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TOPOLOGICAL ENTROPY AND VARIATION FOR TRANSITIVE MAPS

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ABSTRACT. We study the continuous functions which map a compact real interval back into itself. We investigate the relations between two important concepts of the dynamical systems and real analysis for transitive functions, topological entropy and variation.

0. Introduction

This paper is concerned with investigation of relations between the topological entropy and variation for transitive maps.

Topological entropy, denoted $\mathrm{ent}(\cdot)$, is a numerical conjugacy invariant of continuous maps.

Variation of a function f on the interval I, denoted $\mathrm{Var}(f, I)$, is a length of a way of a point $f(x)$ if a point x goes through the interval I.

A continuous map is *transitive* if some point has a dense orbit.

Let $I = [0,1]$ be the closed unit interval and $C(I, I)$ be the set of all continuous functions which map the interval I back into itself.

MAIN THEOREM. *Let (x, y) be a pair of numbers. Then there exists a transitive function $f \in C(I, I)$ such that $(x, y) = \big(\mathrm{Var}(f, I), \mathrm{ent}(f)\big)$ if and only if*

$$(x,y) \in \Big((1,\infty] \times \big(\log\sqrt{2}, \infty]\Big) \cup \Big((1,2) \times \big\{\log\sqrt{2}\,\big\}\Big).$$

In section 1 we give the definitions and some known results. In section 2 we define three transitive maps with prescribed topological entropy and in section 3 we show that all of the cases for a pair of numbers $\big(\mathrm{Var}(f, I), \mathrm{ent}(f)\big)$ from Main Theorem are possible. In section 4 we prove that, up to conjugacy, there

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is only one transitive function $Q \in C(I,I)$ such that $\operatorname{ent}(Q) = \log\sqrt{2}$. The proof of the Main Theorem is a straightforward combination of Lemma 1.2 (6), Lemma 1.3, Lemma 3.1, Lemma 3.2 and Corollary 4.2.

1. Background

Let $I = [0,1]$ be the closed unit interval and $C(I,I)$ be the set of all continuous functions which map the interval I back into itself. For $f \in C(I,I)$ we define f^n inductively by $f^0(x) = x$ and (for $n \geq 1$) $f^n(x) = f(f^{n-1}(x))$. f^n is called the n-th iterate of f. For $x \in I$ the orbit of x under f is $\{f^n(x)\}_{n=0}^{\infty}$. A point x is said to be periodic with period n if $f^n(x) = x$ and $f^i(x) \neq x$ for $0 < i < n$. A fixed point is a periodic point with period 1 and $\operatorname{Fix}(f)$ is the set of all fixed points of f. A map f is called piecewise monotone if there exist $N \geq 0$ and $0 = d_0 < d_1 < \cdots < d_{N+1} = 1$ such that f is strictly monotone on $[d_k, d_{k+1}]$ for each $k = 0, \dots, N$. If $f \in C(I,I)$ is piecewise monotone, then a point $w \in (0,1)$ is called a turning point of f if f is not monotone in any neighbourhood of w. The critical points of f are the turning points of f and the endpoints of the interval I. A set $X \subset I$ is an invariant set under f if $f(X) \subset X$.

If $\mathfrak{P} = \{p_1 < \cdots < p_n\}$ is a finite subset of I, let $f_{\mathfrak{P}}$ be the map defined on $[p_1, p_n]$ which agrees with f on $\mathfrak{P}$ and which is linear on $[p_i, p_{i+1}]$ $(i = 1, \dots, n-1)$.

A map $f \in C(I,I)$ is transitive if the only closed invariant subset of I with non-empty interior is I itself.

Let variation of the $f \in C(I,I)$ at the interval $[a,b]$ be

$$\operatorname{Var}\big(f,\, [a,b]\big) = \sup\left\{\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|\,,\quad a = x_0 < x_1 < \cdots < x_n = b\right\},$$

$$\operatorname{Var}\Big(f, \bigcup_{i\in K} I_i\Big) = \sum_{i\in K} \operatorname{Var}(f, I_i)\,, \qquad \text{where} \quad I_i \cap I_j = \emptyset \quad \text{for} \quad i \neq j\,.$$

In particular, if f is piecewise monotone with critical points $0 = d_0 < d_1 < \cdots < d_N < d_{N+1} = 1$, then clearly $\operatorname{Var}(f, I) = \sum\limits_{k=0}^{N} |f(d_{k+1}) - f(d_k)|$.

A map $f \in C(I,I)$ is Lipschitz with constant $L > 0$ if for every pair $x \neq y \in I$ we have $\left|\dfrac{f(x) - f(y)}{x - y}\right| < L$.

The basic equivalence relation of dynamical systems is topological conjugacy. Two functions $f, g \in C(I,I)$ are topologically conjugate if there is a homeomorphism $h \in C(I,I)$ such that $g = h \circ f \circ h^{-1}$. The topological entropy denoted $\operatorname{ent}(\cdot)$ is a conjugacy invariant of continuous maps.

Now we recall the definition of topological entropy and some useful facts.

DEFINITION 1.1. Let $f\colon M \to M$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $S(f,n,\varepsilon) \subset M$ be the set with maximal possible number of points with property that for every $x,y \in S(f,n,\varepsilon)$, $x \neq y$ there is $i \in \{0,1,\dots,n\}$ such that $|f^i(x) - f^i(y)| > \varepsilon$. Then

$$\operatorname{ent}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card} S(f,n,\varepsilon).$$

LEMMA 1.2. *Let $f,g \in C(I,I)$. Then the following conditions hold.*

(1) *If f is a Lipschitz with constant L, then* $\operatorname{ent}(f) \leq \log L$;
(2) $\operatorname{ent}(f^n) = n \operatorname{ent}(f)$ $(n \geq 0)$;
(3) *if $f(J) \subset J$ and $J \subset I$ is closed, then* $\operatorname{ent}(f) \geq \operatorname{ent}\left(f|_J\right)$;
(4) *if f,g are topologically conjugate, then* $\operatorname{ent}(f) = \operatorname{ent}(g)$;
(5) *if f,g are topologically conjugate and f is transitive, then g is transitive;*
(6) *if f is a transitive, then* $\operatorname{Var}(f,I) > 1$.

LEMMA 1.3. (A. M. Blokh [4]) *Let $f \in C(I,I)$ be a transitive map. Then* $\operatorname{ent}(f) \geq \log\sqrt{2}$.

LEMMA 1.4. (L. Block, J. Guckenheimer, M. Misiurewicz, L. S. Young [3]) *Let $f \in C(I,I)$. If a function f has a periodic orbit of period $n = 2^m p$, $p > 1$ is odd, then $\operatorname{ent}(f) \geq \frac{1}{2^m}\log\lambda_p$, where λ_p is the unique positive root of the polynomial $\lambda^p - 2\lambda^{p-2} - 1$. It is easy to verify that for any odd $p > 1$ we have $\lambda_p > \sqrt{2}$.*

LEMMA 1.5. (M. Barge, J. Martin [1]) *If $f \in C(I,I)$ is transitive but not so f^2, then there exist the intervals J_f, K_f such that $I = J_f \cup K_f$, $J_f \cap K_f = \{p_f\}$, $f(J_f) = K_f$, $f(K_f) = J_f$ and $f^2|_{J_f}$, $f^2|_{K_f}$ are transitive on J_f, K_f respectively.*

LEMMA 1.6. (M. Barge, J. Martin [2]) *Let $f \in C(I,I)$. If f^2 is a transitive map, then f has a point of odd period greater than 1.*

LEMMA 1.7. (E. M. Coven, M. C. Hidalgo [5]) *Let $f \in C(I,I)$ be a transitive map. If f is not piecewise monotone and has at least two fixed points, then* $\operatorname{ent}(f) > \log 2$.

LEMMA 1.8. (E. M. Coven, M. C. Hidalgo [5]) *Let $f \in C(I,I)$ be a transitive map and let $\mathfrak{P}$ be a finite invariant set. Then $\operatorname{ent}(f) \geq \operatorname{ent}(f_{\mathfrak{P}})$, with equality if and only if f is piecewise monotone and $\mathfrak{P}$ contains the critical points of f.*

Lemma 1.9. (W. Parry [7]) *Let* $f \in C(I,I)$ *be a piecewise monotone transitive function with* $\operatorname{ent}(f) = \log\alpha$. *Then* f *is topologically conjugate to a piecewise linear function whose linear pieces have slopes* $\pm\alpha$.

Lemma 1.10. (M. Misiurewicz, W. Szlenk [6]) *Let* $f \in C(I,I)$ *be a piecewise monotone map. Then* $\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Var}(f^n, I) = \operatorname{ent}(f)$.

2. Maps with prescribed entropy

Entropy equal to $\log\sqrt{2}$.

Let

$$Q(x) = x\sqrt{2} + 2 - \sqrt{2} \qquad \text{if} \quad x \in \left[0, 1 - \tfrac{\sqrt{2}}{2}\right],$$
$$Q(x) = \sqrt{2} - x\sqrt{2} \qquad \text{if} \quad x \in \left[1 - \tfrac{\sqrt{2}}{2}, 1\right].$$

Remark. Q is a unimodal map with a constant slope $\pm\sqrt{2}$ (see Figure 1).

Proposition 2.1. *Function* Q *has the following properties:*

(i) $Q \in C(I,I)$,
(ii) $\operatorname{Var}(Q,I) = \sqrt{2}$,
(iii) Q *is transitive,*
(iv) $\operatorname{ent}(Q) = \log\sqrt{2}$.

Proof. The properties (i), (ii) are clear from definition of Q and (iv) follows from Lemma 1.10. Hence it suffices to show that for any interval $J \subset I$ there is a number $n \in \mathbb{N}$ such that $\left[0, 1 - \frac{\sqrt{2}}{2}\right] \subset Q^n(J)$. We have two possible cases. Either $|Q^2(J)| = 2|J|$ or $\{0,1\} \cap Q^2(J) \neq \emptyset$. Hence there is $m \in \mathbb{N}$ such that $\{0,1\} \cap Q^{2m}(J) \neq \emptyset$. Then the fixed point $\dfrac{\sqrt{2}}{\sqrt{2}+1} \in Q^{2m+2}(J)$ and we are easily done. □

Finite entropy greater than $\log\sqrt{2}$.

Let $\alpha > \sqrt{2}$. We will define a map F with infinitely many pieces where it is linear with a slope $\pm\alpha$. Moreover, F will have only one limit point of the critical points and any critical point will be mapped after finite time into a periodic point (see Figure 2).

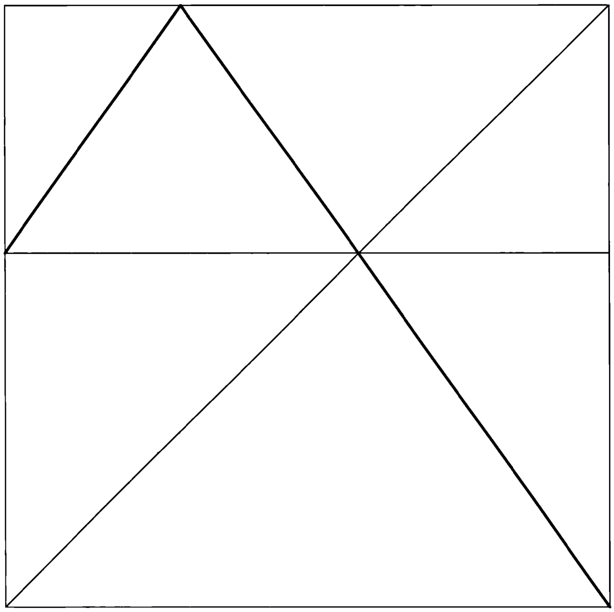

Figure 1. Function Q.

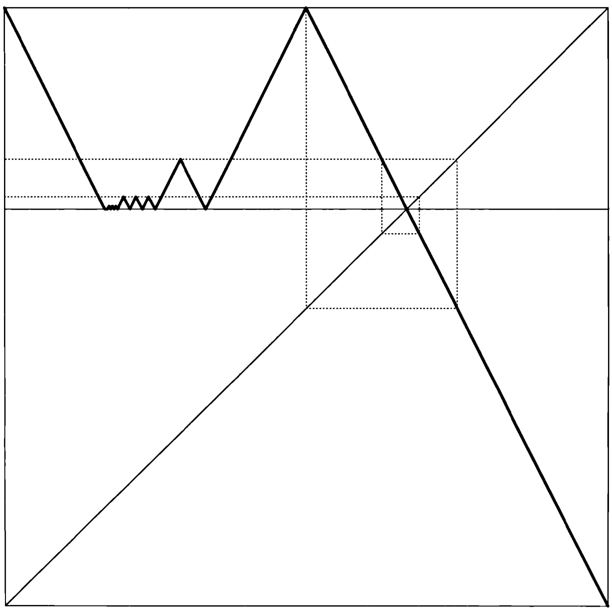

Figure 2. Function F for $\alpha = 2$.

Here is a formal definition of F. Let

$$\begin{aligned} x_0 &= 1\,, & F(x_0) &= 0\,,\\ x_1 &= \frac{\alpha}{\alpha+1} - \frac{1}{\alpha(\alpha+1)}\,, & F(x_1) &= 1\,,\\ x_2 &= \frac{\alpha}{\alpha+1} - \frac{2}{\alpha(\alpha+1)}\,, & F(x_2) &= \frac{\alpha}{\alpha+1}\,. \end{aligned}$$

We have that $x_2 > 0$. Let $n_\alpha \in \{0,1,2,\dots\}$ be the smallest one such that $\frac{1}{\alpha(\alpha+1)\alpha^{2n_\alpha}} < x_2$. Let $F(0) = \sum\limits_{i=0}^{2n_\alpha} \frac{(-1)^i}{\alpha^i}$, $x_\infty = \frac{1}{\alpha(\alpha+1)\alpha^{2n_\alpha}}$ and $F(x_\infty) = \frac{\alpha}{\alpha+1}$.

Now let $p = 1$ and we will define the set $\{x_i\}_{i=3}^{\infty}$ and $F(x_i)$ using induction from $n = 1$.

If $x_{2n} - \frac{2p}{\alpha(\alpha+1)} \le x_\infty$, then we set $p = \frac{p}{\alpha^2}$, else let

$$\begin{aligned} x_{2n+1} &= x_{2n} - \frac{p}{\alpha(\alpha+1)}\,, & F(x_{2n+1}) &= \frac{\alpha}{\alpha+1} + \frac{p}{\alpha+1}\,,\\ x_{2n+2} &= x_{2n} - \frac{2p}{\alpha(\alpha+1)}\,, & F(x_{2n+2}) &= \frac{\alpha}{\alpha+1}\,. \end{aligned}$$

Finally let F be linear on the complementary intervals to the points $\{0, x_\infty\} \cup \{x_i\}_{i=0}^{\infty}$.

PROPOSITION 2.2. *Function F has the following properties:*

(i) $F \in C(I,I)$;

(ii) $x_i > x_{i+1}$ *for* $i \ge 0$, *and* $\lim\limits_{i\to\infty} x_i = x_\infty$;

(iii) F *has slopes* $\pm\alpha$ *on* $[0, x_\infty]$ *and* $[x_{i+1}, x_i]$ *for* $i \ge 0$;

(iv) $\mathrm{Var}(F, I) = \alpha$;

(v) *the orbit* $\{0, F(0), F^2(0), \dots, F^{2n_\alpha+1}(0)\}$ *is periodic with period* $2n_\alpha + 2$;

(vi) *for every* x_i, *either there is a* $k_i \in \mathbb{N}$ *such that* $F^{k_i}(x_i) = 0$ (i *is odd*), *or* $F(x_i) = \frac{\alpha}{\alpha+1}$ *and* $\frac{\alpha}{\alpha+1}$ *is a fixed point;*

(vii) F *is transitive but not so* F^2;

(viii) $\mathrm{ent}(F) = \log\alpha$.

P r o o f . The properties (i) – (vi) are clear from the construction of the function F. Since $F\big(\big[0, \frac{\alpha}{\alpha+1}\big]\big) = \big[\frac{\alpha}{\alpha+1}, 1\big]$ and $F^2\big(\big[0, \frac{\alpha}{\alpha+1}\big]\big) = \big[0, \frac{\alpha}{\alpha+1}\big]$, F^2 cannot be transitive. In order to prove a transitivity of F let us show that for each interval $J \subset I$ there is $n > 0$ such that $F^n(J) \supset \big[0, \frac{\alpha}{\alpha+1}\big]$. It is clear if

$J \cap \left(\{x_\infty, \frac{\alpha}{\alpha+1}\} \cup \{x_{2i}\}_{i=1}^\infty\right) \neq \emptyset$. Assume that $J \cap \left(\{x_\infty, \frac{\alpha}{\alpha+1}\} \cup \{x_{2i}\}_{i=1}^\infty\right) = \emptyset$ and $J \subset \left[0, \frac{\alpha}{\alpha+1}\right]$. Then the definition of F implies that $F^2(J) \subset \left[0, \frac{\alpha}{\alpha+1}\right]$ and $|F^2(J)| \geq \dfrac{|J|\alpha^2}{2}$. Hence $F^{2m}(J) \cap \left(\{x_\infty, \frac{\alpha}{\alpha+1}\} \cup \{x_{2i}\}_{i=1}^\infty\right) \neq \emptyset$ for some $m > 0$ and (vii) is proved.

Finally, it thus only remains to show (viii). By (i), (iii) and Lemma 1.2 (1) we have that $\operatorname{ent}(F) \leq \log \alpha$. Denote $\mathfrak{P}_n = \{0, x_0, x_1, \dots, x_{n-1}, x_n\}$. By (v) and (vi) there is an $m \in \mathbb{N}$ such that the finite set $\widetilde{\mathfrak{P}}_n = \bigcup_{i=0}^{m} F^i(\mathfrak{P}_n)$ is F-invariant. Since $\widetilde{\mathfrak{P}}_n \setminus \mathfrak{P}_n \subset [x_1, x_0]$ we have that $F_{\widetilde{\mathfrak{P}}_n} = F_{\mathfrak{P}_n}$. By Lemma 1.8 $\operatorname{ent}(F) > \operatorname{ent}(F_{\mathfrak{P}_n})$. Hence it is sufficient to show that $\lim_{n\to\infty} \operatorname{ent}(F_{\mathfrak{P}_n}) = \log \alpha$. However, by (iii) the slope of $F_{\mathfrak{P}_n}$ on each interval $[x_i, x_{i-1}]$, $i \in \{1, 2, \dots, n\}$, is equal to $\pm\alpha$ and if we denote β_n the slope of $F_{\mathfrak{P}_n}$ on $[0, x_n]$, then from (ii) we have that $\lim_{n\to\infty} \beta_n = -\alpha$. Now using Lemma 1.10 one can show that $\operatorname{ent}(F_{\mathfrak{P}_n}) \geq \log |\beta_n|$ and the proof is finished. □

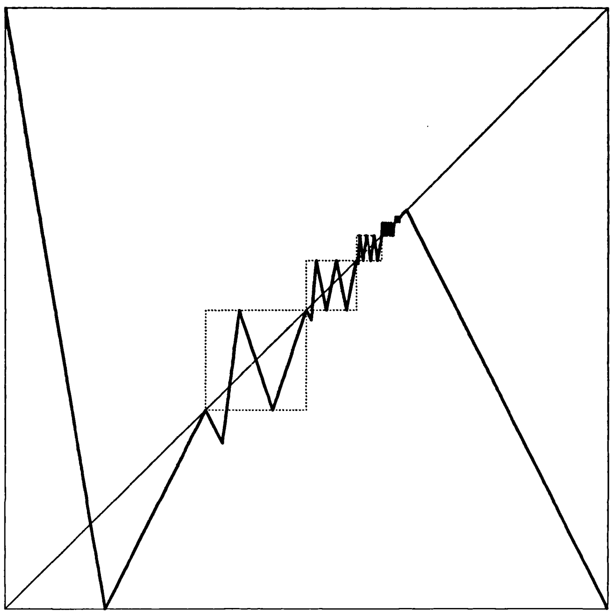

Figure 3. Function G.

Infinite entropy.

Let $x_0 = \frac{1}{3}$, $x_i = \frac{2}{3} - \frac{1}{3 \cdot 2^i}$ for $i = 1, 2, \dots$ and $f_i\Big(\frac{1}{2(2i+1)}\Big) = -1$, $f_i\Big(\frac{2j}{2i+1}\Big) = 0$ and $f_i\Big(\frac{2j+1}{2i+1}\Big) = 1$ for $j = 0, 1, \dots, i$. Now let f_i be linear on the complementary intervals to this points and $h_i(x) = x(x_i - x_{i-1}) + x_{i-1}$.

Now let $G(0) = 1$, $G(\frac{1}{6}) = 0$, $G(\frac{1}{3}) = \frac{1}{3}$, $G(\frac{2}{3}) = \frac{2}{3}$, $G(1) = 0$, G be linear on the intervals $[0, \frac{1}{6}]$, $[\frac{1}{6}, \frac{1}{3}]$, $[\frac{2}{3}, 1]$ and for $x \in [x_{i-1}, x_i]$ let $G(x) = h_i\big(f_i\big(h_i^{-1}(x)\big)\big)$ $(i = 1, 2, \dots)$ (see Figure 3).

PROPOSITION 2.3. *Function G has the following properties:*

(i) $G \in C(I, I)$,
(ii) $\mathrm{Var}(G, I) < \infty$,
(iii) G *is transitive,*
(iv) $\mathrm{ent}(G) = \infty$.

Proof. The property (i) is clear from the construction of G and for the variation we have

$$\begin{aligned}\mathrm{Var}(G, I) &= \mathrm{Var}\left(G, [0, \tfrac{1}{3}]\right) + \mathrm{Var}\left(G, [\tfrac{2}{3}, 1]\right) + \sum_{i=1}^{\infty} \mathrm{Var}\left(G, [x_{i-1}, x_i]\right) \\ &= 2 + \sum_{i=1}^{\infty} \frac{2i+3}{3 \cdot 2^i} < \infty .\end{aligned}$$

Now we will prove (iii). Let $J \subset I$ be an interval. We will show that there is $m \in \mathbb{N}$ such that $G^m(J) = I$. If $J \cap \mathrm{Fix}(G) \cap (x_i, x_{i+1}) = \emptyset$ for all $i \geq 0$, then $|G(J)| > \frac{3|J|}{2}$. So we can assume that $J \cap \mathrm{Fix}(G) \cap (x_{i_0}, x_{i_0+1}) \neq \emptyset$ for some $i_0 \geq 0$. Now, if $x_0 \notin G(J)$, then $|G(J)| > \frac{3|J|}{2}$. Hence we can simply assume that $x_0 \in G(J)$. Now it is easy to see that $G^{2i_0+3}(J) = I$.

Finally, it thus only remains to show (iv). Denote

$$\mathfrak{P}_n = \left\{ x_{n-1} + \frac{(x_n - x_{n-1})j}{2n+1} \right\}_{j=0}^{2n+1} .$$

Now by Lemma 1.8 we have that $\mathrm{ent}(G) > \mathrm{ent}(G_{\mathfrak{P}_n})$ and by Lemma 1.10 we have that $\mathrm{ent}(G_{\mathfrak{P}_n}) = \log(2n+1)$. So we have $\mathrm{ent}(G) = \infty$. □

3. How to get any variation

Let $f, h \in C(I,I)$ and h be a homeomorphism. Then we have that $\mathrm{Var}(h \circ f \circ h^{-1}, I) = \mathrm{Var}(h \circ f, I)$. (It is the same to count variation through all dissections $\{d_i\}_{i=0}^n$ or through all dissections $\{h^{-1}(d_i)\}_{i=0}^n$.) If we compare the graphs of f and $h \circ f$, then we can see that homeomorphism h in the second case only vertically deforms graph of f. We will use this mechanism to change variation of the functions given above to get all the possible variations.

LEMMA 3.1. *For any $K \in (1,2)$ there is a transitive map $f \in C(I,I)$ such that $\mathrm{Var}(f,I) = K$ and $\mathrm{ent}(f) = \log\sqrt{2}$.*

Proof. Let $h \in C(I,I)$ be a homeomorphism such that $h\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right) = 2 - K$ and let $f = h \circ Q \circ h^{-1}$. Then we have $\mathrm{Var}(f,I) = K$ and the rest by Lemma 1.2 (4)(5) and Proposition 2.1. □

LEMMA 3.2. *Let $(K,\alpha) \in (1,\infty] \times (\sqrt{2},\infty]$. Then there is a transitive function $f \in C(I,I)$ such that $(K, \log\alpha) = (\mathrm{Var}(f,I), \mathrm{ent}(f))$.*

Proof. Let $H = F$ if α is finite or $H = G$ if $\alpha = \infty$. The map H is such that we can divide I into the sequence of closed disjoint (except boundary points) intervals $\{J_i\}_{i=1}^\infty$ such that $H^{-1}(J_i)$ is a union of n_i closed disjoint (except boundary points) intervals, each of them mapped homeomorphically by H onto J_i. Moreover $n_1 = 1$ and $\lim\limits_{i\to\infty} n_i = \infty$. Then for every sequence $\{a_i\}_{i=1}^\infty$ of positive numbers with $\sum\limits_{i=1}^{\infty} a_i = 1$ there is a homeomorphism $h \in C(I,I)$ such that $|h(J_i)| = a_i$. Let $f = h \circ H \circ h^{-1}$. We have $\mathrm{Var}(f,I) = \mathrm{Var}(h \circ H, I) = \sum\limits_{i=1}^{\infty} n_i a_i$. Clearly, with the suitable choice of the sequence $\{a_i\}_{i=1}^\infty$ we can get $\sum\limits_{i=1}^{\infty} n_i a_i = K$. Lemma 1.2 (4)(5), Proposition 2.2 and Proposition 2.3 complete the proof. □

4. Maps with entropy equal to $\log\sqrt{2}$

THEOREM 4.1. *Let f be transitive with $\mathrm{ent}(f) = \log\sqrt{2}$. Then f is topologically conjugate to the function Q.*

Proof. Assume that $f \in C(I,I)$ is transitive and $\mathrm{ent}(f) = \log\sqrt{2}$. Then by Lemma 1.4 and Lemma 1.6 we have that f^2 is not transitive. Hence by Lemma 1.5 there exist the intervals J_f, K_f such that $I = J_f \cup K_f$, $J_f \cap K_f$

$= \{p_f\}$, $f(J_f) = K_f$, $f(K_f) = J_f$ and $f^2|_{J_f}$, $f^2|_{K_f}$ are transitive on J_f, K_f respectively. Now we distinguish two cases:

Case I. Suppose that f is piecewise monotone. Then by Lemma 1.9 f is topologically conjugate to a piecewise linear transitive function $q \in C(I,I)$ whose linear pieces have slopes $\pm\sqrt{2}$. Note there is a fixed point $p_q \in I$ such that $q([0,p_q]) = [p_q,1]$ and $q([p_q,1]) = [0,p_q]$. From transitivity we have that there is a point $a \neq p_q$ such that $q(a) = p_q$. We can assume that $a \in [0,p_q]$ (if not we can use topological conjugacy by $h(x) = 1 - x$). Now let $J = \left[a, p_q + \frac{(p_q-a)\sqrt{2}}{2}\right] \cap I$. It is easy to see that $q(J) \subset J$ and from transitivity we have that $a = 0$ and $p_q = \dfrac{\sqrt{2}}{1+\sqrt{2}}$. And finally, because q is piecewise linear with slopes $\pm\sqrt{2}$, and $q([0,p_q]) = [p_q,1]$, $q([p_q,1]) = [0,p_q]$, it is very easy to see that $q = Q$.

Case II. Now let f be not piecewise monotone. Set $g = f^2|_{J_f}$. Then g is transitive not piecewise monotone and $g(p_f) = p_f$. So g has at least two fixed points in J_f (transitivity). Thus by Lemma 1.7 we have $\mathrm{ent}(g) > \log 2$ and by Lemma 1.2 (2)(3)(4) we have $\mathrm{ent}(f) > \log\sqrt{2}$ which is a contradiction. □

COROLLARY 4.2. *Let* $f \in C(I,I)$ *be a transitive function and* $\mathrm{ent}(f) = \log\sqrt{2}$. *Then* $\mathrm{Var}(f,I) \in (1,2)$.

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