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SEVERAL REMARKS TO THE RIESZ REPRESENTATION THEOREM

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ABSTRACT. Given a linear functional T defined on a set $\mathcal{H}$ of bounded real functions may ask under what conditions if is possible to determine a measure m such that $Th = m(h)$ for $h \in \mathcal{H}$. Necessary and sufficient conditions for uniqueness of m are established.

1. The general representation theorem

The general problem can be formulated in the following way: Let $\mathcal{H}$ be a non-empty class of real-valued functions with the range of definition on an abstract non-empty set X. Let T be a functional on $\mathcal{H}$ with the property

$$T(\alpha h + \beta h') = \alpha Th + \beta Th' \qquad \text{for all} \quad \alpha, \beta \in]-\infty, \infty[\,, \ h, h' \in \mathcal{H}$$

(briefly a linear functional) and $\mathcal{E}$ be the smallest ring with respect to which all $h \in \mathcal{H}$ are measurable.

Under what additional conditions can we find a measure $(X, \mathcal{E}, m)$ with the property $Th = m(h)$ for all $h \in \mathcal{H}$? If the measure $(X, \mathcal{E}, m)$ representing T exists, is it determined by T and $\mathcal{H}$ uniquely?

Note that the linearity property assumes that the values of T are known on all the functions of the form $\alpha h + \beta h'$, $\alpha, \beta \in]-\infty, \infty[$, $h, h' \in \mathcal{H}$, although $\alpha h + \beta h'$ itself ought not to be in $\mathcal{H}$.

LEMMA 1.1. *Let T be a bounded linear functional on $\mathcal{H}$ and let $\mathcal{H}$ consist of bounded functions. If $\mathcal{S}(\mathcal{H})$ is the linear span of $\mathcal{H}$, then T has a unique extension to a linear functional with the range of definition on $\mathcal{S}(\mathcal{H})$.*

Proof. Let $\mathcal{H}_0 \subset \mathcal{H}$ be the base for $\mathcal{S}(\mathcal{H})$. Using the standard arguments of the Hahn-Banach theorem [12; sec. IV, §2] one can show that T has an extension to $\mathcal{S}(\mathcal{H})$. Since T is uniquely defined on $\mathcal{H}_0$ its extension from $\mathcal{H}$ to $\mathcal{S}(\mathcal{H})$ is also unique.

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LEMMA 1.2. *Under the assumptions of* 1.1. T *can be extended to a linear functional on the vector lattice containing* $\mathcal{H}$ *and the class* $\{\chi_E \colon E \in \mathcal{E}\}$, *where* $\mathcal{E}$ *is the smallest ring with respect to which all* $h \in \mathcal{H}$ *are measurable.*

The proof of 1.2. follows from 1.1. and from the Hahn-Banach extension theorem. Now we can say that our first problem has an affirmative solution:

THEOREM 1.3. *If* T *is a bounded linear functional on a non-empty class* $\mathcal{H}$ *of bounded real valued functions then there is a measure* $(X, \mathcal{E}, m)$ *representing* T *on* $\mathcal{H}$.

To avoid a misunderstanding we note that the function f on X is bounded if the supremum norm $\|\cdot\|$ of f is finite. T is bounded if there is a real constant $\gamma \in]0, \infty[$ with the property $|Tf| \le \gamma \|f\|$ for each function f on X.

By a measure we understand a finite, real-valued, finitely additive set function m which is defined on a ring $\mathcal{E}$ of subsets of X. If we want to be exact, we speak about the measure $(X, \mathcal{E}, m)$. Each measure can be written in the form $m = m^+ - m^-$, where m^+ and m^- are the smallest among all the nonnegative measures on $\mathcal{E}$ for which the decomposition of m holds.

For arbitrary $E \subset X$ we denote by χ_E the characteristic function of E. The system of all simple $\mathcal{E}$-measurable functions is denoted by $s(\mathcal{E})$, i.e. $s(\mathcal{E}) = \{s \colon s = \sum \alpha_i \chi_{E_i}$, where $\alpha_i \in]-\infty, \infty[$, $E_i \in \mathcal{E}$ are pairwise disjoint and $i = 1, 2, \dots, n\}$.

The integral $m(s)$ of any simple $\mathcal{E}$ measurable function s is defined by $m(s) = \sum \alpha_i m E_i$, where $s = \sum \alpha_i \chi_{E_i} \in s(\mathcal{E})$ and we summarize over all $i = 1, 2, \dots, n$. If m is a nonnegative measure and f a bounded nonnegative function, then f is said to be m-integrable if $\sup\{m(s) \colon 0 \le s \le f, s \in s(\mathcal{E})\} = \inf\{m(s) \colon f \le s, s \in s(\mathcal{E})\}$. The number in the last equation is said to be the integral of f and it is denoted by $m(f)$. The function f on X is said to be m-integrable if the positive and negative parts of f are m integrable. The integral $m(f)$ is then defined by $m(f) = m(f^+) - m(f^-)$. Generally, if m is a signed measure and f a bounded function on X, then f is m-integrable if it is m^+- and m^--integrable. The integral $m(f)$ of f is now defined by $m(f) = m^+(f) - m^-(f)$.

The definition of the integral given above can be extended (using the m zero sets) to functions with infinite values, however, for our purposes it is sufficient. The details related to the integration theory are in [2; 3; 6- 9, 11; 13]

P r o o f o f t h e T h e o r e m 1 3. In virtue of 1.2. T can be extended to a linear functional on the vector lattice $\mathcal{L}(\mathcal{H})$ which is generated by H and by the class $\{\chi_E \colon E \in \mathcal{E}\}$, where $\mathcal{E}$ is the smallest ring with respect to which all $h \in \mathcal{H}$ are measurable. Since T is assumed to be bounded, the functional L

extending T from $\mathcal{H}$ to $\mathcal{L}(\mathcal{H})$ can be assumed also bounded. The set function m defined on $\mathcal{E}$ by $mE = L\chi_E$ for all $E \in \mathcal{E}$ is a measure on $\mathcal{E}$. Each $h \in \mathcal{H}$ can be written in the form $h = \alpha_1 h_1 + \alpha_2 h_2$, where $\alpha_i \in]-\infty, \infty[$ and $h_i \in \mathcal{H}$, $i = 1,2$. $L = L_1 - L_2$, where L_1 and L_2 are nonnegative bounded linear functionals on $\mathcal{L}(\mathcal{H})$. Of course, $m = m_1 + m_2$, where $m_i E = L_i \chi_E$ for all $E \in \mathcal{E}$. Whence, for the verification of our theorem it is sufficient to show that $Th = m(h)$ in the case when T and L are nonnegative and $0 \le h < 1$, $h \in \mathcal{H}$.

Let L be nonnegative and $h \in \mathcal{H}$ be chosen so that $0 \le h < 1$. For each natural k the sets $G_i = \left\{x\colon h(x) > \frac{i}{k}\right\}$, $i = 1,2,\dots,k$ are in $\mathcal{E}$. Put $s = \frac{1}{k}\sum_{i=1}^{k}(i-1)\chi_{G_i - G_{i-1}}$. Then $s \in s(\mathcal{E})$, $s = \frac{1}{k}\sum_{i=1}^{k}\chi_{G_i}$ and consequently, $s \le h \le \frac{1}{k}\chi_{G_0} + s$. Since

$$-\frac{1}{k}L\chi_{G_0} + Lh \le m(s) \le m(h) \le \frac{1}{k}mG_0 + m(s) \le \frac{1}{k}L\chi_{G_0} + Lh$$

and k can tend to infinity, the relation $Th = Lh = m(h)$ holds.

It is impossible to obtain some results about the uniqueness of the extension in 1.3. without additional assumptions. The Theorem 1.4. leads to the following partial problem which was studied by J. P. R. Christiensen [4] and other authors.

Let X, ϱ be a separable metric space and Ba be the system of all balls $B(x,r) = \{y\colon \varrho(x,y) < r\}$, where $x \in X$, r is a rational number. If $\mathcal{H}$ is the class $\{\chi_{B(x,r)}\colon B(x,r) \in Ba\}$ and if T is a bounded monotone σ-additive linear functional on $\mathcal{H}$, then we can consider the values $mB(x,r) = T\chi_{B(x,r)}$ as values of a nonnegative σ-additive set function m. Now there is a question what properties must X, ϱ have in order that m has a unique extension to a Borel measure. More about this problem can be found in [4].

2. The uniqueness of the integral representation

If T is a bounded monotone linear functional on a class $\mathcal{H}$ of nonnegative bounded functions on X and if $0 \in \mathcal{H}$, then the relation $T_* f = \sup\{Th\colon h \le f,\ h \in \mathcal{H}\}$ defines a bounded monotone functional T_* on $[0,\infty]^X$. Moreover

Lemma 2.1. *The subclass $\mathcal{C}_*$ of $[0,\infty[^X$ defined by*

$$\mathcal{C}_* = \{f\colon T_* g = T_* g \wedge f + T_*(g-f)^+ \quad \textit{for all} \quad g \in [0,\infty[^X,\ \|f\| < \infty\}$$

is closed with respect to the operation of addition and T_* restricted to $\mathcal{C}_*$ is a bounded monotone additive functional.

Proof. Of course we assume that $\mathcal{C}_* \neq \emptyset$. We fix $g \in [0,\infty[^X$ and $f_1, f_2 \in \mathcal{C}_*$.

$$g \wedge (f_1 + f_2) \wedge f_1 = g \wedge f_1, \qquad (g \wedge (f_1 + f_2) - f_1)^+ = ((g - f_1) \wedge f_2)^+$$

whence by the definition of f_1 and f_2

$$\begin{aligned} T_* g \wedge (f_1 + f_2) &= T_* g \wedge f_1 + T_*(g - f_1)^+ \wedge f_2 \\ T_*(g - f_1)^+ &= T_*(g - f_1)^+ \wedge f_2 + T_*(g - (f_1 + f_2))^+. \end{aligned}$$

Combining the last two equations with the definition of f_1 we obtain

$$T_* g \wedge (f_1 + f_2) + T_*(g - (f_1 + f_2))^+ = T_* g,$$

that is, $f_1 + f_2 \in \mathcal{C}_*$. The additivity of T_* on $\mathcal{C}_*$ is clear.

Let $\mathcal{F}$ be a non-empty class of subsets of X containing $\emptyset$ and $\mathcal{E}$ be the algebra generated by $\mathcal{F}$. If $\mathcal{H} = \{\chi_F \colon F \in \mathcal{F}\}$, then it can be easily proved that $\{\chi_E \colon E \in \mathcal{E}\} \subset \mathcal{C}_*$ and T_* defines a nonnegative measure on $\mathcal{E}$. The Lemma 2.1. so extends the idea of Carathéodory [3]. This fact was used in [11] and also our next considerations are based on it.

Let us consider the axioms

a) $0 \le h < \infty$, $1 \wedge h \in \mathcal{H}$, $\alpha h \in \mathcal{H}$ for each $\alpha \in [0,\infty[$, $h \in \mathcal{H}$
b) $h_1 \vee h_2$, $h_1 \wedge h_2 \in \mathcal{H}$ for each $h_1, h_2 \in \mathcal{H}$
c) $h_1 + h_2 \in \mathcal{H}$ for each $h_1, h_2 \in \mathcal{H}$
d) $h_1 - h_2 \in \mathcal{H}$ whenever $h_1 \ge h_2$ are in $\mathcal{H}$.

The class $\mathcal{H}$ with properties a), b) and c) is said to be a $(0, \vee f, \wedge f)$ convex cone. If $\mathcal{H}$ is a convex cone which satisfies d), then $\mathcal{H}$ is said to be a $(0, \vee f, \wedge f, \setminus)$ convex cone. The bounded monotone functional T defined on a lattice $\mathcal{H}$ of nonnegative functions on X is said to be tight if

i) $T_*(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 T h_1 + \alpha_2 T h_2$ for $\alpha_i \in [0,\infty[$, $h_i \in \mathcal{H}$ and
ii) $Th_1 - Th_2 = T_*(h_1 - h_2)$ for $h_1 \ge h_2$ in $\mathcal{H}$.

THEOREM 2.2. *If $\mathcal{H}$ is a class of functions with properties* a), b) *and if T is a tight functional on $\mathcal{H}$, then the class $\mathcal{C}_*$ defined by*

$$\mathcal{C}_* = \{f \colon Th = T_* h \wedge f + T_*(h - f)^+ \text{ for all } h \in \mathcal{H},\ \|f\| < \infty\}$$

is a $(0, \vee f, \wedge f, \setminus)$ convex cone containing 1 and T_* restricted to $\mathcal{C}_*$ is a bounded, monotone, homogeneous and additive functional.

Proof. Due to the tightness condition $\mathcal{H} \subset \mathcal{C}_*$. Clearly $\mathcal{C}_*$ has the property a) from the definition of a $(0, \vee f, \wedge f, \setminus)$ convex cone. The i) part of the tightness condition implies that $\mathcal{C}_*$ is identical with the same system denoted in Lemma 2.1. Therefore $\mathcal{C}_*$ has the property c). As for the proof of b) and d) we refer the reader to [11; part I, §3].

A class $\mathcal{G}$ of subsets of X is said to be a $(\emptyset, \cup f, \cap f)$ paving if $\emptyset \in \mathcal{G}$ and $G_1 \cup G_2$, $G_1 G_2 \in \mathcal{G}$ whenever $G_1, G_2 \in \mathcal{G}$. The set function $m_0 \colon \mathcal{G} \to]-\infty, \infty[$ is modular if $m_0 G_1 \cup G_2 + m_0 G_1 G_2 = m_0 G_1 + m_0 G_2$ for all $G_1, G_2 \in \mathcal{G}$. If moreover $m_0 \emptyset = 0$, then m_0 is an evaluation. B. J. Pettis [10; Theorem 1.2.] proved

THEOREM 2.3. *If $\mathcal{E}$ is the ring generated by a $(\emptyset, \cup f, \cap f)$ paving $\mathcal{G}$ and if m_0 is an evaluation on $\mathcal{G}$, then m_0 has a unique extension to a measure on the ring $\mathcal{E}$. If the evaluation m_0 is monotone on $\mathcal{G}$, then the measure extending m_0 is nonnegative.*

In the context of our main problem the Theorem 2.3. establishes that if $\mathcal{H} = \{\chi_G \colon G \in \mathcal{G}\}$ and $T\chi_G = m_0 G$ for $G \in \mathcal{G}$, if $\mathcal{G}$ and m_0 have the properties which are assumed in 2.3., then the problem has an affirmative solution.

Let $\mathcal{C}_*$ be the class of functions which was defined in 2.2. If $\mathcal{L}^+(\mathcal{C}_*)$ is the system of all norm bounded functions in the $\vee c$ closure of $\mathcal{C}_*$ (i.e. $f \in \mathcal{L}^+(\mathcal{C}_*)$, if there is a sequence $\{f_n\} \subset \mathcal{C}_*$ for which $f = \vee f_n$, $f_n \uparrow f$ and $\|f\| < \infty$), then $\mathcal{L}^+(\mathcal{C}_*)$ is a $(0, \vee f, \wedge f)$ convex cone. By $\mathcal{G}(\mathcal{C}_*)$ we denote the class $\mathcal{G}(\mathcal{C}_*) = \{G \colon G = \{x \colon f(x) > 0\},\ f \in \mathcal{C}_*\}$. In virtue of a) and b) $\mathcal{G}(\mathcal{C}_*)$ is a $(\emptyset, \cup f, \cap f)$ paving and $\{\chi_G \colon G \in \mathcal{G}(\mathcal{C}_*)\} \subset \mathcal{L}^+(\mathcal{C}_*)$ ($\chi_G = \lim_{n \to \infty} 1 \wedge nf$ if $G = \{x \colon f(x) > 0,\ f \in \mathcal{C}_*\}$). The ring generated by $\mathcal{G}(\mathcal{C}_*)$ is denoted by $\mathcal{E}(\mathcal{C}_*)$. By $\mathcal{F}(\mathcal{C}_*)$ we denote the class $\{F \colon F = \{x \colon f(x) = 0\},\ f \in \mathcal{C}_*\}$.

We say that the function $f \in [0, \infty[^X$ is continuous with respect to $\mathcal{G}(\mathcal{C}_*)$ if $f^{-1}(U) \in \mathcal{G}(\mathcal{C}_*)$ for each open subset U of the real line with the usual topology. If $\mathcal{G}(\mathcal{C}_*)$ is closed under the formation of countable unions (i.e. if $\mathcal{G}(\mathcal{C}_*)$ is a $(\emptyset, \cup c, \cap f)$ paving), then the system $C^+(X, \mathcal{G}(\mathcal{C}_*))$ of all bounded nonnegative $\mathcal{G}(\mathcal{C}_*)$ continuous functions is a $(0, \vee f, \wedge f, \setminus)$ convex cone. Since for each $f \in \mathcal{C}_*$ and rational r $\{x \colon f(x) > r\} = \{x \colon (f - f \wedge r)(x) > 0\} \in \mathcal{G}(\mathcal{C}_*)$, $\{x \colon f(x) < r\} = \{x \colon (r - f \wedge r)(x) > 0\} \in \mathcal{G}(\mathcal{C}_*)$, all the functions in $\mathcal{C}_*$ are $\mathcal{G}(\mathcal{C}_*)$ continuous.

The measure m on $\mathcal{E}(\mathcal{C}_*)$ is said to be regular if $mE = \sup\{mF \colon F \subset E,\ F \in \mathcal{F}(\mathcal{C}_*)\}$.

THEOREM 2.4. *Let $\mathcal{H}$ be a class of functions with the properties* a) *and* b), *T be a tight functional on $\mathcal{H}$. Let T_* be additive on $\mathcal{L}^+(\mathcal{C}_*)$. Then there is a unique regular measure $\big(X, \mathcal{E}(\mathcal{C}_*), m\big)$ representing T_* on $\mathcal{C}_*$.*

Proof. In virtue of the additivity of T_* on $\mathcal{L}^+(\mathcal{C}_*)$ and due to 2.3. m_0 defined on $\mathcal{E}(\mathcal{C}_*)$ by $m_0 G = T_* \chi_G$ is an evaluation defining a unique nonnegative measure m on $\mathcal{E}(\mathcal{C}_*)$. Since each $f \in \mathcal{C}_*$ is continuous, the sets $G = \{x\colon f(x) > r\}$ are in $\mathcal{G}(\mathcal{C}_*)$ for all rational r and we can prove as in 1.3. that $T_* f = m(f)$ for each $f \in \mathcal{C}_*$.

We prove that m is regular. We fix some $G \in \mathcal{G}(\mathcal{C}_*)$ and $\varepsilon > 0$. Then we choose $f \in \mathcal{C}_*$, $f \leq \chi_G$ for which $mG < Tf + \varepsilon$. T is a bounded functional, whence there is a $\gamma \in \,]0, \infty[$ and a natural number k that $T_* f \leq \gamma \|f\|$ for all $f \in \mathcal{C}_*$ and $\frac{\gamma}{k} < \varepsilon$. The set F defined by $F = \left\{x\colon f(x) \geq \frac{1}{k}\right\}$ is in $\mathcal{F}(\mathcal{C}_*)$, $F \subset G$ and

$$mG < T_* f \wedge \frac{1}{k} + T_*\Big(f - \frac{1}{k}\Big)^+ + \varepsilon \leq \frac{1}{k} T_* 1 \wedge kf + mF + \varepsilon < mF + 2\varepsilon.$$

This proves that $mG = \sup\{mF\colon F \subset G,\ F \in \mathcal{F}(\mathcal{C}_*)\}$. Now let $G_1, G_2 \in \mathcal{G}(\mathcal{C}_*)$, $G_1 \supset G_2$. To the $\varepsilon > 0$ we can choose $F \in \mathcal{F}(\mathcal{C}_*)$ for which $mG_1 < mF + \varepsilon$, $F \subset G_1$. Since $mG_1 - G_2 < mF - G_2 + \varepsilon$, $F - G_2 \subset G_1 - G_2$ and $F - G_2 \in \mathcal{F}(\mathcal{C}_*)$, $mG_1 - G_2 = \sup\{mF\colon F \subset G_1 - G_2,\ F \in \mathcal{F}(\mathcal{C}_*)\}$. The rest of this part of proof follows easily from the fact that each $E \in \mathcal{E}(\mathcal{C}_*)$ can be written as a union of a finite sequence of pairwise disjoint sets of the form $G_1 - G_2$, where $G_1 \supset G_2$ are in $\mathcal{G}(\mathcal{C}_*)$.

Finally we have to prove that m is the unique regular measure on $\mathcal{E}(\mathcal{C}_*)$ with the property $T_* f = m(f)$ for all $f \in \mathcal{C}_*$. Suppose that there is another regular measure m', on $\mathcal{E}(\mathcal{C}_*)$ with the property $T_* f = m'(f)$. It is easy to observe that $mG \leq m'G$ for each $G \in \mathcal{G}(\mathcal{C}_*)$. $1 \in \mathcal{C}_*$, thus $T_* 1 = mX = m'X$ and $m'F \leq mF$ for all $F \in \mathcal{F}(\mathcal{C}_*)$. Now it is easy to show, using the regularity of m', that $m = m'$ on $\mathcal{E}(\mathcal{C}_*)$.

COROLLARY 2.5. *Under the assumptions of* 2.4. *T determines a unique measure $(X, \mathcal{E}(\mathcal{C}_*), m)$ representing T on $\mathcal{H}$, which coincides on $\mathcal{E}(\mathcal{C}_*)$ with a regular measure.*

The proof of 2.5 follows from 2.4 and from the fact that the small st ring with respect to which all $h \in \mathcal{H}$ are measurable is in $\mathcal{E}(\mathcal{C}_*)$. The ideas used in the proof of 2.4. are quite close to that used in [11; part I., §3] Also the following idea can be found in [11].

Let $\mathcal{U}^+(\mathcal{C}_*)$ be the $\wedge c$ closure of $\mathcal{C}_*$. We say that $\mathcal{C}_*$ has the "in between" property if to each $u \in \mathcal{U}^+(\mathcal{C}_*)$ and $l \in \mathcal{L}^+(\mathcal{C}_*)$ for which $u \leq l$ there is $f \in \mathcal{C}_*$ such that $u \leq f \leq l$.

LEMMA 2.6. *If $\mathcal{C}_*$ has the "in between" property, then T_* is additive on $\mathcal{L}^+(\mathcal{C}_*)$.*

Proof. Let $l_1, l_2 \in \mathcal{L}^+(\mathcal{C}_*)$ and $\varepsilon > 0$ be given. Choose $f \in \mathcal{C}_*$, $f \leq l_1 + l_2$ and $f' \in \mathcal{C}_*$ with $Tf + \varepsilon > T_*(l_1 + l_2)$, $(f - l_2)^+ \leq f' \leq l_1$. Now $(f - f')^+ \leq l_2$ and

$$T_*(l_1 + l_2) - \varepsilon \leq Tf = T_* f \wedge f' + T_*(f - f')^+ \leq T_* l_1 + T_* l_2.$$

This means that $T_*(l_1 + l_2) \leq T_* l_1 + T_* l_2$. The reverse inequality holds as well, thus the lemma is proved.

LEMMA 2.7. *If T_* is additive on $\mathcal{L}^+(\mathcal{C}_*)$, then $\mathcal{C}_*$ has the "in between" property.*

Proof. Let us assume $u \in \mathcal{U}^+(\mathcal{C}_*)$ and $l \in \mathcal{L}^+(\mathcal{C}_*)$ such that $u \leq l$. Of course there are $\{f'_n\} \subset \mathcal{C}_*$ and $\{f''_n\} \subset \mathcal{C}_*$ such that $f'_n \downarrow u$ and $f''_n \uparrow l$. Put $f_n = \bigvee_{j \leq n} (f'_j \wedge f''_j)$, $g_n = f_{n-1} \vee f'_n$ for $n = 1, 2, \dots$. $\{f_n\}, \{g_n\}$ are contained in $\mathcal{C}_*$ and since

$$0 \leq g_n - f_n = f_{n-1} \vee f'_n - f_{n-1} \vee \left(f'_n \wedge f''_n\right) \leq \left(f'_n - f''_n\right)^+$$

for each $n = 1, 2 \dots$. Consequently $f = \bigvee f_n = \bigwedge g_n$ is a function with the property $u \leq f \leq l$.

To prove that $f \in \mathcal{C}_*$ note that for each $h \in \mathcal{H}$

$$h \wedge f = \bigvee (h \wedge f_n), \qquad (h - f)^+ = \left(h - \bigwedge g_n\right)^+ = \bigvee (h - g_n)^+.$$

$h \wedge f_n, (h - g_n)^+ \in \mathcal{C}_*$ for all $n = 1, 2, \dots$, which implies that $h \wedge f$, $(h - f)^+ \in \mathcal{L}^+(\mathcal{C}_*)$. T_* is additive on $\mathcal{L}^+(\mathcal{C}_*)$, thus

$$Th = T_*\big(h \wedge f + (h - f)^+\big) = T_* h \wedge f + T_*(h - f)^+$$

and we can conclude that $f \in \mathcal{C}_*$.

THEOREM 2.8. *Let $\mathcal{H}$ be a class of functions with the properties a) and b), T be a tight functional on $\mathcal{H}$. Then the evaluation m_0 defined on $\mathcal{G}(\mathcal{C}_*)$ by $m_0 G = T_* \chi_G$ determines the unique regular measure representing T_* on $\mathcal{C}_*$ if and only if $\mathcal{C}_*$ has the "in between" property.*

Proof. In virtue of 2.6. and 2.4. we know that the "in between" property implies that m_0 determines the unique regular measure representing T_*.

Conversely if m is the regular measure representing T_* on $\mathcal{C}_*$ with the property $mG = m_0 G$ for all $G \in \mathcal{G}(\mathcal{C}_*)$, then each $l \in \mathcal{L}^+(\mathcal{C}_*)$ is m-integrable and $m(l) = T_* l$. However, this means that T_* is additive on $\mathcal{L}^+(\mathcal{C}_*)$ and in virtue of 2.7. $\mathcal{C}_*$ has the "in between" property.

COROLLARY 2.9. *If $\mathcal{H}$ is a $(0, \vee f, \wedge f, \backslash)$ convex cone and if T is a bounded monotone linear functional, then T_* restricted to $\{\chi_G : G \in \mathcal{G}(\mathcal{C}_*)\}$ determines the unique regular measure representation of T_* on $\mathcal{C}_*$ if and only if $\mathcal{C}_*$ has the "in between" property.*

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