Mathematica Slovaca

Milan Jasem

Weak isometries in directed groups

Mathematica Slovaca, Vol. 44 (1994), No. 1, 39--43

Persistent URL: http://dml.cz/dmlcz/131688

Math. Slovaca, **44** (1994), No. 1, 39–43

Mathematica Slovaca

WEAK ISOMETRIES IN DIRECTED GROUPS

MILAN JASEM [1]

(*Communicated by Tibor Katriňák*)

ABSTRACT. The main result of this paper is that every stable weak isometry in a directed group is an involutory group automorphism.

In [11], S w a m y introduced the concept of an isometry in an abelian lattice ordered group G as a surjection $f\colon G \to G$ such that

$$|x - y| = |f(x) - f(y)| \qquad \text{for each} \quad x, y \in G \tag{1}$$

and proved that every isometry g in an abelian lattice ordered group G can be written uniquely as $g(x) = T(x) + a$, where a is a fixed element of G and T is an involutory isometric group automorphism. J a k u b í k [2], [3] proved that for every isometry f in a lattice ordered group (l-group) G such that $f(0) = 0$ there exists a uniquely determined direct decomposition $G = A \times B$ of G such that $f(x) = x_A - x_B$ for each $x \in G$ (x_A and x_B are components of x in the direct factors A and B, respectively) and extended the above mentioned S w a m y 's result to non-abelian l-groups. Isometries in l-groups investigated also H o l l a n d [1]. R a c h ů n e k [10] generalized the notion of the isometry for any partially ordered group (po-group) and studied isometries in a certain class of Riesz groups. In [5], [6], [8], [9], was J a k u b í k 's result on the relation between isometries and direct decompositions of l-groups extended to some types of po-groups.

In [4], J a k u b í k defined a weak isometry in an l-group G as a mapping $f\colon G \to G$ satisfying the condition (1) and proved that each weak isometry in a representable l-group is a bijection. Analogous result concerning weak isometries in isolated Riesz groups and distributive multilattice groups (and hence also in l-groups) was obtained in [7], [9].

AMS Subject Classification (1991): Primary 06F15.
Key words: Weak isometry, Isometry, Directed group, Involutory group automorphism.
[1] Supported by SAV grant 362/91 .

In this paper, it is proved that every stable weak isometry in a directed group G is an involutory group automorphism, and that every weak isometry f in G can be written as $f(x) = T(x)+a$, where a is a fixed element of G, and T is an involutory isometric group automorphism. From this it follows that every weak isometry in a directed group is a bijection. This generalizes the above mentioned extension S w a m y 's result by J a k u b í k and some results of [7], [9].

First we recall some notions and notations used in the paper.

Let G be a po-group. The group operation will be written additively. We denote $G^+ = \{x \in G;\ x \geq 0\}$. If a, b are elements of G, then we denote by $U(a,b)$ and $L(a,b)$ the set of all upper bounds and the set of all lower bounds of the set $\{a,b\}$ in G, respectively. If for $a,b \in G$ there exists the least upper bound (greatest lower bound) of the set $\{a,b\}$ in G, then it will be denoted by $a \vee b$ ($a \wedge b$). For each $a \in G$, $|a| = U(a,-a)$.

If G is a po-group, then a mapping $f\colon G \to G$ is called a *weak isometry* if $|f(x) - f(y)| = |x - y|$ for each $x, y \in G$. A weak isometry f is called a *stable weak isometry* if $f(0) = 0$.

A po-group G is called *directed* if $U(x,y) \neq \emptyset$ and $L(x,y) \neq \emptyset$ for each $x, y \in G$.

1. THEOREM. *Let G be a po-group, and let f be a stable weak isometry in G. Let $x \in G^+$, $x_1 = x + f(x)$, $x_2 = -f(x) + x$. Then $x_1 = f(2x) \vee 0$, $x_2 = -f(2x) \vee 0$, $2x = x_1 + x_2 = x_2 + x_1 = x_1 \vee x_2$, $x_1 \wedge x_2 = 0$, $f(2x) = x_1 - x_2 = 2f(x)$, $f(x_1) = x_1$, $f(x_2) = -x_2$, $f^2(x) = x$, $f^2(-x) = -x$, $x + f(x) = f(x) + x$, $f(-x) = -f(x)$.*

P r o o f. Let $x \in G^+$. Then from $|x| = |f(x)|$ we get $x = -f(x) \vee f(x)$. Thus $x + f(x) \geq 0$, $-f(x) + x \geq 0$. From $|2x| = |f(2x)|$ we obtain $2x = -f(2x) \vee f(2x)$. Since $|x| = |2x - x| = |f(2x) - f(x)|$, we have $x \geq f(2x) - f(x)$, $x \geq f(x) - f(2x)$. This implies $x + f(x) \geq f(2x)$, $-f(x) + x \geq -f(2x)$, $x + f(2x) \geq f(x)$, $-f(2x) + x \geq -f(x)$. Hence $x_1 \in U\big(f(2x), 0\big)$, $x_2 \in U\big(-f(2x), 0\big)$, $2x + f(2x) \geq x + f(x)$, $-f(2x) + 2x \geq -f(x) + x$. Let $t \in U\big(-f(2x), 0\big)$. Then $x_1 + t \in U\big(f(2x), -f(2x)\big)$. Thus $x_1 + t \geq -f(2x) \vee f(2x) = 2x = x_1 + x_2$. This implies $t \geq x_2$. Therefore $x_2 = -f(2x) \vee 0$. Analogously, we can show that $x_1 = f(2x) \vee 0$. Clearly, $x_1 + x_2 \in U(x_1, x_2)$. Let $z \in U(x_1, x_2)$. Then $z \in U\big(-f(2x), f(2x)\big)$. This yields $z \geq 2x = x_1 + x_2$. Hence $x_1 \vee x_2 = x_1 + x_2$. Then we can easily get $x_1 \wedge x_2 = 0$, $x_1 + x_2 = x_2 + x_1$. Since $-x_2 = f(2x) \wedge 0$ and $f(2x) = f(2x) \vee 0 + f(2x) \wedge 0$, we have $f(2x) = x_1 - x_2 = x + f(x) - x + f(x)$.

The relation $|x_1| = |f(x_1)|$ yields $x_1 \geq f(x_1)$, $x_1 \geq -f(x_1)$. Then $f(x_1)+x_2 \geq -x_1+x_2 = x_2-x_1 = -f(2x)$. Further, from $|x_2| = |x_1+x_2-x_1| = |2x - x_1| = |f(2x) - f(x_1)| = |x_1 - x_2 - f(x_1)|$ we obtain $x_2 \geq x_1 - x_2 - f(x_1)$. Then $f(x_1) + x_2 \geq -x_2 + x_1 = x_1 - x_2 = f(2x)$. Hence $f(x_1) + x_2 \geq -f(2x) \vee f(2x) = x_1 + x_2$. This implies $f(x_1) \geq x_1$. Therefore $f(x_1) = x_1$.

From $|x_2| = |f(x_2)|$ we get $x_2 \geq f(x_2)$, $x_2 \geq -f(x_2)$. Hence $-f(x_2)+x_1 \geq -x_2 + x_1 = x_1 - x_2 = f(2x)$. From $|x_1| = |2x - x_2| = |f(2x) - f(x_2)|$ we get $x_1 \geq f(x_2) - f(2x)$. Then $-f(x_2) + x_1 \geq -f(2x)$. Thus $-f(x_2) + x_1 \geq -f(2x) \vee f(2x) = x_2 + x_1$. This yields $-f(x_2) \geq x_2$. Thus $-f(x_2) = x_2$, and hence $f(x_2) = -x_2$.

From $|x| = |x+f(x)-f(x)| = \big|f\big(x+f(x)\big)-f^2(x)\big| = |x+f(x)-f^2(x)|$ we get $x \geq x+f(x)-f^2(x)$. Thus $f^2(x) \geq f(x)$. Then from $|f(x)-x| = |f^2(x)-f(x)|$ we get $x - f(x) = f^2(x) - f(x)$. Therefore $f^2(x) = x$.

From $|x| = |-x| = |f^2(-x)|$ we obtain $x \geq f^2(-x)$. Since $|2x| = |x-(-x)| = |f^2(x) - f^2(-x)| = |x - f^2(-x)|$, we have $2x = x - f^2(-x)$. Therefore $f^2(-x) = -x$.

Since $|2x - f(x)| = |f(2x) - f^2(x)|$, we have $2x - f(x) \geq x - f(2x)$. Then $x - f(x) \geq -f(2x)$. Because of $x - f(x) \geq 0$, we obtain $x - f(x) \geq -f(2x) \vee 0 \geq -f(x)+x$. This implies $f(x)+x \geq x+f(x)$, $x-f(x)+x_1 \geq -f(2x) \vee f(2x) = 2x$. From the last relation we have $x+f(x) \geq f(x)+x$. Hence $x+f(x) = f(x)+x$. Then $f(2x) = 2f(x)$.

From $|x| = |-x| = |f(-x)|$ we get $x \geq f(-x)$. Since $|f(x) - (-x)| = |f^2(x) - f(-x)| = |x - f(-x)|$, we have $x + f(x) = x - f(-x)$. Thus $f(-x) = -f(x)$.

2. THEOREM. *Let G be a po-group, and let f be a stable weak isometry in G. Let $x_1, x_2 \in G^+$. Then*

$$f(x_1 + x_2) = f(x_1) + f(x_2), \qquad f(x_1 - x_2) = f(x_1) - f(x_2),$$
$$f(-x_1 + x_2) = -f(x_1) + f(x_2).$$

P r o o f. Let x_1, $x_2 \in G^+$. In view of 1, we have $x_1 + x_2 - f(x_2) \geq 0$, $|x_1 + x_2 - f(x_2)| = |f(x_1 + x_2) - f^2(x_2)| = |f(x_1 + x_2) - x_2|$. Hence $x_1 + x_2 - f(x_2) \geq x_2 - f(x_1 + x_2)$. This implies $f(x_1 + x_2) + x_1 + x_2 - f(x_2) \geq f(x_1+x_2)+x_2-f(x_1+x_2) \geq 0$. According to 1, $x_1+x_2+f(x_1) \geq x_1+f(x_1) \geq 0$, $|x_1+x_2+f(x_1)| = |x_1+x_2-f(-x_1)| = |f(x_1+x_2)-f^2(-x_1)| = |f(x_1+x_2)+x_1| = |f(x_1 + x_2) + x_1 + x_2 - x_2| = \big|f\big(f(x_1 + x_2) + x_1 + x_2\big) - f(x_2)\big| = |f(x_1 + x_2) + x_1 + x_2 - f(x_2)|$. This yields $x_1 + x_2 + f(x_1) = f(x_1 + x_2) + x_1 + x_2 - f(x_2)$. Then from 1 it follows that $f(x_1) + f(x_2) = f(x_1 + x_2)$.

From $|x_1 + x_2 - x_1| = |x_1 - (x_1 - x_2)| = |f(x_1) - f(x_1 - x_2)|$ we obtain $x_1 + x_2 - x_1 \geq f(x_1 - x_2) - f(x_1)$. According to 1, $-f(x_1) + x_1 \geq 0$. Then $x_1 + x_2 \geq f(x_1 - x_2) - f(x_1) + x_1 \geq f(x_1 - x_2)$, and hence $x_1 + x_2 - f(x_1 - x_2) \geq 0$. In view of 1, we also have $|x_1 + x_2 + f(x_2) - f(x_1)| = \big|f\big(x_1 + x_2 + f(x_2)\big) - f^2(x_1)\big| = \big|f(x_1) + f\big(x_2 + f(x_2)\big) - x_1\big| = |f(x_1) + x_2 + f(x_2) - x_1| = |f(x_1) + f(x_2) + x_2 - x_1| = |f(x_1 + x_2) - (x_1 - x_2)| = |f^2(x_1 + x_2) - f(x_1 - x_2)| = |x_1 + x_2 - f(x_1 - x_2)|$. Since 1 yields $x_1 + x_2 + f(x_2) - f(x_1) \geq 0$, we have $x_1 + x_2 + f(x_2) - f(x_1) = x_1 + x_2 - f(x_1 - x_2)$. Therefore $f(x_1 - x_2) = f(x_1) - f(x_2)$.

By 1, $x_1 + x_2 - f(x_2) + f(x_1) \geq 0$. From $|x_1| = |x_2 - (-x_1 + x_2)| = |f(x_2) - f(-x_1 + x_2)|$ we get $x_1 \geq f(-x_1 + x_2) - f(x_2)$. In view of 1, we have $x_1 + x_2 \geq f(-x_1 + x_2) - f(x_2) + x_2 \geq f(-x_1 + x_2)$. Then, according to 1, we obtain $|x_1 + x_2 - f(x_2) + f(x_1)| = |x_1 + x_2 - f(x_2) - f(-x_1)| = \big|f\big(x_1 + x_2 - f(x_2)\big) - f^2(-x_1)\big| = \big|f(x_1) + f\big(x_2 - f(x_2)\big) + x_1\big| = |f(x_1) + f(x_2) - x_2 + x_1| = |f(x_1 + x_2) - (-x_1 + x_2)| = |f^2(x_1 + x_2) - f(-x_1 + x_2)| = |x_1 + x_2 - f(-x_1 + x_2)|$. This implies $x_1 + x_2 - f(x_2) + f(x_1) = x_1 + x_2 - f(-x_1 + x_2)$. Therefore $f(-x_1 + x_2) = -f(x_1) + f(x_2)$.

3. Theorem. *Each stable weak isometry in a directed group is an involutory group automorphism.*

P r o o f . Let H be a directed group, and let f be a stable weak isometry in H. It is easy to see that f is an injection. Let $x, y \in H$. Then $x = x_1 - x_2$, $y = y_1 - y_2$, where x_1, x_2, y_1, $y_2 \in H^+$. In view of 1 and 2, we have $f(x+y) = f\big(x_1 + y_1 - (y_2 - y_1 + x_2 + y_1)\big) = f(x_1 + y_1) - f(y_2 - y_1 + x_2 + y_1) = f(x_1) + f(y_1) - f(-y_1 + x_2 + y_1) - f(y_2) = f(x_1) + f(y_1) - f(x_2 + y_1) + f(y_1) - f(y_2) = f(x_1) - f(x_2) + f(y_1) - f(y_2) = f(x) + f(y)$. From this and 1 it follows that $f^2(x) = f^2(x_1 - x_2) = f^2(x_1) - f^2(x_2) = x_1 - x_2 = x$. Therefore f is an involutory group automorphism.

If f is a weak isometry in a po-group G, then the mapping g defined by $g(x) = f(x) - f(0)$ for each $x \in G$ is a stable weak isometry in G. Hence we have the following two corollaries.

4. Corollary. *For each weak isometry f in a directed group H there exists just one involutory isometric group automorphism g such that*

$$f(x) = g(x) + f(0) \qquad \textit{for each} \quad x \in H .$$

5. Corollary. *Each weak isometry in a directed group is a bijection.*

6. THEOREM. *Let G be a directed group and let f be a stable weak isometry in G. Then*

$$x + f(x) = f(x) + x \qquad \text{for each} \quad x \in G\,.$$

Proof. Let $x \in G$. Then $x = x_1 - x_2$, where $x_1, x_2 \in G^+$. In view of 1 and 2. we have $x + f(x) = x_1 - x_2 + f(-x_2 + x_2 + x_1 - x_2) = x_1 - x_2 - f(x_2) + f(x_2 + x_1 - x_2) = x_1 - f(x_2) - x_2 + f(x_2 + x_1 - x_2) = x_1 - f(x_2) - x_1 - x_2 + x_2 + x_1 - x_2 + f(x_2 + x_1 - x_2) = x_1 - f(x_2) - x_1 - x_2 + f(x_2 + x_1 - x_2) + x_2 + x_1 - x_2 = x_1 - f(x_2) - (x_2 + x_1) + f(x_2 + x_1) - f(x_2) + x_2 + x_1 - x_2 = x_1 - f(x_2) + f(x_2) + f(x_1) - (x_2 + x_1) + x_2 - f(x_2) + x_1 - x_2 = x_1 + f(x_1) - x_1 - x_2 + x_2 - f(x_2) + x_1 - x_2 = f(x_1) + x_1 - x_1 - f(x_2) + x_1 - x_2 = f(x_1) - f(x_2) + x_1 - x_2 = f(x_1 - x_2) + x_1 - x_2 = f(x) + x$.

REFERENCES

[1] HOLLAND, CH.: *Intrinsic metrics for lattice ordered groups*, Algebra Universalis **19** (1984), 142–150.

[2] JAKUBÍK, J.: *Isometries of lattice ordered groups*, Czechoslovak Math. J. **30** (1980), 142–152.

[3] JAKUBÍK, J.: *On isometries of non-abelian lattice order ed groups*, Math. Slovaca **31** (1981), 171–175.

[4] JAKUBÍK, J.: *Weak isometries of lattice ordered groups*, Math. Slovaca **38** (1988), 133–138.

[5] JAKUBÍK, J.—KOLIBIAR, M.: *Isometries of multilattice groups*, Czechoslovak Math. J. **33** (1983), 602–612.

[6] JASEM, M.: *Isometries in Riesz groups*, Czechoslovak Math. J. **36** (1986), 35–43.

[7] JASEM, M.: *On weak isometries in multilattice groups*, Math. Slovaca **40** (1990), 337–340.

[8] JASEM, M.: *Isometries in non-abelian multilattice groups*, Czechoslovak Math. J. (To appear).

[9] JASEM, M.: *Weak isometries and isometries in partially ordered groups.* (Submitted).

[10] RACHŮNEK, J.: *Isometries in ordered groups*, Czechoslovak Math. J. **34** (1984), 334–341.

[11] SWAMY, K. L. N.: *Isometries in autometrized lattice order ed groups*, Algebra Universalis **8** (1978), 59–64.

Received August 25, 1992

Department of Mathematics
Faculty of Chemistry
Slovak Technical University
SK-812 37 Bratislava
Slovakia