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REMARK TO TRANSFORMATIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

VÁCLAV TRYHUK

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ABSTRACT. Effective sufficient and necessary conditions are given that the functional-differential equation

$$y'(x) = \sum_{i=0}^{n} a_i(x)b_i\big(y(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(y(\xi_j(x))\big) + q(x)y(x)\,, \qquad x \in I \subseteq \mathbb{R}\,,$$

with m $(m \geq 1)$ delays be globally transformable into an equation of the form

$$u'(s) = \sum_{i=0}^{n} a_i b_i\big(u(s)\big) \prod_{j=1}^{m} \delta_{ij}\big(u(s+c_j)\big) + qu(s)\,, \qquad s \in \mathbb{R}\,,$$

with constant coefficients and deviations.

Here b_i, δ_{ij} are nontrivial solutions, in the class of functions continuous at a point, of Cauchy's functional equation $b(uv) = b(u)b(v)$ $(u, v \in \mathbb{R} - \{0\})$.

1. Introduction

The theory of global transformations converting any homogeneous linear differential equation of the nth order into another equation of the same kind and order on the whole interval of their definition was developed in the monograph of F. Neuman [9] (see historical remarks, definitions, results and some applications). The most general form of global pointwise transformations for homo-

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geneous linear differential equations of the nth order $(n \geq 2)$ is

$$z(t) = L(t)y\bigl(\varphi(t)\bigr)\,,$$

where φ is a bijection of an interval J onto an interval I $(J \subseteq \mathbb{R},\ I \subseteq \mathbb{R})$ and $L(t)$ is a nonvanishing function on J, i.e. this global transformation consist of a change of the independent variable and of a nonvanishing factor L. Global transformations may serve for example for investigation of oscillatory behavior of solutions from certain classes of linear differential equations because each global pointwise transformation preserves distribution of zeros of solutions of a linear differential equation.

The form of the most general pointwise transformation of homogeneous linear differential equations with deviating arguments was derived in [2], [8], [10], [11], [12]. This form coincides for arbitrary order with the form considered for linear differential equations of the nth $(n \geq 2)$ order without deviation. Transformations and canonical forms of linear functional-differential equations were studied in [6], [7], [10], oscillatory behavior of solutions of functional-differential equations and functional-differential equations with constant coefficients and deviations in [3], [4], [5], for example.

Effective criterion for transformations of linear functional-differential equations of the first order into canonical forms with constant coefficients and deviations was derived in [14].

Functional-differential equations considered in [5] are nonlinear, such as

$$y'(x) + p(x)\bigl|y\bigl(\tau(x)\bigr)\bigr|^{\lambda}\operatorname{sign} y\bigl(\tau(x)\bigr) = 0\,, \qquad \lambda \geq 0\,;$$

$$y'(x) = \frac{(x-1)^3}{x^2(x-2)^2}y(x-1)^3\,, \qquad x \geq 3\,;$$

$$y'(x) = 2^{1-x}y(2x)^{1/3}y(3x)y(4x)^{1/3}\,;$$

$$y'(x) = \frac{|y(x+\sin x)|^{\alpha_1}\operatorname{sign} y(x+\sin x)|y(x+\cos x)|^{\alpha_2}\operatorname{sign} y(x+\cos x)}{x^{\beta}|\ln(x+\sin x)|^{\alpha_1}|\ln(x+\cos x)|^{\alpha_2}}\,,$$

$x \geq 2\pi$, $\alpha_i > 0$, $\alpha = \alpha_1 + \alpha_2 > 1$, $\beta < 1$. In [13], a general form

$$y'(x) = \sum_{i=0}^{n} a_i(x)b_i\bigl(y(x)\bigr)\prod_{j=1}^{m}\delta_{ij}\bigl(y\bigl(\xi_j(x)\bigr)\bigr) + q(x)y(x)\,, \qquad x \in I \subseteq \mathbb{R},$$

where b_i, δ_{ij} are nontrivial solutions, in the class of functions continuous at a point, of Cauchy's functional equation $b(uv) = b(u)b(v)$ $(u, v \in \mathbb{R} - \{0\})$, was derived. This form is generally nonlinear and allows global transformations $z(t) = L(t)y\bigl(\varphi(t)\bigr)$ such that convert any equation into another equation of the same kind and order on the whole intervals of definitions. The transformation $z(t) = L(t)y\bigl(\varphi(t)\bigr)$ is the most general pointwise transformation for this general form of functional-differential equations. An effective criterion of equivalence of two such equations is derived in [13].

The general solutions continuous at a point of Cauchy's functional equation are given by

$$g(x) = 0\,, \qquad g(x) = |x|^c\,, \qquad g(x) = |x|^c \operatorname{sign} x\,,$$

$c \in \mathbb{R}$ is an arbitrary constant (see A c z é l [1]) and we can investigate the illustrated examples of nonlinear functional-differential equations.

Transformations of differential equations and systems with deviating arguments converting equations into equations with constant deviations were investigated in [7]. The necessary conditions for the existence such transformations are derived (for $m = 1$ the sufficient conditions are derived too).

In this paper we derive a criterion for transformations of the above functional-differential equation into an equation of the same kind and order with constant coefficients and deviations. The criterion that we give is effective, i.e. it is verifiable for any considered equation.

2. Notations, basic definitions

Consider two functional-differential equations with m $(m \geq 1)$ deviating arguments

$$y'(x) = \sum_{i=0}^{n} a_i(x) b_i\big(y(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(y(\xi_j(x))\big) + q(x)y(x)\,, \qquad x \in I \subseteq \mathbb{R}, \quad (1)$$

and

$$z'(t) = \sum_{i=0}^{n} A_i(t) b_i\big(z(t)\big) \prod_{j=1}^{m} \delta_{ij}\big(z(\eta_j(t))\big) + Q(t)z(t)\,, \qquad t \in J \subseteq \mathbb{R}. \quad (2)$$

Here $y \in C^1(I)$, $I \subseteq \mathbb{R}$ being an interval; b_i, δ_{ij} are continuous solution (in the class of functions continuous at a point) of Cauchy's functional equation $b(uv) = b(u)b(v)$ $(u, v \in \mathbb{R} - \{0\})$; $a_i \in C^0(I)$ are nonvanishing functions on I, $q \in C^0(I)$. For deviations of (1) we give the following assumptions.

ASSUMPTIONS. Each ξ_j is a C^1 diffeomorphism of I onto I and $\xi_j'(x) > 0$ on I $(j = 1, \dots, m)$; $\xi_0 = \mathrm{id}_I$, $\xi_j(x) \neq \xi_k(x)$ for $j \neq k$ on $I = (a, b) \subseteq \mathbb{R}$, $j, k \in \{0, \dots, m\}$, $m, n \in \mathbb{N} = \{1, 2, \dots\}$;

$$\lim_{x \to a_+} \xi_j(x) = a\,, \qquad \lim_{x \to b_-} \xi_j(x) = b$$

for $j = 1, \dots, m$; $a \geq -\infty$, $b \leq \infty$.

Similar assumption we consider for equation (2).

DEFINITION. We say that (1) is *globally transformable* into (2) if there exist two functions φ, L such that

— the function L is of the class $C^1(J)$ and is nonvanishing on J;
— the function φ is a C^1 diffeomorphism of the interval J onto the interval I;

and the function

$$z(t) = L(t)y\big(\varphi(t)\big) \tag{3}$$

is a solution of (2) whenever $y(x)$ is a solution of (1).

If (1) is globally transformable into (2) then (see [2], [6], [8], [10], [11])

$$\xi_j\big(\varphi(t)\big) = \varphi\big(\eta_j(t)\big) \tag{4}$$

is satisfied on J for deviations ξ_j, η_j $(j = 1,\dots,m)$ and we say that (1), (2) are *equivalent equations.*

If we require that the delayed arguments are converted again into the delayed ones (or the advanced into advanced), then we need

$$\varphi'(t) > 0 \qquad \text{on} \quad J\,. \tag{5}$$

We restrict the pointwise transformations namely to this case of the increasing changes of the independent variable.

DEFINITION. We say that (3) is a *stationary transformation* of (1) if it globally transforms an equation (1) into itself on I, i.e. if $L \in C^1(I)$, $L(x) \neq 0$ on I, φ is a C^1 diffeomorphism of I onto $I = \varphi(I)$ and the function $z(x) = L(x)y\big(\varphi(x)\big)$ is the solution of

$$z'(x) = \sum_{i=0}^{n} a_i(x)b_i\big(z(x)\big)\prod_{j=1}^{m}\delta_{ij}\big(z\big(\xi_j(x)\big)\big) + q(x)z(x)$$

whenever y is the solution of

$$y'(x) = \sum_{i=0}^{n} a_i(x)b_i\big(y(x)\big)\prod_{j=1}^{m}\delta_{ij}\big(y\big(\xi_j(x)\big)\big) + q(x)y(x)\,, \qquad x \in I\,.$$

In this situation,

$$\xi_j\big(\varphi(x)\big) = \varphi\big(\xi_j(x)\big)$$

is satisfied on I for deviations ξ_j $(j = 1,\dots,m)$.

PROPOSITION 1. ([13]) *An equation* (1) *is globally transformable into* (2) *if and only if the functions* L, φ ($L \in C^1(J)$, $L(x) \neq 0$ *on* J, φ *is a* C^1

diffeomorphism of J onto I) satisfy relations

$$\xi_j\big(\varphi(t)\big) = \varphi\big(\eta_j(t)\big)\,, \qquad j = 1,2,\dots,m\,; \;\; \varphi(J) = I\,,$$

$$Q(t) = \frac{L'(t)}{L(t)} + q\big(\varphi(t)\big)\varphi'(t)\,,$$

$$A_i(t) = \frac{a_i\big(\varphi(t)\big)\varphi'(t)L(t)}{b_i\big(L(t)\big)\prod_{j=1}^{m}\delta_{ij}\big(L(\eta_j(t))\big)}\,, \qquad i = 1,\dots,n\,,$$

on J.

As a corollary we get:

Proposition 2. *Transformation* (3) *is a stationary transformation of equation* (1) *if and only if φ is a simultaneous solution of*

$$\xi_j\big(\varphi(x)\big) = \varphi\big(\xi_j(x)\big)\,, \qquad j = 1,2,\dots,m\,; \;\; \varphi(I) = I\,,$$

and

$$a_i\big(\varphi(x)\big)\varphi'(x)L(x) = a_i(x)b_i\big(L(x)\big)\prod_{j=1}^{m}\delta_{ij}\big(L(\xi_j(x))\big)\,, \qquad i = 1,\dots,n\,;$$

$$L'(x) = \big(q(x) - q(\varphi(x))\varphi'(x)\big)L(x)\,, \qquad x \in I\,.$$

Definition. If all η_j in (2) are of the form

$$\eta_j(t) = t + c_j\,,$$

c_j being nonzero constants, equation (2) is said to be *with constant deviations* also *with discrete deviations*, see e.g. [6], [7], [8].

Using (4), equation (1) is globally transformable into an equation

$$z'(t) = \sum_{i=0}^{n} A_i(t)b_i\big(z(t)\big)\prod_{j=1}^{m}\delta_{ij}\big(z(t+c_j)\big) + Q(t)z(t)\,, \qquad t \in \mathbb{R}, \tag{6}$$

with constant deviations if and only if the following conditions for transformation (3), (4) are satisfied

$$\eta_j(t) = t + c_j \iff h\big(\xi_j(x)\big) = h(x) + c_j \tag{7}$$

for $j = 1,\dots,m$; where

$$x = \varphi(t) \iff t = h(x)\,,$$

i.e. $h = \varphi^{-1}$ is the inverse function to φ $(t \in \mathbb{R},\ x \in I)$.

Proposition 3. ([6]) *If there exists a solution $h \in C^1$, $h' \neq 0$ of a system of functional equations* (7) *with ξ_i $(1 \le i \le m)$ then each ξ_i and ξ_j commute, and for any (positive, negative, or* 0*) integers r and s either $\xi_i^r \equiv \xi_j^s$, or $\xi_i^r \neq \xi_j^s$ everywhere, where the expressions are defined.*

3. Results

Lemma 1. *Equation* (1) *is equivalent to an equation*

$$u'(s) = \sum_{i=0}^{n} a_i b_i\big(u(s)\big) \prod_{j=1}^{m} \delta_{ij}\big(u(s+c_j)\big) + qu(s)\,, \qquad s \in \mathbb{R}, \tag{8}$$

with constant coefficients and discrete deviations $\eta_j(s) = s + c_j$, $c_j \neq 0$, $j = 1,\dots,m$, *if and only if each* $\xi_j, \xi_k \in \{\xi_1,\dots,\xi_m\}$ *commute and to every function* $\xi \in \{\xi_1,\dots,\xi_m\}$ *there exists a function* $L \in C^1(I)$, $L(x) \neq 0$ *on* I *such that*

$$z(x) = L(x)y\big(\xi(x)\big)\,, \quad \xi_j\big(\xi(x)\big) = \xi\big(\xi_j(x)\big)\,, \qquad x \in I\,,$$

is a stationary transformation of equation (1).

P r o o f. An equation (1) is equivalent to (8) if and only if there exists a global transformation

$$u(s) = L_1(s)y\big(\varphi(s)\big)\,, \qquad \xi_j\big(\varphi(s)\big) = \varphi\big(\eta_j(s)\big) = \varphi(s+c_j)\,, \\ j = 1,\dots,m\,, \quad s \in \mathbb{R},$$

converting (1) into (8). Here $L_1 \in C^1(\mathbb{R})$, $L_1(s) \neq 0$ on $\mathbb{R}$, φ is a C^1 diffeomorphism of $\mathbb{R}$ onto I. Without loss of generality we assume that $L_1(s) = f\big(\varphi(s)\big)$, $f \in C^1(I)$ being a nonzero function on I, i.e.

$$u(s) = f\big(\varphi(s)\big)y\big(\varphi(s)\big)\,, \qquad \xi_j\big(\varphi(s)\big) = \varphi\big(\eta_j(s)\big) = \varphi(s+c_j)\,, \\ j = 1,\dots,m\,, \quad s \in \mathbb{R}. \tag{9}$$

Thus, each $\xi_j, \xi_k \in \{\xi_1,\dots,\xi_m\}$ commute in accordance with Proposition 3.

There exists the inverse transformation

$$y(x) = \frac{1}{f(x)}u\big(\varphi^{-1}(x)\big)\,, \quad \xi_j(x) = \varphi\big(\eta_j\big(\varphi^{-1}(x)\big)\big)\,, \qquad j = 1,\dots,m\,, \ x \in I\,, \tag{10}$$

converting (8) into (1) in accordance with the definition of global transformations.

Each function $\psi(\tau) = \eta_k(\tau) = \tau + c_k$, $\tau \in \mathbb{R}$ is a C^1 diffeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ and a transformation

$$v(\tau) = u\big(\psi(\tau)\big) = u\big(\eta_k(\tau)\big) = u(\tau + c_k)\,, \\ \eta_j\big(\psi(\tau)\big) = \eta_j\big(\eta_k(\tau)\big) = \eta_k(\tau) + c_j = \tau + c_k + c_j = \eta_k\big(\eta_j(\tau)\big) = \psi\big(\eta_j(\tau)\big)\,, \\ j = 1,\dots,m\,, \quad \tau \in \mathbb{R}, \tag{11}$$

$k \in \{1,\dots,m\}$ being fixed, is a stationary transformation of equation (8). Thus

$$v'(\tau) = \big(u(\psi(\tau))\big)' = u'(\tau + c_k) = \sum_{i=0}^{n} a_i b_i\big(v(\tau)\big) \prod_{j=1}^{m} \delta_{ij}\big(v(\tau + c_j)\big) + qv(\tau)\,, \qquad \tau \in \mathbb{R}. \tag{12}$$

We apply the inverse transformation (10) onto (12) as

$$z(t) = \frac{1}{f(t)} v\big(\varphi^{-1}(t)\big)\,, \quad \xi_j(t) = \varphi\big(\eta_j\big(\varphi^{-1}(t)\big)\big)\,, \qquad j = 1,\dots,m\,, \;\; t \in I\,, \tag{13}$$

and we use the composition of transformations (9), (11), (13)

$$z(t) = L(t)y\big(\tilde{\varphi}(t)\big)\,, \qquad \xi_j\big(\tilde{\varphi}(t)\big) = \tilde{\varphi}(t)\,, \tag{14}$$

where $L(t) = \frac{f(\tilde{\varphi}(t))}{f(t)}$ and $\tilde{\varphi}(t) = \varphi \circ \psi \circ \varphi^{-1}(t) = \varphi \circ \eta_k \circ \varphi^{-1}(t) = \xi_k(t)$, $t \in I$, $k \in \{1,\dots,m\}$ being an arbitrary fixed. This transformation (14) is a stationary transformation of (1) and the assertion of Lemma 1 is proved. □

Lemma 2. *Let each $\xi_j, \xi_k \in \{\xi_1,\dots,\xi_m\}$ commute. To every function $\xi \in \{\xi_1,\dots,\xi_m\}$ there exists a function $L \in C^1(I)$, $L(x) \neq 0$ on I, such that*

$$z(x) = L(x)y\big(\xi(x)\big)\,, \quad \xi_j\big(\xi(x)\big) = \xi\big(\xi_j(x)\big)\,, \qquad x \in I\,,$$

is a stationary transformation of equation (1) *if and only if the relations*

$$\begin{gathered} a_i\big(\varphi(x)\big)\varphi'(x)L(x) = a_i(x)b_i\big(L(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(L\big(\xi_j(x)\big)\big)\,, \qquad i = 1,\dots,n\,; \\ \frac{L'(x)}{L(x)} = q(x) - q\big(\varphi(x)\big)\varphi'(x) \end{gathered} \tag{15}$$

are satisfied on I for the functions L, ξ and the coefficients of (1).

Moreover,

$$L(x) = \frac{f(x)}{f\big(\xi(x)\big)} \qquad \textit{where} \quad \frac{f'(x)}{f(x)} = q(x)\,, \;\; x \in I\,.$$

Proof. If conditions (15) are fulfilled on I then there exists a function $L \in C^1(I)$, $L(x) \neq 0$ on I, to every function $\xi \in \{\xi_1,\dots,\xi_m\}$. The assertion of Lemma 2 follows from Proposition 2. □

Lemma 3. *Let each $\xi_j, \xi_k \in \{\xi_1, \dots, \xi_m\}$ commute and to every function $\xi \in \{\xi_1, \dots, \xi_m\}$ there exists a function $L \in C^1(I)$, $L(x) \neq 0$ on I, such that relations* (15) *hold on I. Then equation* (1) *is equivalent to an equation* (2) *with constant coefficients and discrete deviations.*

Proof. Let the assumptions of Lemma 3 be satisfied. Consider a transformation

$$y(x) = f(x)v(x)\,, \tag{16}$$

where $f \in C^1(I)$, $f(x) \neq 0$ on I. This transformation converts any equation (1) into some equation

$$\begin{aligned} v'(x) &= \sum_{i=0}^{n} A_i(x) b_i\big(v(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(v(\xi_j(x))\big) + \left(q(x) - \frac{f'(x)}{f(x)}\right) v(x)\,, \\ A_i(x) &= \frac{1}{f(x)} a_i(x) b_i\big(f(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(f(\xi_j(x))\big)\,, \qquad i = 1, \dots, n\,, \end{aligned} \tag{17}$$

and we put

$$q(x) = \frac{f'(x)}{f(x)}\,, \qquad x \in I\,. \tag{18}$$

Then $L(x) = c\frac{f(x)}{f(\xi(x))}$, $c \in \mathbb{R} - \{0\}$ and we can take $c = 1$. Thus there exist functions

$$L_k(x) := \frac{f(x)}{f\big(\xi_k(x)\big)} > 0 \text{ on } I, \quad L_k \in C^1(I)\,, \qquad k \in \{1, \dots, m\}\,. \tag{19}$$

Moreover, (17) becomes

$$\begin{aligned} v'(x) &= \sum_{i=0}^{n} A_i(x) b_i\big(v(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(v(\xi_j(x))\big)\,, \\ A_i(x) &= \frac{1}{f(x)} a_i(x) b_i\big(f(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(f(\xi_j(x))\big)\,, \qquad i = 1, \dots, n\,. \end{aligned} \tag{20}$$

Functions b_i, δ_{ij} are nontrivial continuous solutions of Cauchy's functional equation $b(uv) = b(u)b(v)$ and we have

$$b(u) = b\Big(\frac{u}{v} v\Big) = b\Big(\frac{u}{v}\Big) b(v)\,, \qquad u, v \in \mathbb{R} - \{0\}\,. \tag{21}$$

Using (20), (19), (15) and (21),

$$\begin{aligned} A_i(\xi_k)\xi_k' &= \frac{f a_i(\xi_k)\xi_k' b_i\big(f(\xi_k)\big) \prod \delta_{ij}\big(f(\xi_j(\xi_k))\big)}{f(\xi_k) a_i b_i(f) \prod \delta_{ij}\big(f(\xi_j)\big)} A_i \\ &= a_i(\xi_k)\xi_k' L_k \frac{1}{a_i b_i(L_k) \prod \delta_{ij}\big(L_k(\xi_j)\big)} A_i = A_i\,, \end{aligned}$$

i.e.

$$A_i\big(\xi_k(x)\big)\xi'_k(x) = A_i(x) \neq 0\,, \qquad i = 1,\dots,n\,,\ \ k \in \{1,\dots,m\} \tag{22}$$

on I because each two functions $\xi_j, \xi_k \in \{\xi_1,\dots,\xi_m\}$ commute.

Now we define a transformation

$$x = \varphi(t) \iff t = h(x) := \int\limits_{x_0}^{x} |A_r(s)|\, \mathrm{d}s + a_1\,, \tag{23}$$

where $a_1 \in \mathbb{R}$, $x_0 \in I$ and $r \in \{1,\dots,m\}$ is fixed, $A_r \neq 0$ on I. Then $h'(x) = |A_r(x)| > 0$ and from (22) we get

$$\big(h(\xi(x)) - h(x)\big)' = \big|A_r\big(\xi(x)\big)\big|\xi'(x) - \big|A_r\big(\xi(x)\big)\big| = \big|A_r\big(\xi(x)\big)\xi'(x) - A_r(x)\big| = 0$$

for $\xi \in \{\xi_1,\dots,\xi_m\}$ because $\xi'_j(x) > 0$ on I $(j = 1,\dots,m)$. Hence $h\big(\xi(x)\big) = h(x) + c$, $c \in \mathbb{R}$, and (7) gives

$$\eta_j(t) = t + c_j \iff h\big(\xi_j(x)\big) = h(x) + c_j \tag{24}$$

and

$$\xi_j(x) = \xi_j\big(\varphi(x)\big) = h^{-1}(t + c_j) = \varphi(t + c_j) \tag{25}$$

hold for all $j \in \{1,\dots,m\}$.

If we define a transformation

$$z(t) = v\big(\varphi(t)\big) = v(x)\,, \qquad \xi_j\big(\varphi(t)\big) = \varphi(t + c_j)\,, \quad j = 1,\dots,m\,, \tag{26}$$

we obtain

$$\begin{aligned} v\big(\xi_j(x)\big) = v\big(\xi_j\big(\varphi(t)\big)\big) &= v\big(\varphi(t + c_j)\big)\,, \qquad j = 1,\dots,m\,,\\ v'(x) = \big(z\big(h(x)\big)\big)' &= z'\big(h(x)\big)h'(x)\,, \end{aligned}$$

where $h'(x) = |A_r(x)| > 0$ on I. Transformation (23), (26) globally transforms equation (20) into the equation

$$z'(t) = \sum_{i=0}^{n} A_i\big(\varphi(t)\big)\varphi'(t) b_i\big(z(t)\big) \prod_{j=1}^{m} \delta_{ij}\big(z(t + c_j)\big) \tag{27}$$

with discrete deviations.

Define the functions

$$H_i(x) := H_i\big(\varphi(t)\big) = A_i\big(\varphi(t)\big)\varphi'(t) = \frac{A_i(x)}{|A_r(x)|}\,, \qquad x \in I\,, \ \ i = 1,\dots,n\,. \tag{28}$$

Then using (22),

$$H_i\big(\xi(x)\big) = \frac{A_i\big(\xi(x)\big)}{\big|A_r\big(\xi(x)\big)\big|}\frac{\xi'(x)}{\xi'(x)} = \frac{A_i(x)}{|A_r(x)|} = H_i(x)\,, \qquad i = 1,\dots,n\,,$$

and
$$\xi'(x) > 0\,, \qquad \xi(x) \neq x$$
hold for all $\xi \in \{\xi_1, \dots, \xi_m\}$ on I in accordance with the assumptions for equation (1). Due to the condition $\lim\limits_{x\to b_-} \xi(x) = b$, the nth iterate $\xi^{[n]}$ of $\xi \in \{\xi_1, \dots, \xi_m\}$ exists for all positive or negative integers n accordingly to $\xi(x) > x$ or $\xi(x) < x$ on $I = (a,b)$ and
$$\lim_{n\to\infty} \xi^{[n]}(x) = b \quad \text{for } \xi(x) > x\,, \qquad \lim_{n\to-\infty} \xi^{[n]}(x) = b \quad \text{if } \xi(x) < x$$
on I. Hence
$$H\big(\xi^{[n]}(x)\big) = H\big(\xi^{[n-1]}(x)\big) = \dots = H(x)\,, \qquad x \in I\,,$$
gives
$$H_i(x) = H_i(b_-) \in \mathbb{R}, \qquad x \in I\,,$$
i.e. coefficients $A_i\big(\varphi(t)\big)\varphi'(t) = H_i\big(\varphi(t)\big) = H_i(x)$ of equation (27) are constant functions $(i = 1, \dots, n)$.

Repeating arguments given by F. Neuman [6], [8] we can prove that $h(I) = (-\infty, \infty) = \mathbb{R}$ in accordance with assumptions
$$\lim_{x\to a_+} \xi_j(x) = a\,, \quad \lim_{x\to b_-} \xi_j(x) = b\,, \qquad j = 1, \dots, m\,.$$
Here ξ_j is a C^1 diffeomorphism of I onto I, $\xi_j'(x) > 0$ on I $(j = 1, \dots, m)$, $\xi_j(x) \neq \xi_k(x)$ for $j \neq k$ on I, $j,k \in \{0, \dots, m\}$.

The assertion of Lemma 2 is proved. □

THEOREM 1. *Let Assumptions for equation* (1) *be satisfied. Then the following assertions are equivalent*

(a) *Equation* (1) *is equivalent to an equation* (2) *with constant coefficients and discrete deviations.*

(b) *Each* $\xi_j, \xi_k \in \{\xi_1, \dots, \xi_m\}$ *commute and to every function* $\xi \in \{\xi_1, \dots \dots, \xi_m\}$ *there exists a function* $L \in C^1(I)$, $L(x) \neq 0$ *on* I *such that*
$$z(x) = L(x)y\big(\xi(x)\big)\,, \quad \xi_j\big(\xi(x)\big) = \xi\big(\xi_j(x)\big)\,, \qquad x \in I\,,$$
is a stationary transformation of equation (1).

(c) *There exist functions* $L_k \in C^1(I)$, $L_k(x) \neq 0$ *on* I *such that the relations*
$$a_i\big(\xi_k(x)\big)\xi_k'(x)L_k(x) = a_i(x)b_i\big(L_k(x)\big)\prod_{j=1}^{m}\delta_{ij}\big(L_k\big(\xi_j(x)\big)\big)\,, \qquad i = 1, \dots, n\,;$$
$$\frac{L_k'(x)}{L_k(x)} = q(x) - q\big(\xi_k(x)\big)\xi_k'(x)$$

hold on I for functions L_k, ξ_k, $\xi_k \in \{\xi_1,\dots,\xi_m\}$, $k \in \{1,\dots,m\}$, and coefficients of (1). *Moreover,*

$$L_k(x) = \frac{f(x)}{f\big(\xi_k(x)\big)} \qquad \textit{where} \quad \frac{f'(x)}{f(x)} = q(x)\,, \quad x \in I\,.$$

Proof. We have:

(a) $\iff$ (b) using Lemma 1,

(b) $\iff$ (c) by means of Lemma 2. □

Remark 1. Let the assertion (a) of Theorem 1 be fulfilled. A global transformation converting equation (1) into an equation (2) with constant coefficients and discrete deviations there is the composition of transformations

$$y(x) = f(x)v(x)\,, \qquad \frac{f'(x)}{f(x)} = q(x)\,, \tag{29}$$

$$x = \varphi(t) \iff t = \varphi^{-1}(x) = h(x) := \int \frac{1}{f(x)} a_r(x) b_r\big(f(x)\big) \prod_{j=1}^{m} \delta_{ij}\big(f\big(\xi_j(x)\big)\big)\,\mathrm{d}x, \tag{30}$$

a_r being an arbitrary coefficient of (1),

$$z(t) = v\big(\varphi(t)\big) = v(x)\,, \quad \eta_j(t) = \varphi^{-1}\big(\xi_j\big(\varphi(t)\big)\big)\,, \qquad t \in \mathbb{R}\,. \tag{31}$$

Constructions of the above transformations are obtained in the proof of Lemma 3.

EXAMPLE. The equation

$$\begin{aligned} y'(x) = {} & \frac{1}{x}\exp\{\mu x\} y(x)^{\alpha} |y(x/2)|^{\beta} |y(8x)|^{\gamma} \operatorname{sign} y(8x) \\ & -\frac{3}{x}\exp\{\tau x\}\big(\operatorname{sign} y(x)\big) y(x/2) + y(x)\,, \end{aligned} \tag{32}$$

$x \in I = (0,\infty)$; $\alpha,\beta,\gamma,\mu,\tau \in \mathbb{R} - \{0\}$ is an equation (1). Here $n = m = 2$,

$$\begin{array}{llll} a_1(x) = \frac{1}{x}\exp\{\mu x\}\,, & b_1(y) = y^{\alpha}\,, & \delta_{11}(y) = |y|^{\beta}\,, & \delta_{12}(y) = |y|^{\gamma} \operatorname{sign} y\,; \\ a_2(x) = -\frac{3}{x}\exp\{\tau x\}\,, & b_2(y) = \operatorname{sign} y\,, & \delta_{21}(y) = y\,, & \delta_{22}(y) = 1\,; \end{array}$$
$$q(x) \equiv 1 \qquad \text{on} \quad I\,.$$

The deviation $\xi_1(x) = x/2$ ($\xi_2(x) = 8x$) is a C^1 diffeomorphism of $I = (0,\infty)$ onto I, $\xi_1'(x) > 0$ ($\xi_2'(x) > 0$) on I. Moreover, $\xi_1(x) \neq x$, $\xi_2(x) \neq x$, $\xi_1(x) \neq \xi_2(x)$ on I and $\lim\limits_{x\to 0_+} \xi_i(x) = 0$, $\lim\limits_{x\to\infty} \xi_i(x) = \infty$ for $i = 1,2$. The assumptions given for equation (1) are fulfilled. We see that functions ξ_1, ξ_2 commute.

Solve equations

$$L_k(x) = \frac{f(x)}{f\big(\xi_k(x)\big)} \qquad \text{where} \quad \frac{f'(x)}{f(x)} = q(x)\,,$$

i.e.

$$\frac{L_k'(x)}{L_k(x)} = q(x) - q\big(\xi_k(x)\big)\xi_k'(x)\,,$$

$$a_i\big(\xi_k(x)\big)\xi_k'(x)L_k(x) = a_i(x)b_i\big(L_k(x)\big)\prod_{j=1}^{m}\delta_{ij}\big(L_k\big(\xi_j(x)\big)\big)\,; \tag{33}$$

$x \in I$, $k = 1,2$; $i = 1,2$. We get

$$\frac{f'(x)}{f(x)} = q(x) \equiv 1 \iff f(x) = c\exp\{x\}\,, \quad c \in \mathbb{R} - \{0\}\,,$$

and

$$\left.\begin{aligned} L_1(x) &= \frac{f(x)}{f(x/2)} = \frac{c\exp\{x\}}{c\exp\{x/2\}} = \exp\{x/2\}\,,\\ L_2(x) &= \frac{f(x)}{f(8x)} = \exp\{-7x\}\,, \end{aligned}\right. \qquad \text{on} \quad I\,.$$

Relations (33) are equivalent to

$$\tau = 1/2\,, \qquad 2\alpha + \beta + 2\mu + 16\gamma - 2 = 0\,. \tag{34}$$

Hence, equation (32) is equivalent to an equation (2) with constant coefficients and discrete deviations if and only if (34) is satisfied.

Transformations

$$\left.\begin{aligned} z(x) &= L_1(x)y\big(\xi_1(x)\big) = \exp\{x/2\}y(x/2)\,,\\ z(x) &= L_2(x)y\big(\xi_2(x)\big) = \exp\{-7x\}y(8x)\,, \end{aligned}\right. \qquad x \in I\,,$$

are stationary transformations of equation (32) if and only if (34) holds, in accordance with (b) of Theorem 1.

Using Remark 1 and (34), the transformation

$$y(x) = f(x)v(x) = c\exp\{x\}v(x) = \exp\{x\}v(x) \text{for } c = 1$$

converts (32) into

$$v'(x) = \frac{1}{x}v(x)^{\alpha}|v(x/2)|^{\beta}|v(8x)|^{\gamma}\operatorname{sign} v(8x) - \frac{3}{x}\big(\operatorname{sign} v(x)\big)v(x/2)\,, \qquad x \in I\,.$$

We have (for example)

$$\begin{aligned} & x = \varphi(t)\\ \iff & t = \varphi^{-1}(x) = h(x) = \int \frac{1}{f(x)}a_1(x)b_1\big(f(x)\big)\delta_{11}\big(f\big(\xi_1(x)\big)\big)\delta_{12}\big(f\big(\xi_2(x)\big)\big)\,\mathrm{d}x\\ &= \int \frac{1}{x}\exp\big\{(2\alpha+\beta+2\mu+16\gamma-2)x/2\big\}\,\mathrm{d}x = \int\frac{1}{x}\,\mathrm{d}x = \ln x + c_1\,,\\ &\qquad c_1 \in \mathbb{R}, \quad x \in I\,, \end{aligned}$$

and choosing $c_1 = 1$ (for simplification) we get

$$x = \varphi(t) = \exp\{t\} \iff t = \varphi^{-1}(x) = h(x) = \ln x .$$

The transformation

$$z(t) = v(\varphi(t)) = v(\exp\{t\}), \qquad \eta_j(t) = \varphi^{-1}(\xi_j(\varphi(t))), \quad j = 1, 2,$$

i.e.

$$\eta_1(t) = t - \ln 2, \qquad \eta_2(t) = t + \ln 8 = t + 3\ln 2,$$

converts equation (35) into the equation

$$z'(t) = z(t)^\alpha |z(t-\ln 2)|^\beta |z(t+3\ln 2)|^\gamma \operatorname{sign} z(t+3\ln 2) - 3(\operatorname{sign}\ z(t))z(t-\ln 2),$$
$$t \in \mathbb{R},$$

$$2\alpha + \beta + 2\mu + 16\gamma - 2 = 0,$$

with constant coefficients and discrete deviations.

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