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ON β-ALGEBRAS

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ABSTRACT. In this paper, we discuss a class of algebras related to both groups and sets in a rather natural way. This class contains a wide variety of other objects but seems analyzable in somewhat traditional ways nevertheless.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3], [4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1], [2], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. The present authors [7] introduced the notion of d-algebras, i.e.,

(i) $x * x = 0$,
(ii) $0 * x = 0$,
(iii) $x * y = 0$ and $y * x = 0$ imply $x = y$,

which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim [5] introduced a new notion, called an BH-algebra, i.e., (i), (iii) and

(iv) $x * 0 = x$,

which is a generalization of $BCH/BCI/BCK$-algebras. They defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. Recently J. Neggers and

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H. S. Kim [8] introduced another class related to some of the previous ones, viz., B-algebras and studied some of its properties. In this paper we introduce the notion of β-algebra where two operations are coupled in such a way as to reflect the natural coupling which exists between the usual group operation and its associated B-algebra which is naturally defined by it. This class turns out to be both much wider and possessing sufficient general structural properties so as to enable one to proceed with the details.

2. β-algebras

A *β-algebra* is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

(I) $x - 0 = x$,

(II) $(0 - x) + x = 0$,

(III) $(x - y) - z = x - (z + y)$

for all x, y, z in X.

EXAMPLE 2.1. Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$-$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X; +, -, 0)$ is a β-algebra.

EXAMPLE 2.2. Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$-$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X; +, -, 0)$ is a β-algebra.

EXAMPLE 2.3. Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

+	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

−	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Then $(X; +, -, 0)$ is a β-algebra.

We observe that the three axioms (I), (II) and (III) are independent. Let $X := \{0, 1, 2\}$ be a set with the following tables:

+	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

−	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

Then the axioms (II) and (III) hold, but not (I), since $2 - 0 = 0 \neq 2$. Similarly, let $X := \{0, 1, 2\}$ be a set with the following tables:

+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

−	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

Then the axioms (I) and (III) hold, but not (II), since $(0 - 2) + 2 = 2 \neq 0$. Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

+	0	1	2	3
0	0	1	0	3
1	1	1	0	2
2	2	0	1	0
3	3	2	0	2

−	0	1	2	3
0	0	2	1	2
1	1	0	0	0
2	2	0	0	1
3	3	0	0	0

Then the axioms (I) and (II) hold, but not (III), since $(2-3)-2=0\neq 2=2-(2+3)$.

PROPOSITION 2.4. *Let $(G;\cdot,e)$ be a group. If we define $x+y:=x\cdot y$, $x-y:=x\cdot y^{-1}$, $0:=e$ for any $x,y\in G$, then $(G;+,-,0)$ is a β-algebra, called a group-derived β-algebra and denoted by $A(G)$.*

Proof. Straightforward. □

Example 2.1 is a group-derived β-algebra from the Klein 4-group, and Example 2.2 is also a group-derived β-algebra from the group $\mathbb{Z}_4$.

PROPOSITION 2.5. *Let S be a set. If we define $x+y:=x$; $x-y:=x$ and $0\in S$, then $(S;+,-,0)$ is a β-algebra, called a left β-algebra and denoted by A_S.*

Proof. Straightforward. □

Example 2.3 is a left β-algebra. Given β-algebras $(X;+,-,0_X)$ and $(Y;+,-,0_Y)$, where X is a group-derived β-algebra and Y is a left β-algebra, let $(x_1,y_1)+(x_2,y_2):=(x_1+x_2,y_1+y_2)$, $(x_1,y_1)-(x_2,y_2):=(x_1-x_2,y_1-y_2)$ for any $(x_1,y_1),(x_2,y_2)\in X\times Y$. Then $\big(X\times Y;+,-,(0_X,0_Y)\big)$ is a β-algebra which is neither group-derived nor a left β-algebra, and denoted by $A(G)\times A_S$. There are other examples of β-algebras which are neither a group-derived β-algebra nor a left β-algebra.

The class of β-algebras appear to be an interesting class of algebras in that it contains both "groups" and "sets" under one heading, including other structures such as $A(G)\times A_S$ for example.

We note that if a β-algebra is either $A(G)$ or A_S, then it is also the case that

(IV) $x+y=x-(0-y)$.

Hence the condition (IV) holds for β-algebras of the type $A(G)\times A_S$ as well.

Group-derived and left β-algebras part ways via the following conditions:

($\mathrm{V_a}$) $x-x=0$ (group derived),
($\mathrm{V_b}$) $x-x=x$ (left).

We list two classes of β-algebras of special interest. A β-algebra X is said to be a *B^*-algebra* if (IV) and ($\mathrm{V_a}$) hold; and an *L^*-algebra* if (IV) and ($\mathrm{V_b}$) hold.

J. Neggers and H. S. Kim [8] introduced the notion of B-algebra, and obtained various properties, especially that a B-algebra can be derived from any group using the notion of zero adjoint mapping.

An algebra $(X; -, 0)$ is said to be a *B-algebra* ([8]) if it satisfies (I), ($\mathrm{V_a}$) and

(VI) $(x - y) - z = x - \big(z - (0 - y)\big)$

for any $x, y, z \in X$.

EXAMPLE 2.6. Let $X := \{0, 1\}$ be a set with the following tables:

+	0	1
0	0	1
1	0	1

−	0	1
0	0	0
1	1	1

Then $(X; +, -, 0)$ is an L^*-algebra which is neither group-derived nor a left β-algebra.

PROPOSITION 2.7. *Let $(X; -, 0)$ be a B-algebra. If we define $x + y := x - (0 - y)$, $x, y \in X$, then $(X; +, -, 0)$ is a B^*-algebra.*

Proof. For any $x \in X$, by (IV) and ($\mathrm{V_a}$), we obtain $(0 - x) + x = (0 - x) - (0 - x) = 0$. Using (IV) and (VI) it follows that

$$\begin{aligned} x - (z + y) &= x - \big(z - (0 - y)\big) \\ &= (x - y) - z \end{aligned}$$

for any $x, y, z \in X$, proving the proposition. □

PROPOSITION 2.8. *If $(X; +, -, 0)$ is a B^*-algebra, then $(X; -, 0)$ is a B-algebra.*

Proof. It follows that

$$\begin{aligned} (x - y) - z &= x - (z + y) && \text{(by (III))} \\ &= x - \big(z - (0 - y)\big) && \text{(by (IV))} \end{aligned}$$

for any $x, y, z \in X$. □

Hence we shall usually identify B^*-algebras with B-algebras as being "essentially the same".

Let $(X; +, -, 0)$ be a β-algebra and let $0 \in A \subseteq X$. A is said to be a *β-subalgebra* of X if $x+y, x-y \in A$ for any $x, y \in A$.

In Example 2.2, the set $I_1 := \{0, 2\}$ is a β-subalgebra of X, but $I_2 := \{0, 1\}$ is not a β-subalgebra of X, since $1 + 1 = 2 \notin I_2$.

PROPOSITION 2.9. *Let $(X;+,-,0)$ be an L^*-algebra. Then $x+x=x$ for any $x \in X$.*

Proof. For any $x \in X$ we have

$$\begin{aligned} x &= x-0 && \text{(by (I))} \\ &= x-\big((0-x)+x\big) && \text{(by (II))} \\ &= (x-x)-(0-x) && \text{(by (III))} \\ &= x-(0-x) && \text{(by (V}_{\mathrm{b}}\text{))} \\ &= x+x && \text{(by (IV))}, \end{aligned}$$

proving the proposition. □

A β-subalgebra A of a β-algebra X is said to be an *L^*-subalgebra* of X if it is an L^*-algebra.

Given a β-algebra X, let $I(X)$ be the collection of all *β-idempotents*, i.e., elements x of X such that $x-x=x+x=x$.

PROPOSITION 2.10. *Let $(X;+,-,0)$ be a β-algebra. If A is an L^*-subalgebra of X, then $A \subseteq I(X)$.*

Proof. If follows from Proposition 2.9 and $(\mathrm{V_b})$. □

3. Bi-abelian and β-algebras

A β-algebra X is said to be *negative abelian* if $x-y=y-x$ for any $x,y \in X$.

The β-algebra in Example 2.1 is negative abelian, while the β-algebra in Example 2.2 is not negative abelian.

PROPOSITION 3.1. *Let X be a negative abelian β-algebra. If A is an L^*-subalgebra of X, then A is a trivial β-subalgebra.*

Proof. Since X is negative abelian, $x=x-0=0-x$ for any $x \in X$. It follows from Proposition 2.11 that $x=x+x=(0-x)+x=0$, i.e., $A=\{0\}$. □

THEOREM 3.2. *Let $A(G)$ be a negative abelian β-algebra which is group-derived from a group $(G;\cdot,e)$. Then the group $(G;\cdot,e)$ is abelian and all of its elements are of order 2.*

Proof. Since $A(G)$ is group-derived, $x-y=x\cdot y^{-1}$ and $y-x=y\cdot x^{-1}$ for any $x,y \in X$, and since $A(G)$ is negative abelian, we see that $x\cdot y^{-1}=y\cdot x^{-1}$. If we let $y:=e$, then $x=x^{-1}$ for any $x \in A(G)$. This means that $x^2=e$ and $x\cdot y=x\cdot y^{-1}=y\cdot x^{-1}=y\cdot x$ for any $x \in A(G)$, proving the theorem. □

A β-algebra X is said to be *positive abelian* if $x+y=y+x$ for any $x,y \in X$.

PROPOSITION 3.3. *Any β-algebra $A(G)$ which is derived from an abelian group G is positive abelian.*

Proof. Straightforward. □

PROPOSITION 3.4. *If a group-derived β-algebra $A(G)$ from a group G is negative abelian, then it is also positive abelian, i.e., bi-abelian.*

Proof. By applying Theorem 3.2 the underlying group G is abelian. Hence $x+y = x\cdot y = y\cdot x = y+x$ for any $x,y \in G$. □

4. Dissimilarity algebras

Let $(X;+,-,0)$ be a β-algebra. An element θ of X is said to be a *null element* of X if $x-\theta = x$, $(\theta - x)+x = \theta$ for any $x \in X$.

Obviously the constant 0 is a null element of X. We let $N(X)$ be the collection of all null elements of X.

EXAMPLE 4.1. Let $X := \{0,1,2,3\}$ be a set with the following tables:

+	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	1	1	0	0
3	0	0	1	1

−	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Then $(X;+,-,0)$ is a β-algebra and $N(X) = \{0,1\}$.

THEOREM 4.2. *If X is a β-algebra, then $N(X)$ is a left β-algebra as a β-subalgebra.*

Proof. If $\theta_1, \theta_2 \in N(X)$, then $\theta_1 - \theta_2 = \theta_1$ and hence $\theta_1 = (\theta_1 - \theta_2) + \theta_2 = \theta_1 + \theta_2$. It is easy to show that $N(X)$ is a β-subalgebra. □

Notice that $A(\theta) := (X;+,-,\theta)$ is a β-algebra for any $\theta \in N(X)$.

Given elements θ_1 and θ_2 of $N(X)$, we shall consider them *similar* if $A(\theta_1) \cong A(\theta_2)$, i.e., if there is a bijection $\varphi\colon A(\theta_1) \to A(\theta_2)$ such that $\varphi(x+y) = \varphi(x)+\varphi(y)$, $\varphi(x-y) = \varphi(x)-\varphi(y)$ and $\varphi(\theta_1) = \theta_2$, where $x,y \in X$, called a *similarity mapping*.

A mapping $\varphi\colon X \to X$ defined by $\varphi(0) = 1$, $\varphi(1) = 0$, $\varphi(2) = 3$, $\varphi(3) = 2$ in Example 4.1 is a similarity mapping.

PROPOSITION 4.3. *If $\varphi\colon A(\theta_1) \to A(\theta_2)$ is a similarity mapping, then $\varphi^{-1}\colon A(\theta_2) \to A(\theta_1)$ is also a similarity mapping.*

Proof. For any $x, y \in X$, let $\varphi(x) := a$ and $\varphi(y) := b$. Then $\varphi^{-1}(a \pm b) = x \pm y = \varphi^{-1}(a) \pm \varphi^{-1}(b)$. Clearly $\varphi^{-1}(\theta_2) = \theta_1$, proving the proposition. □

Let X be a β-algebra and let $\theta_1, \theta_2 \in N(X)$. Define $\theta_1 \sim \theta_2$ by $A(\theta_1) \cong A(\theta_2)$. Then $\sim$ is an equivalence relation on $N(X)$. Let $[\theta_1] := \{\theta \in N(X) : \theta \sim \theta_1\}$ be an equivalence class of θ_1. Since $\theta_1 \pm \theta_2 = \theta_1$, the relation $\sim$ is a congruence relation on $N(X)$. If $N(X)/\sim$ denotes the collection of all equivalence classes in $N(X)$, then it follows that $[\theta_1] + [\theta_2] = [\theta_1] - [\theta_2] = [\theta_1]$ defines a left β-algebra in a natural way. This algebra is a "measure" of the "dissimilarity" of the null elements of X. Thus we may consider $N(X)/\sim$ to be the *dissimilarity algebra* of X.

PROPOSITION 4.4. *Let A_S be a β-algebra derived from a set S. Then the dissimilarity algebra $N(A_S)/\sim$ of A_S is the trivial β-algebra.*

Proof. In A_S any bijection $\varphi\colon S \to S$ has the property that $\varphi(x + y) = \varphi(x - y) = \varphi(x) = \varphi(x) + \varphi(y) = \varphi(x) - \varphi(y)$. Since $N(A_S) = A_S$, it follows that $A(\theta_1) \cong A(\theta_2)$ for any $\theta_1, \theta_2 \in N(A_S)$, whence $N(A_S)/\sim = \{[\theta]\}$ is the trivial β-algebra. □

THEOREM 4.5. *Let X be a negative abelian β-algebra. Then the dissimilarity algebra $N(X)/\sim$ is the trivial β-algebra.*

Proof. It is easy to see that $N(X) \subseteq I(X)$. Let θ be a null element of X. For any $x \in I(X)$, $x = x - \theta = \theta - x$, since X is negative abelian. It follows that $x = x + x = (\theta - x) + x = \theta$, proving that $I(X) = \{\theta\}$. This means that $N(X)/\sim = \{[\theta]\}$, the trivial β-algebra. □

Let $(X; +, -, \theta)$ and $(Y; +, -, \theta')$ be a β-algebras. A mapping $\varphi\colon X \to Y$ is said to be a *β-homomorphism* if $\varphi(x+y) = \varphi(x) + \varphi(y)$, $\varphi(x-y) = \varphi(x) - \varphi(y)$ for any $x, y \in X$.

EXAMPLE 4.6. Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

$+$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	0	2	3
3	3	3	3	3

$-$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Then $(X;+,-,0)$ is a β-algebra and $N(X) = \{0,1\}$. If we define a map $\varphi\colon X \to X$ by $\varphi(0) = 1$, $\varphi(1) = 0$, $\varphi(2) = \varphi(3) = 1$, then φ is a β-homomorphism of X.

THEOREM 4.7. *If $\varphi\colon X \to Y$ is a β-homomorphism of β-algebras, then $\varphi(N(X)) \subseteq N(\operatorname{Im}\varphi)$ and $(\operatorname{Im}\varphi;+,-,\varphi(\theta))$, $\theta \in N(X)$, is also a β-algebra.*

Proof. It is enough to show that $\varphi(\theta)$ is a null element in $\operatorname{Im}\varphi$ for any $\theta \in N(X)$. In fact, $\varphi(\theta)-\varphi(x) = \varphi(\theta - x) = \varphi(\theta)$ and $\varphi(\theta) = \varphi((\theta - x)+x) = (\varphi(\theta) - \varphi(x)) + \varphi(x)$ for any $x \in X$, whence $\varphi(\theta) \in N(\operatorname{Im}\varphi)$, and hence $(\operatorname{Im}\varphi;+,-,\varphi(\theta))$ is a β-algebra. □

Let $\varphi\colon X \to Y$ be a β-homomorphism of β-algebras. We define a set by $(\operatorname{Ker}\varphi)_\theta := \varphi^{-1}(\{\varphi(\theta)\})$, where $\theta \in N(X)$, and call it the *kernel of φ at θ*.

For example, $(\operatorname{Ker}\varphi)_0 = \{1\}$ and $(\operatorname{Ker}\varphi)_1 = \{0\}$ in Example 4.6. Observe that if $x_1, x_2 \in (\operatorname{Ker}\varphi)_\theta$, $\theta \in N(X)$, then $\varphi(x_1 - x_2) = \varphi(x_1)-\varphi(x_2) = \theta-\theta = \theta$ and $\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2) = (\theta - \theta) + \theta = \theta$. Hence $x_1 \pm x_2 \in (\operatorname{Ker}\varphi)_\theta$. This means that $(\operatorname{Ker}\varphi)_\theta$ is a β-subalgebra of X, where $\theta \in N(X)$.

5. Quotient β-algebras

Let $\varphi\colon X \to Y$ be a β-homomorphism of β-algebras and $x, y \in X$. Define an equivalence relation $\equiv_\varphi$ on X by $x \equiv_\varphi y \iff \varphi(x) = \varphi(y)$. Let $[x]_\varphi := \{y \in X : x \equiv_\varphi y\}$ be an equivalence class of X and let $X/\varphi := \{[x]_\varphi : x \in X\}$. Define $[x]_\varphi + [y]_\varphi := [x+y]_\varphi$ and $[x]_\varphi - [y]_\varphi := [x-y]_\varphi$. Then $(X/\varphi;+,-,[\theta]_\varphi)$, $\theta \in N(X)$, is a β-algebra, called the *quotient β-algebra determined by φ*.

In fact, if $[x]_\varphi = [x']_\varphi$ and $[y]_\varphi = [y']_\varphi$, then $\varphi(x) = \varphi(x')$ and $\varphi(y) = \varphi(y')$, and hence $\varphi(x \pm y) = \varphi(x) \pm \varphi(y) = \varphi(x') \pm \varphi(y') = \varphi(x' \pm y')$, i.e., $[x + y]_\varphi = [x' + y']_\varphi$ and $[x - y]_\varphi = [x' - y']_\varphi$. Hence the operations are well-defined. Checking three axioms for β-algebra is elementary and we omit the proof. We summarize:

THEOREM 5.1. *Let $\varphi\colon X \to Y$ be a β-homomorphism of β-algebras. If we define $x \equiv_\varphi y \iff \varphi(x) = \varphi(y)$, $x, y \in X$, then $X/\varphi := \{[x]_\varphi : x \in X\}$ is a β-algebra, where $[x]_\varphi$ is the equivalence class of x.*

Obviously the map $\pi\colon X \to X/\varphi$ defined by $\pi(x) := [x]_\varphi$, $x \in X$, is a β-homomorphism, called the *canonical β-epimorphism*.

THEOREM 5.2. *Let $\varphi\colon X \to Y$ be a β-epimorphism of β-algebras. Then there exists a β-isomorphism $\nu\colon X/\varphi \to Y$ such that $\varphi = \nu \circ \pi$.*

Proof. Define a map $\nu\colon X/\varphi \to Y$ by $\nu([x]_\varphi) := \varphi(x)$. Then ν is a β-homomorphism, since $\nu([x]_\varphi \pm [y]_\varphi) = \nu([x \pm y]_\varphi) = \varphi(x \pm y) = \varphi(x) \pm \varphi(y) = \nu([x]_\varphi) \pm \nu([y]_\varphi)$. Clearly, ν is onto. Let $[x]_\varphi, [y]_\varphi \in X/\varphi$ with $\nu([x]_\varphi) = \nu([y]_\varphi)$. Then $\varphi(x) = \varphi(y)$ and hence $x \equiv_\varphi y$, i.e., $[x]_\varphi = [y]_\varphi$. This means that $X/\varphi \cong Y$. For any $x \in X$, $(\nu \circ \pi)(x) = \nu(\pi(x)) = \nu([x]_\varphi) = \varphi(x)$, i.e., $\nu \circ \pi = \varphi$. □

Let $(X; +, -, \theta)$ and $(Y; +, -, \theta')$ be β-algebras and $x \in X$. If a mapping $\varphi\colon X \to Y$ is a β-homomorphism, then we denote by $x - (\operatorname{Ker}\varphi)_\theta := \{x - n : n \in (\operatorname{Ker}\varphi)_\theta\}$.

PROPOSITION 5.3. *If $\varphi\colon X \to Y$ is a β-homomorphism of β-algebras and $x \in X$, then*

(i) $x - (\operatorname{Ker}\varphi)_\theta \subseteq [x]_\varphi$;

(ii) $[x]_\varphi = \bigcup_{x_1 \in [x]_\varphi} (x_1 - (\operatorname{Ker}\varphi)_\theta)$;

(iii) *if* $(x_1 - (\operatorname{Ker}\varphi)_\theta) \cap (x_2 - (\operatorname{Ker}\varphi)_\theta) \neq \emptyset$, $x_1, x_2 \in X$, *then* $[x_1]_\varphi = [x_2]_\varphi$.

Proof.

(i) Since $\varphi(\theta)$ is a null element in $\operatorname{Im}\varphi$ by Theorem 4.7, for any $x - n \in x - (\operatorname{Ker}\varphi)_\theta$, $\varphi(x - n) = \varphi(x) - \varphi(n) = \varphi(x) - \varphi(\theta) = \varphi(x)$. This means that $x - n \equiv_\varphi x$, i.e., $[x - n]_\varphi = [x]_\varphi$. Hence $x - n \in [x]_\varphi$.

(ii) If follows immediately from (i).

(iii) If $(x_1 - (\operatorname{Ker}\varphi)_\theta) \cap (x_2 - (\operatorname{Ker}\varphi)_\theta) \neq \emptyset$, $x_1, x_2 \in X$, then $x_1 - n_1 = x_2 - n_2$ for some $n_1, n_2 \in (\operatorname{Ker}\varphi)_\theta$. It follows that $\varphi(x_1) = \varphi(x_1) - \varphi(\theta) = \varphi(x_1) - \varphi(n_1) = \varphi(x_1 - n_1) = \varphi(x_2 - n_2) = \varphi(x_2) - \varphi(n_2) = \varphi(x_2) - \varphi(\theta) = \varphi(x_2)$ and hence $[x_1]_\varphi = [x_2]_\varphi$. □

Let φ be a β-homomorphism on X and $\theta \in N(X)$. We construct a graph $\Gamma(\varphi)$ as follows: $V = X$ is the set of vertices, and there is an edge between vertices x and y, denoted by $x \longleftrightarrow y$, provided $(x - (\operatorname{Ker}\varphi)_\theta) \cap (y - (\operatorname{Ker}\varphi)_\theta) \neq \emptyset$, whence also $[x]_\varphi = [y]_\varphi$. If there is a path $x \longleftrightarrow u_1 \longleftrightarrow u_2 \longleftrightarrow \cdots \longleftrightarrow u_n \longleftrightarrow y$ $(n \geq 0)$ in $\Gamma(\varphi)$, it follows that $[x]_\varphi = [y]_\varphi$. Hence, if $\langle x \rangle_\varphi$ denotes the component of x in $\Gamma(\varphi)$, we obtain $[x]_\varphi = \bigcup_{y \in [x]_\varphi} \langle y \rangle_\varphi$. The chain of structures associated with φ is:

$$\{x\} \subseteq x - (\operatorname{Ker}\varphi)_\theta \subseteq \langle x \rangle_\varphi \subseteq [x]_\varphi .$$

Indeed, if $y \in x - (\operatorname{Ker}\varphi)_\theta$, then $y - \theta \in y - (\operatorname{Ker}\varphi)_\theta$ so that $y \in (x - (\operatorname{Ker}\varphi)_\theta) \cap (y - (\operatorname{Ker}\varphi)_\theta)$, and $y \longleftrightarrow x$, i.e., $y \in \langle x \rangle_\varphi$.

PROPOSITION 5.4. *Let $\varphi\colon X \to Y$ be a β-homomorphism of β-algebras and let $\theta \in N(X)$, $\theta' \in N(Y)$. If X is a group-derived β-algebra and $x \in X$, then $x - (\operatorname{Ker}\varphi)_\theta = [x]_\varphi$.*

Proof. If $y \in [x]_\varphi$, then $\varphi(x) = \varphi(y)$. Since X is group-derived, $\varphi(xy^{-1}) = \varphi(y^{-1}x) = \theta'$ and hence $y = (xx^{-1})y = x(y^{-1}x)^{-1} = x - (y^{-1}x) \in x - (\operatorname{Ker}\varphi)_\theta$. Thus $[x]_\varphi \subseteq x - (\operatorname{Ker}\varphi)_\theta$. By Proposition 5.3 we obtain $[x]_\varphi = x - (\operatorname{Ker}\varphi)_\theta$. □

PROPOSITION 5.5. *Let $\varphi\colon (X;+,-,\theta) \to (Y;+,-,\theta')$ be a β-homomorphism of β-algebras. If X is a left β-algebra, then:*

(i) $\operatorname{Im}\varphi \subseteq I(Y)$;
(ii) $(\operatorname{Im}\varphi;+,-,\varphi(\theta))$ *is a left β-algebra;*
(iii) *if $I(Y) = \{\theta'\}$, then φ is the zero mapping.*

Proof. Straightforward. □

6. $N(X)$ and X^*

The set $N(X)$ appears to be an important substructure of a β-algebra X. We explore this idea a little further still.

THEOREM 6.1. *Let X be a β-algebra and $\theta_1, \theta_2 \in N(X)$. If $\theta_2 \in (\theta_1 - X) \cap N(X)$, then $\theta_1 = \theta_2$.*

Proof. Since $\theta_2 \in \theta_1 - X$, there exists $x \in X$ such that $\theta_2 = \theta_1 - x$ and $\theta_2 + x = (\theta_1 - x) + x = \theta_1$. It follows from (III) that $\theta_2 = \theta_2 - \theta_2 = (\theta_1 - x) - \theta_2 = \theta_1 - (\theta_2 + x) = \theta_1 - \theta_1 = \theta_1$, completing the proof. □

Of course $\theta_1 = \theta_1 - \theta_1 \in (\theta_1 - X) \cap N(X)$, we obtain the corollaries:

COROLLARY 6.2. *Let X be a β-algebra and $\theta \in N(X)$. Then $|(\theta - X) \cap N(X)| = 1$.*

COROLLARY 6.3. *Let X be a β-algebra. If $\theta - X = X$ for any $\theta \in N(X)$, then $|N(X)| = 1$ and $N(X) = \{\theta\}$.*

Let X be a β-algebra. We define a set $X^* := \bigcup_{\theta \in N(X)} (\theta - X)$, and investigate some relations with $N(X)$. Note that $N(X) \subseteq X^*$. If $u = \theta_1 - x, v = \theta_2 - y \in X^*$, then $u - v = (\theta_1 - x) - (\theta_2 - y) = \theta_1 - \big((\theta_2 - y) + x\big) \in \theta_1 - X \subseteq X^*$, i.e., X^* is a *closed system* with respect to $-$. Suppose now that we look for X^* to be a closed system with respect to $+$ as well. Thus, we need conditions:

(VII) For any $\theta_1 - x, \theta_2 - y \in X$, there exists $\theta_3 - z \in X$ such that $(\theta_1 - x) + (\theta_2 - y) = \theta_3 - z$, where $\theta_i \in N(X)$, $i = 1,2,3$, and $x, y, z \in X$.

(VIII) For any $x, y \in X$, there exists $z \in X$ such that $x + y = x - z$.

Note that, since $X = A(G)$ or $X = A_S$ or $X = A(G) \times A_S$ satisfying (IV), they satisfy (VIII) also. If a β-algebra X satisfies (VIII), then it satisfies (VII). Indeed, for any $\theta_1 - x, \theta_2 - y \in X$, where $\theta_1, \theta_2 \in N(X)$ and $x, y \in X$, there exists $\alpha \in X$ such that $(\theta_1 - x) + (\theta_2 - y) = (\theta_1 - x) - \alpha = \theta_1 - (\alpha + x)$.

PROBLEMS.

1. Under which conditions on the β-algebra X are rules (VII) and (VIII) equivalent?
2. If rule (VII) holds, then $(X^*; +, -, \theta)$ is a β-algebra.

Let X be a β-algebra. If $X = X^*$, then $x \in X$ implies $x = \theta - y$ for some $\theta \in N(X)$ and $y \in X$. This means that $x + y = (\theta - y) + y = 0$, i.e., *every element of X has an additive right inverse* for certain null elements of $N(X)$. Also, if $X = X^*$, then $(\theta_1 - x) + (\theta_2 - y) = w \in X$, where $w = \theta_3 - z \in X^*$, so that the condition (VII) holds.

Let X be a β-algebra. Note that $N(X) \subseteq N(X^*)$. We denote X^{**} by $\bigcup_{\theta^* \in N(X^*)} (\theta^* - X^*)$. Then we have the following theorem:

THEOREM 6.4. *If X is a β-algebra, then $X^{**} \subseteq X^*$ and $N(X^*) \subseteq N(X^{**})$.*

Proof. If $z \in X^{**}$, then $z = \theta^* - x$ for some $\theta^* \in N(X^*)$ and some $x \in X^*$. Since $x \in X^*$, there exists $\theta_1 \in N(X)$ and $y_1 \in X$ such that $x = \theta_1 - y_1$. Moreover, $\theta^* \in N(X^*)$ implies $\theta^* = \theta_2 - y_2$ for some $\theta_2 \in N(X)$ and $y_2 \in X$. Hence $z = \theta^* - x = (\theta_2 - y_2) - (\theta_1 - y_1) = \theta_2 - \big((\theta_1 - y_1) - y_2\big) \in X^*$, i.e., $X^{**} \subseteq X^*$. It follows immediately that $N(X^*) \subseteq N(X^{**})$. □

From Theorem 6.4 we construct a tower:

$$N(X) \subseteq N(X^*) \subseteq N(X^{**}) \subseteq \cdots \subseteq N\big(X^{*(i)}\big) \subseteq N\big(X^{*(i+1)}\big) \subseteq \cdots$$

with $\sqrt{N(X)} = \bigcup_{i=0}^{\infty} N(X^{*(i)})$. Similarly we have a descending chain:

$$X \supseteq X^* \supseteq X^{**} \supseteq \cdots \supseteq X^{*(i)} \supseteq X^{*(i+1)} \supseteq \cdots$$

with $X^{*(w)} = \bigcap_{i=0}^{w} X^{*(i)}$. Notice that if X is a finite β-algebra, then we expect that $\sqrt{N(X)} = N\big(X^{*(i)}\big)$ for some $i \in \mathbb{Z}$ and thus also that $X^{*(w)} = X^{*(i+1)}$ according to the definition above.

THEOREM 6.5. *Let X be a β-algebra and let $\theta_1, \theta_2 \in N(X)$. If $u \in (\theta_1 - X) \cap (\theta_2 - X)$, then $\theta_1 = \theta_2$.*

Proof. Since $u \in (\theta_1 - X) \cap (\theta_2 - X)$, there exist $x, y \in X$ such that $u = \theta_1 - x = \theta_2 - y$. Hence $u - u = (\theta_1 - x) - (\theta_1 - x) = \theta_1 - \big((\theta_1 - x) + x\big) = \theta_1 - \theta_1 = \theta_1$. Similarly $u - u = \theta_2$, whence $\theta_1 = \theta_2$. □

COROLLARY 6.6. *If X is a β-algebra, then $\{\theta - X\}_{\theta\in N(X)}$ is a partition of X^*.*

P r o o f. It follows from Theorem 6.5. □

PROPOSITION 6.7. *Let X be a β-algebra. If $X = X^*$, then $x-x = \theta \in N(X)$ for any $x \in X$.*

P r o o f. If $X = X^*$, then we know that every element x of X has an additive right inverse for certain null elements of $N(X)$, i.e., there exists $\theta \in N(X)$ and $y \in X$ such that $x = \theta - y$ and $x + y = \theta$. Hence $x - x = (\theta - y) - x = \theta - (x + y) = \theta - \theta = \theta \in N(X)$. □

THEOREM 6.8. *Let X be a β-algebra and let $\theta_1 \in N(X)$. If $\theta_2 \in (\theta_1 + X) \cap N(X)$, then $\theta_1 = \theta_2$.*

P r o o f. Since $\theta_2 \in \theta_1 + X$, $\theta_2 = \theta_1 + x$ for some $x \in X$. Hence $\theta_1 = \theta_1 - \theta_2 = \theta_1 - (\theta_1 + x) = (\theta_1 - x) - \theta_1 = \theta_1 - x$. It follows that $\theta_2 = \theta_1 + x = (\theta_1 - x) + x = \theta_1$, proving the theorem. □

We obtain the following corollaries from Theorem 6.8 and omit the proof.

COROLLARY 6.9. *Let X be a β-algebra and let $\theta \in N(X)$. Then $|(\theta + X) \cap N(X)| = 1$.*

COROLLARY 6.10. *Let X be a β-algebra and let $\theta \in N(X)$. If $\theta + X = X$, then $|N(X)| = 1$.*

PROPOSITION 6.11. *Let X be a β-algebra satisfying* (VIII) *and let $\theta_1, \theta_2 \in N(X)$. If $u \in (\theta_1 + X) \cap (\theta_2 + X)$, then $\theta_1 = \theta_2$.*

P r o o f. Since $u \in (\theta_1 + X) \cap (\theta_2 + X)$, $u = \theta_1 + x = \theta_2 + y$ for some $x, y \in X$. Since X satisfies (VIII), $\theta_1 + x = \theta_1 - p$, $\theta_2 + y = \theta_2 - q$ for some $p, q \in X$. Hence $u = \theta_1 - p = \theta_2 - q \in (\theta_1 - X) \cap (\theta_2 - X)$. By Theorem 6.5, $\theta_1 = \theta_2$. □

PROPOSITION 6.12. *Let X be a β-algebra satisfying* (VIII) *and let $\theta, \theta' \in N(X)$. Then*

(i) *$\theta + X = \theta - X$;*
(ii) *if $\theta \neq \theta'$, then $(\theta + X) \cap (\theta' + X) = \emptyset$.*

P r o o f. It follows immediately from (VIII) and Theorem 6.5. □

PROPOSITION 6.13. *Let X be a β-algebra satisfying* (VIII) *and let $X^\circ := \bigcup_{\theta\in N(X)} (\theta + X)$. Then $(\theta - X) \cap X^\circ = \theta + X$ for any $\theta \in N(X)$.*

Proof. By (VIII) it is clear that $\theta + X \subseteq (\theta - X) \cap X^\circ$. If $a \in (\theta - X) \cap X^\circ$, then $a \in \theta - X$ and $a \in \theta' + X$ for some $\theta' \in N(X)$. By Proposition 6.12(i), $a \in (\theta - X) \cap (\theta' - X)$. It follows immediately from Theorem 6.5 that $\theta = \theta'$ and hence $a \in \theta + X$. □

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