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CONVERGENCES AND COMPLETE DISTRIBUTIVITY OF LATTICE ORDERED GROUPS

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C. J. Everett and S. Ulam [2] investigated the order convergence of sequences in an abelian lattice ordered group G. Some other types of convergences in G were studied by F. Papangelou [10]. An axiomatic treatement of sequential convergences on G was performed by M. Harminc [3], [4], [5] (in some results of [5] the lattice ordered group G need not be abelian).

Higher degrees of distributivity in lattice ordered groups (including complete distributivity) were studied by several authors (cf. e.g., Weinberg [8] and the author [6]).

Let $\operatorname{Conv} G$ be the system of all sequential convergences on G (for the definition, cf. below). The system $\operatorname{Conv} G$ is partially ordered by inclusion. In [5] it was shown that $\operatorname{Conv} G$ need not be a lattice and it was proved (without assuming the commutativity of G) that the following conditions are equivalent:

(i) $\operatorname{Conv} G$ has a greatest element.

(ii) $\operatorname{Conv} G$ is a lattice.

(iii) $\operatorname{Conv} G$ is a complete lattice.

In the present paper it will be shown that each archimedean completely distributive lattice ordered group satisfies the condition (i).

1. Preliminaries

Throughout the paper, G denotes a lattice ordered group. For denotations, cf. the monographs of P. Conrad [1] and V. M. Kopytov [7]. The group operation will be denoted additively.

Let N be the set of all positive integers. The direct product $\Pi_{n \in N} G_n$, where $G_n = G$ for each $n \in N$, will be denoted by G^N. The elements of G^N will be denoted by $(g_n)_{n \in N}$, or simply (g_n). If there exists $g \in G$ such that $g_n = g$ for each $n \in N$, then we denote $(g_n) = \operatorname{const} g$.

(g_n) is said to be a sequence in G. The notion of a subsequence has the usual meaning.

Let α be a convex normal subsemigroup of $(G^N)^+$ such that the following conditions are satisfied:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II) Let $(g_n) \in (G^N)^+$. If each subsequence of (g_n) has a subsequence belonging to α, then (g_n) belongs to α.

(III) Let $g \in G$. Then const g belongs to α if and only if $g = 0$.

Under these assumptions α is said to be a convergence in G. The system of all convergences in G will be denoted by Conv G; this system is partially ordered by inclusion. (Cf. [5], Definition 1.4 and Lemma 1.9.)

For $(g_n) \in G^N$ and $g \in G$ we put $g_n \to_\alpha g$ if and only if $(|g_n - g|) \in \alpha$.

Let $A \subseteq (G^N)^+$. We denote by δA the system of all subsequences of sequences belonging to A. The convex closure (in G^N) of the set $A \cup \{\text{const}\, 0\}$ will be denoted by $[A]$. Next let $\langle A \rangle$ be the subsemigroup of G^N generated by the set A. The symbol A^* will denote the set of all sequences in G for which each subsequence has a subsequence belonging to A.

1.1. Proposition. (Cf. [5], Theorem 1.18.) *Let $\emptyset \neq A \subseteq (G^N)^+$. Assume that G is abelian. Then the following conditions are equivalent.*

(*a*) *There exist $\alpha \in \text{Conv}\, G$ such that $A \subseteq \alpha$.*

(*b*) *If $g \in G$, $\text{const}\, g \in [\langle \delta A \rangle]$, then $g = 0$.*

2. Complete distributivity

For the notion of complete distributivity of lattice ordered groups cf. [8] or [6].

2.1. Theorem. (Cf. [8].) *Let G be a completely distributive archimedean lattice ordered group. Then there exist linearly ordered groups G_i $(i \in I)$ and a complete isomorphism of G into $\Pi_{i \in I} G_i$.*

Throughout this section we assume that G is a completely distributive archimedean lattice ordered group. In view of 2.1, we can suppose without loss of generality that G is an l-subgroup of a lattice ordered group $\Pi_{i \in I} G_i$, where each G_i is linearly ordered and all joins and meets in G are performed component-wise. Moreover, we can assume that for each $i \in I$ and each $x^i \in G_i$ there exists $g \in G$ such that the i-th component of g is x^i.

2.2. Lemma. *Let $i \in I$. Let α_i be a non-discrete convergence on G_i. Let (x_n) be a sequence in G_i, $x_n \geqq 0$ for $n = 1, 2, \ldots$. Then the following conditions are equivalent:*

(i) $x_n \to_{\alpha_i} 0$.

(ii) *If $0 < a^i \in G_i$, then there exists a positive integer m such that $x_n < a^i$ for each $n \geqq m$.*

(iii) *The sequence (x_n) o-converges to 0 in G_i.*

Proof. According to [5], Theorem 2.10, (i)$\Leftrightarrow$(ii). The equivalence (ii)$\Leftrightarrow$ $\Leftrightarrow$(iii) is obvious.

2.3. Lemma. *Let* $\alpha \in \operatorname{Conv} G$, $0 < a \in G$, $i \in I$. *Assume that* $a(i) > 0$. *Let* (x_n) *be a sequence in* G *such that* $x_n \to_\alpha 0$. *Then there is a positive integer* m *such that* $x_n(i) < a(i)$ *for each* $n \geqq m$.

Proof. By way of contradiction, assume that the assertion to be proved fails to hold. Then there is a subsequence of (x_n) such that the i-th component of each member of this sequence is greater than or equal to $a(i)$. For simplifying the denotation, let us suppose that (x_n) coincides with the subsequence under consideration. Put $y_n = x_n \wedge a$. Hence $y_n \to_\alpha 0$ and $y_n(i) = a(i)$ for each positive integer n.

Denote $z_n = y_1 \wedge y_2 \wedge y_3 \wedge \dots \wedge y_n$ for each positive integer n. Then $0 \leqq \leqq z_n \leqq y_n$, hence $z_n \to_\alpha 0$. Moreover, $z_1 \geqq z_2 \geqq \dots \geqq z_n \geqq \dots$. Hence we must have $\wedge_{n=1}^{\infty} z_n = 0$. Since G is a closed sublattice of $\Pi_{j \in I} G_j$ we infer that $\wedge_{n=1}^{\infty} z_n(i) = 0$. But $z_n(i) = a(i) > 0$ for each positive integer n, which is a contradiction.

Since for each $i \in I$ there exists $0 < a \in G$ with $a(i) > 0$, from 2.2 and 2.3 we obtain:

2.4. Corollary. *Let* $\alpha \in \operatorname{Conv} G$, $i \in I$. *Let* (x_n) *be a sequence in* G *such that* $x_n \to_\alpha 0$. *Then* $(x_n(i))$ *o-converges to* 0 *in* G_i.

Let us denote by α_0 the system of all sequences (x_n) in G^+ such that for each $i \in I$, $(x_n(i))$ o-converges to 0 in G_i.

2.5. Lemma. $\alpha_0 \in \operatorname{Conv} G$.

Proof. From the definition of α_0 we obtain that for $\alpha_0 = A$ we have $[\langle \delta A \rangle]^* = A$ and that the condition (b) from 1.1 is satisfied. Hence according to [5], Thm. 1.18 we obtain $\alpha_0 \in \operatorname{Conv} G$.

Now, according to 2.4 we have $\alpha \leqq \alpha_0$ for each $\alpha \in \operatorname{Conv} G$. Thus we have arrived at the following result:

2.6. Theorem. *Let G be an archimedean completely distributive lattice ordered group. Then the partially ordered set* Conv*G possesses a greatest element.*

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СХОДИМОСТЬ И ПОЛНАЯ ДИСТРИБУТИВНОСТЬ РЕШЕТОЧНО УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

В статье доказано, что упорядоченное множество Conv G всех сходимостей на вполне дистрибутивной архимедовой решеточно упорядоченной группе G является полной решеткой.