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Zygfryd Kominek

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A FEW REMARKS ON ALMOST C-POLYNOMIAL FUNCTIONS

Zygfryd Kominek

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ABSTRACT. We give some sufficient conditions for a function transforming a commutative semigroup to a commutative group to be a polynomial function. Some stability results are also given.

Introduction

Let $(X,+)$ be a commutative semigroup and let $(Y,+)$ be a commutative group. If $f\colon X \to Y$ is a function and $h \in X$, then we define the *difference operator* Δ_h in the following way

$$\Delta_h f(x) := f(x+h) - f(x)\,, \qquad x \in X\,.$$

The superposition of several operators $\Delta_{h_1}, \dots, \Delta_{h_p}$ will be denoted briefly by

$$\Delta_{h_1,\dots,h_p} := \Delta_{h_1} \cdots \Delta_{h_p}\,, \qquad p = 1,2,\dots\,.$$

If $h_1 = \cdots = h_p = h$, we will write Δ_h^p instead of $\Delta_{h_1,\dots,h_p}$. It is well known ([4], for example) that if $f,g\colon X \to Y$, $u,v,x \in X$, then

$$\Delta_{u,v} = \Delta_{v,u}\,, \quad \Delta_{-u} f(x) = -\Delta_u f(x-u)\,, \quad \Delta_u (f+g) = \Delta_u f + \Delta_u g\,.$$

A function $f\colon X \to Y$ is called *strongly polynomial function of pth order* if and only if

$$\Delta_{h_1,\dots,h_{p+1}} f(x) = 0 \tag{1}$$

for all $x, h_1, \dots, h_{p+1} \in X$. If we assume that condition (1) holds for all $x, h \in X$ and $h_1 = h_2 = \cdots = h_{p+1} = h$, i.e.

$$\Delta_h^{p+1} f(x) = 0\,, \tag{2}$$

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then f is said to be a *polynomial function of pth order*. Let C be a subset of X. A function $f\colon X \to Y$ is called *strongly C-polynomial function of pth order* if and only if condition (1) is satisfied for every $x \in X$ and all $h_1, \dots, h_{p+1} \in C$. Analogously, f is said to be *C-polynomial function of pth order* if and only if condition (2) is satisfied for every $x \in X$ and each $h \in C$.

It follows from Djokovič's theorem ([2; Corollary 1], also [5]) that if Y has the property

$$(\forall y)\Big([y \in Y \wedge ((p+1)!)y = 0] \implies y = 0\Big),$$

then $f\colon X \to Y$ is a polynomial function of pth order if and only if it is strongly polynomial function of pth order as well. We say that $f\colon X \to Y$ is a *polynomial of pth order* if there exist a constant a_0 and symmetric i-additive functions $a_i\colon X^i \to Y$, $i = 1, \dots, p$ (i.e. additive in each variable) such that $f(x) = a_0 + \sum_{i=1}^{p} a_i(x, \dots, x)$, $x \in X$.

1. C-polynomial functions

In [3] it is proven that if X and Y are uniquely divisible by $(p+1)!$, $C - C = X$, $C + C \subset C$ and $\frac{1}{(p+1)!}C \subset C$, then every C-polynomial function of pth order is a polynomial of pth order. In this part of the paper, we will obtain some other results of this type. We start with the following lemma.

Lemma 1. *Let X be a commutative semigroup and let Y be a commutative group. If $f\colon X \to Y$ is a function, then for arbitrary $x, h_j^i \in X$, $j = 1, 2, \dots, p$, $i = 0, 1$, we have*

$$\Delta_{h_1^0+h_1^1, \dots, h_p^0+h_p^1} f(x) = \sum_{\varepsilon_1, \dots, \varepsilon_p = 0}^{1} \Delta_{h_1^{\varepsilon_1}, \dots, h_p^{\varepsilon_p}} f\Big(x + \sum_{k=1}^{p} (1 - \varepsilon_k) h_k^1\Big). \tag{3}$$

If, moreover, X is a group, then

$$\Delta_{h_1^0-h_1^1, \dots, h_p^0-h_p^1} f(x) = \sum_{\varepsilon_1, \dots, \varepsilon_p = 0}^{1} (-1)^{\varepsilon_1+\dots+\varepsilon_p} \Delta_{h_1^{\varepsilon_1}, \dots, h_p^{\varepsilon_p}} f\Big(x - \sum_{k=1}^{p} h_k^1\Big). \tag{4}$$

Proof. Induction. As an example, we give a proof of equality (4). For $p = 1$ we have

$$\begin{aligned}
\Delta_{h_1^0-h_1^1} f(x) &= f(x + h_1^0 - h_1^1) - f(x - h_1^1) + f(x - h_1^1) - f(x) \\
&= \Delta_{h_1^0} f(x - h_1^1) + \Delta_{-h_1^1} f(x) = \Delta_{h_1^0} f(x - h_1^1) - \Delta_{h_1^1} f(x - h_1^1) \\
&= \sum_{\varepsilon_1=0}^{1} (-1)^{\varepsilon_1} \Delta_{h_1^{\varepsilon_1}} f(x - h_1^1).
\end{aligned}$$

Assume (4) and take arbitrary $x, h_j^i \in X$, $j = 1, \dots, p{+}1$, $i = 0, 1$. Then,

$$\begin{aligned}
&\Delta_{h_1^0 - h_1^1, \dots, h_{p+1}^0 - h_{p+1}^1} f(x) \\
&= \Delta_{h_1^0 - h_1^1, \dots, h_p^0 - h_p^1} \left(\sum_{\varepsilon_{p+1}=0}^{1} (-1)^{\varepsilon_{p+1}} \Delta_{h_{p+1}^{\varepsilon_{p+1}}} f\big(x - h_{p+1}^1\big) \right) \\
&= \sum_{\varepsilon_{p+1}=0}^{1} (-1)^{\varepsilon_{p+1}} \Delta_{h_{p+1}^{\varepsilon_{p+1}}} \sum_{\varepsilon_1, \dots, \varepsilon_p = 0}^{1} (-1)^{\varepsilon_1 + \dots + \varepsilon_p} \Delta_{h_1^{\varepsilon_1}, \dots, h_p^{\varepsilon_p}} f\Big(x - h_{p+1}^1 - \sum_{k=1}^{p} h_k^1\Big) \\
&= \sum_{\varepsilon_1, \dots, \varepsilon_{p+1} = 0}^{1} \Delta_{h_1^{\varepsilon_1}, \dots, h_{p+1}^{\varepsilon_{p+1}}} f\Big(x - \sum_{k=1}^{p+1} h_k^1\Big),
\end{aligned}$$

which ends the proof. □

The next two lemmas are consequences of Lemma 1.

Lemma 2. *Let X be a commutative semigroup and let Y be a commutative group. If $C \subset X$ satisfies the condition*

$$C + C = X\,, \tag{5}$$

then every strongly C-polynomial function of pth order $f\colon X \to Y$ is strongly polynomial of pth order.

Proof. Fix $x, h_1, \dots, h_{p+1} \in X$. According to (5), there exist $h_j^i \in C$, $j = 1, \dots, p{+}1$, $i = 0, 1$, such that $h_j = h_j^0 + h_j^1$, $j = 1, \dots, p{+}1$. By virtue of (3) of Lemma 1 and our assumption we obtain

$$\begin{aligned}
\Delta_{h_1, \dots, h_{p+1}} f(x) &= \Delta_{h_1^0 + h_1^1, \dots, h_{p+1}^0 + h_{p+1}^1} f(x) \\
&= \sum_{\varepsilon_1, \dots, \varepsilon_{p+1} = 0}^{1} \Delta_{h_1^{\varepsilon_1}, \dots, h_{p+1}^{\varepsilon_{p+1}}} f\Big(x + \sum_{k=1}^{p+1} (1 - \varepsilon_k) h_k^1\Big) = 0\,,
\end{aligned}$$

which finishes the proof. □

In a similar way one can prove the following lemma.

Lemma 3. *Let X and Y be commutative groups. If $C \subset X$ satisfies the condition*

$$C - C = X\,, \tag{6}$$

then every strongly C-polynomial function of pth order $f\colon X \to Y$ is strongly polynomial of pth order.

Let m be a fixed positive integer. We say that a group X has a $(m{-}C)$*-property* if and only if each element $h \in X$ has a representation $h = \sum_{i=1}^{m} h_i$, where

$h_i \in C \cup (-C)$, $i = 1, \dots, m$. Note that if $f \colon X \to Y$ is a strongly C-polynomial function of pth order, then it is also strongly $\big(C \cup (-C)\big)$-polynomial function of pth order.

THEOREM 1. *Let X and Y be commutative groups. If X has the $(m-C)$-property with some positive integer m, then every strongly C-polynomial function of pth order $f \colon X \to Y$ is strongly polynomial of pth order.*

Proof. Fix $x, h_1, \dots, h_{p+1} \in X$. There exist a positive integer m and $h_{j,k} \in C \cup (-C)$, $j = 1, \dots, p{+}1$, $k = 1, \dots, m$, such that $h_j = \sum_{k=1}^{m} h_{j,k}$. Thus

$$\begin{aligned}
&\Delta_{h_1,\dots,h_{p+1}} f(x) \\
&= \Delta_{\sum_{k=1}^{m} h_{1,k},\dots,\sum_{k=1}^{m} h_{p+1,k}} f(x) \\
&= \Delta_{\sum_{k=1}^{m} h_{1,k},\dots,\sum_{k=1}^{m} h_{p,k}} \sum_{j_{p+1}=1}^{m} \Big[f\Big(x + \sum_{k=1}^{j_{p+1}} h_{p+1,k}\Big) - f\Big(x + \sum_{k=1}^{j_{p+1}-1} h_{p+1,k}\Big)\Big] \\
&= \Delta_{\sum_{k=1}^{m} h_{1,k},\dots,\sum_{k=1}^{m} h_{p,k}} \sum_{j_{p+1}=1}^{m} \Delta_{h_{p+1,j_{p+1}}} f\Big(x + \sum_{k=1}^{j_{p+1}-1} h_{p+1,k}\Big) \\
&\;\vdots \\
&= \sum_{j_1=1}^{m} \cdots \sum_{j_{p+1}=1}^{m} \Delta_{h_{1,j_1},\dots,h_{p+1,j_{p+1}}} f\Big(x + \sum_{k=1}^{j_1-1} h_{1,k} + \dots + \sum_{k=1}^{j_{p+1}-1} h_{p+1,k}\Big) = 0\,,
\end{aligned}$$

because f is strongly $\big(C \cup (-C)\big)$-polynomial function of pth order. This ends the proof. □

2. Stability in the sense of Ulam and Hyers

Assume X is a commutative semigroup and Y is a real Banach space. Let us fix $\varepsilon \ge 0$ and let $f \colon X \to Y$ be a function. We are interested in solutions to the inequalities

$$\|\Delta_{h_1,\dots,h_{p+1}} f(x)\| \le \varepsilon\,, \qquad x \in X\,,\ \ h_1, \dots, h_{p+1} \in C\,, \tag{7}$$

and

$$\|\Delta_h^{p+1} f(x)\| \le \varepsilon\,, \qquad x \in X\,,\ \ h \in C\,, \tag{8}$$

where C is a subset of X. In the case of $C = X$, the problem was considered by many authors. In particular, M. Albert and J. A. Baker [1] have proved the following theorem.

THEOREM A-B. *Let X be a commutative semigroup with zero and let Y be a real Banach space. If $f\colon X \to Y$ satisfies condition (7) with $C = X$, then there exists a unique (up to an additive constant) polynomial $g\colon X \to Y$ of pth order such that*

$$\|f(x) - g(x)\| \le \varepsilon\,, \qquad x \in X\,.$$

The first theorem in this section reads as follows.

THEOREM 2. *Let X be a commutative semigroup with zero and let Y be a real Banach space. If $f\colon X \to Y$ satisfies condition (7) where $C \subset X$ satisfies one of conditions (5) or (6), then there exists a unique (up to an additive constant) polynomial $g\colon X \to Y$ of pth order such that*

$$\|f(x) - g(x)\| \le 2^{p+1}\varepsilon\,, \qquad x \in X\,.$$

Proof. Assume (5) (if (6) is satisfied, then the proof is similar). Let $x, h_1, \dots, h_{p+1} \in X$ be arbitrary fixed. According to (5), there exist $h_j^i \in C$, $j = 1, \dots, p{+}1$, $i = 0, 1$, such that $h_j = h_j^0 + h_j^1$, $j = 1, \dots, p{+}1$. By Lemma 1 and (7) we get

$$\|\Delta_{h_1,\dots,h_{p+1}} f(x)\| \le \sum_{\varepsilon_1,\dots,\varepsilon_{p+1}=0}^{1} \Big\|\Delta_{h_1^{\varepsilon_1},\dots,h_{p+1}^{\varepsilon_{p+1}}} f\Big(x + \sum_{k=1}^{p+1} (1-\varepsilon_k) h_k^1\Big)\Big\| \le 2^{p+1}\varepsilon\,.$$

Our assertion follows now from Theorem A-B. □

J. H. B. Kemperman ([4; p. 369]) noticed that if X is a commutative group admitting division by $(p+1)!$, then we can express values of the operator $\Delta_{h_1,\dots,h_{p+1}}$ as linear combinations of iterates of the $(p+1)$th order of difference operators depending only on one span. More precisely, if $x, h_1, \dots, h_{p+1} \in X$ and $f\colon X \to Y$ is a function, then

$$\Delta_{h_1,\dots,h_{p+1}} f(x) = \sum_{\varepsilon_1,\dots,\varepsilon_{p+1}=0}^{1} (-1)^{\varepsilon_1+\dots+\varepsilon_{p+1}} \Delta^{p+1}_{h'_{\varepsilon_1,\dots,\varepsilon_{p+1}}} f\big(x + h''_{\varepsilon_1,\dots,\varepsilon_{p+1}}\big)\,,$$

where

$$h'_{\varepsilon_1,\dots,\varepsilon_{p+1}} = -\sum_{j=1}^{p+1} \frac{\varepsilon_j}{j} h_j\,,$$

and

$$h''_{\varepsilon_1,\dots,\varepsilon_{p+1}} = \sum_{j=1}^{p+1} \varepsilon_j h_j\,.$$

The next theorem refers to inequality (8).

THEOREM 3. *Let X be a commutative group admitting division by $(p+1)!$, let Y be a real Banach space. Assume $\frac{1}{(p+1)!}C \subset C$, $C + C \subset C$ and (6). If $f\colon X \to Y$ satisfies condition (8), then there exists a unique (up to an additive constant) polynomial $g\colon X \to Y$ of pth order such that*

$$\|f(x) - g(x)\| \le 4^{p+1}\varepsilon, \qquad x \in X.$$

P r o o f. Fix arbitrary $x, h_1, \dots, h_{p+1} \in X$. There exist $h_j^i \in C$, $j = 1, \dots, p+1$, $i = 0, 1$, such that $h_j = h_j^0 - h_j^1$, $j = 1, \dots, p+1$. For arbitrary $\varepsilon_j, \delta_j \in \{0, 1\}$, $j = 1, \dots, p+1$, let us define

$$h_{\delta_1,\dots,\delta_{p+1}}^{\varepsilon_1,\dots,\varepsilon_{p+1}} := \sum_{j=1}^{p+1} \delta_j \frac{h_j^{\varepsilon_j}}{j},$$

$$z_{\delta_1,\dots,\delta_{p+1}}^{\varepsilon_1,\dots,\varepsilon_{p+1}} := x + \sum_{j=1}^{p+1} (1 - \varepsilon_j) h_j^1 + \sum_{j=1}^{p+1} \delta_j h_j^{\varepsilon_j} + h_{\delta_1,\dots,\delta_{p+1}}^{\varepsilon_1,\dots,\varepsilon_{p+1}}.$$

According to Lemma 1 we obtain

$$\begin{aligned}
&\Delta_{h_1,\dots,h_{p+1}} f(x) \\
&= \Delta_{h_1^0 - h_1^1,\dots,h_{p+1}^0 - h_{p+1}^1} f(x) \\
&= \sum_{\varepsilon_1,\dots,\varepsilon_{p+1}=0}^{1} (-1)^{\varepsilon_1+\dots+\varepsilon_{p+1}} \Delta_{h_1^{\varepsilon_1},\dots,h_{p+1}^{\varepsilon_{p+1}}} f\Big(x - \sum_{j=1}^{p+1} (1 - \varepsilon_j) h_j^1\Big) \\
&= -\sum_{\varepsilon_1,\dots,\varepsilon_{p+1}=0}^{1} (-1)^{\varepsilon_1+\dots+\varepsilon_{p+1}} \sum_{\delta_1,\dots,\delta_{p+1}=0}^{1} (-1)^{\delta_1+\dots+\delta_{p+1}} \Delta^{p+1}_{h_{\delta_1,\dots,\delta_{p+1}}^{\varepsilon_1,\dots,\varepsilon_{p+1}}} f\big(z_{\delta_1,\dots,\delta_{p+1}}^{\varepsilon_1,\dots,\varepsilon_{p+1}}\big).
\end{aligned}$$

Hence

$$\|\Delta_{h_1,\dots,h_{p+1}} f(x)\| \le 4^{p+1} \big\|\Delta^{p+1}_{h_{\delta_1,\dots,\delta_{p+1}}^{\varepsilon_1,\dots,\varepsilon_{p+1}}} f\big(z_{\delta_1,\dots,\delta_{p+1}}^{\varepsilon_1,\dots,\varepsilon_{p+1}}\big)\big\|,$$

which together with (8) implies that

$$\|\Delta_{h_1,\dots,h_{p+1}} f(x)\| \le 4^{p+1}\varepsilon.$$

Now our assertion follows from Theorem A-B. □

As a final remark note that we are able to repeat the argumentation used in the proof of Theorem 3 to obtain the following theorem.

THEOREM 4. *Let X be a commutative group admitting division by $(p+1)!$ and let Y be a commutative group. If C is a subset of X such that*

$$\tfrac{1}{(p+1)!}C \subset C\,, \qquad C + C \subset C \qquad \text{and} \qquad C - C = X\,,$$

then every C-polynomial function of pth order is a strongly polynomial function of pth order.

Remark. Recall that ([2; Theorem 3]) if, moreover, Y is a commutative group such that for every $y \in Y$

equation $(p!)x = y$ has a unique solution $x = \frac{y}{p!}$,

then every polynomial function $f\colon X \to Y$ of pth order is a polynomial of pth order, too.

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Institute of Mathematics
Silesian University
Bankowa 14
PL-40-007 Katowice
POLAND

E-mail: zkominek@ux2.math.us.edu.pl