmatrix} \theta \\ f \end{bmatrix} \in P_{\Gamma(T)} \ (f \in K) \implies \beta = T(T\alpha)^* = T \cdot \overline{\alpha T^*},$$

If $f \in D(T^*T)$, then $\begin{bmatrix} \alpha & (T\alpha)^* \\ T\alpha & T(T\alpha)^* \end{bmatrix} \begin{bmatrix} f \\ Tf \end{bmatrix} = \begin{bmatrix} \alpha(1+T^*T)f \\ T\alpha(1+T^*T)f \end{bmatrix} = \begin{bmatrix} f \\ Tf \end{bmatrix}$ implies $\alpha = (I + T^*T)^{-1}$. □

17. The proof of Proposition 15. In the notation of Paragraph 4, we have $J(p_\pi, q_\pi) = \bigcup_{n=0}^{\infty} J_n(p_\pi, q_\pi)$, where

$$J_0(p_\pi, q_\pi) = \{p_\pi, q_\pi, 0\},$$
$$J_n(p_\pi, q_\pi) = \{r \in M_\pi^{\mathrm{pr}} : (\exists s, t \in J_{n-1}(p_\pi, q_\pi))(st = 0 \ \& \ r \le s + t)\} \quad (n \in \mathbb{N}).$$

We will prove the following assertion: the projections in $J_n(p_\pi, q_\pi)$ are subprojections of projections onto graphs of compact operators which are functions of T. In addition, Proposition 15 holds for the corresponding null-space.

Let us make the matrix structure of such subprojections more precise. Given a nonnegative compact operator V which is a function of T, we have, according to Proposition 16, $r = \left[\begin{smallmatrix} r_{11} & r_{11}V \\ Vr_{11} & Vr_{11}V \end{smallmatrix}\right]$. We shall seek r_{11} in the form $r_{11} = x\Delta_V$, where $x \in \mathcal{N}$ is an unknown operator. As $\mathcal{N}$ is commutative,

$$x\Delta_V = \Delta_V x = x^*\Delta_V \implies x = x^*,$$

and

$$x\Delta_V = r_{11} = r_{11}(I_K + V^2)r_{11} = x\Delta_V\Delta_V^{-1}x\Delta_V = x^2\Delta_V \implies x = x^2.$$

Thus, there exists $\omega \in \mathfrak{B}(\mathbb{R})$ such that $r_{11} = e(\omega)\Delta_V$. Hence

$$r = \begin{bmatrix} \Delta_V e(\omega) & V\Delta_V e(\omega) \\ V\Delta_V e(\omega) & V^2\Delta_V e(\omega) \end{bmatrix}. \tag{$*$}$$

We will prove the above assertion by induction on n. It can easily be seen that it is true for every projection in $J_0(p_\pi, q_\pi)$. Suppose it holds for the projections in $J_k(p_\pi, q_\pi)$ $(0 \le k \le n)$. Consider $s, t \in J_n(p_\pi, q_\pi)$ such that $st = 0$. Write s and t in the matrix form:

$$s = \begin{bmatrix} \Delta_V e(\omega) & V\Delta_V e(\omega) \\ V\Delta_V e(\omega) & V^2\Delta_V e(\omega) \end{bmatrix}, \qquad t = \begin{bmatrix} \Delta_W e(\delta) & W\Delta_W e(\delta) \\ W\Delta_W e(\delta) & W^2\Delta_W e(\delta) \end{bmatrix}.$$

Since s and t are orthogonal, we conclude that $e(\omega)e(\delta) = e(\omega \cap \delta) = 0$ (as $V, W \ge 0$, by the assumption of induction). Without loss of generality, we can suppose that $\omega \cap \delta = \emptyset$. Then an easy computation shows that the projection $r = s + t \in J_{n+1}(p_\pi, q_\pi)$ is a subprojection of the projection onto the graph of the non-negative compact operator $X = Ve(\omega) + We(\delta)$. By the inductive assumption, it follows that

$$\begin{aligned} \operatorname{Ker}(I_K - r_{11}) &= \{f \in \mathcal{R}(e(\omega \cup \delta)) : \ \Delta_V e(\omega)f + \Delta_W e(\delta)f = f\} \\ &= \{f \in \mathcal{R}(e(\omega \cup \delta)) : \ \Delta_V e(\omega)f = e(\omega)f, \ \Delta_W e(\delta)f = e(\delta)f\} \\ &= \{f \in \mathcal{R}(e(\omega \cup \delta)) : \ e(\omega)f = e(\delta)f = \theta\} = \{\theta\}. \end{aligned}$$

The proof is complete.

REFERENCES

[1] SEGAL, I.: *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401–457.

[2] GLEASON, A. M.: *Measures on the closed subspaces of a Hilbert space*, J. Math. Mech. **6** (1957), 885–894.

[3] CHRISTENSEN, E.: *Measures on projections and physical states*, Comm. Math. Phys. **86** (1982), 113–115.

[4] YEADON, F.: *Finitely additive measures on projections in finite W^*-algebras.* Bull. London Math. Soc. **16** (1984), 145–150.

[5] MATVEJCHUK, M. S.: *Description of finite measures in semifinite algebras* (Russian), Funktsional. Anal. i Prilozhen. **15** (1981), 41–53. [English translation: Functional Anal. Appl. **15** (1982), 187–197].

[6] LUGOVAYA, G. D.—SHERSTNEV, A. N.: *On the linearity problem for unbounded measures on projections*, Funktsional. Anal. (Ul'yanovsk) no. 23 (1984), 76–81. (Russian)

[7] MATVEJCHUK, M. S.: *The extension of an unbounded measure to a weight*, Izv. Vyssh. Uchebn. Zaved. Mat. **4** (1987), 47–51. (Russian)

[8] LUGOVAYA, G. D.—SHERSTNEV, A. N.: *On a problem of the non-commutative measure theory.* In: Algebra and Analysis (Abstracts of the Conf. Dedicated to the 100-al B. M. Gagaeff), Kaz. Math. Soc., Kazan, 1997, p. 139. (Russian)

[9] LUGOVAYA, G. D.—SHERSTNEV, A. N.: *On the extension problem for unbounded measures on projections to weights*, Dokl. Akad. Nauk **365** (1999), 165–166. (Russian)

[10] SHERSTNEV, A. N.: *On certain problems in the theory of unbounded measures on projections*, Atti Sem. Mat. Fis. Univ. Modena **42** (1994), 357–366.

[11] DIXMIER, J.: *Les algèbres d'opérateurs dans l'espace Hilbertien (algèbres de von Neumann).* In: Cahiers scientifiques. Fasc. XXV, 2e ed. revue et augmentee (Gauthier-Villars, ed.), Paris, 1969. (French)

Received February 23, 1999

Department of Math. and Mechanics
Kazan State University
Ul. Kremlevskaya 18
Kazan, 420008
RUSSIA

E-mail: Anatolij.Sherstnev@ksu.ru