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DIRECT PRODUCT DECOMPOSITIONS OF PSEUDO EFFECT ALGEBRAS

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ABSTRACT. In this paper we deal with internal direct product decompositions of a pseudo effect algebra satisfying a certain interpolation property. This property was investigated by Dvurečenskij and Vetterlein.

1. Introduction

Pseudo effect algebras were introduced and studied by Dvurečenskij and Vetterlein [2], [3], [4].

Some interpolation properties for pseudo effect algebras were dealt with in the mentioned papers. It was shown that each pseudo effect algebra $\mathcal{A}$ satisfying the interpolation property RDP_1 can be represented as the interval $[0,u]$ of some partially ordered group G with a strong unit u (for detailed definitions cf. Section 2 below). The notation $\mathcal{A} = \Gamma(G,u)$ is applied in this situation. The analogous notation has been used for MV-algebras; cf. Cignoli, D'Ottaviano and Mundici [1].

We denote by $\mathcal{D}$ the class of all pseudo MV-algebras satisfying the interpolation property RDP_1.

Let $\mathcal{A} \in \mathcal{D}$. Similarly as in the case of groups (cf. e.g., Kurosh [12]) we introduce the notion of an internal direct product decomposition of $\mathcal{A}$; we apply the notation $\mathcal{A} = (\mathrm{int}) \prod\limits_{i\in I} \mathcal{A}_i$, or $\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ if the set I is finite (in this case we speak about a finite internal direct product decomposition). For the notion of an internal direct product decomposition of an ordered group cf. [8], [9]. Analogously we can introduce this notion for partially ordered sets having the least element.

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Let G be as above, i.e., $\mathcal{A} = \Gamma(G, u)$. The case $u = 0$ being trivial for our purposes, we assume that $u > 0$. Then without loss of generality it suffices to suppose that all $\mathcal{A}_i$ (under the notation as above) are non-zero.

We investigate the relations between internal direct product decompositions of $\mathcal{A}$ and those of $\ell(\mathcal{A})$, where $\ell(\mathcal{A})$ is the underlying partially ordered set of $\mathcal{A}$. We generalize some results on direct product decompositions of MV-algebras and pseudo MV-algebras; cf. [10], [11]. We prove that there exists a one-to-one correspondence between finite internal direct product decompositions of $\mathcal{A}$ and internal direct product decomposition of G. (In general, $\mathcal{A}$ can have infinite internal direct product decompositions; on the other hand, each internal direct product decomposition of G is finite.)

Namely, let $\mathrm{ID}_{\mathrm{f}}(\mathcal{A})$ be the set of all finite internal direct product decompositions of $\mathcal{A}$ and let $\mathrm{ID}(G)$ be the set of all internal direct product decomposition of G. Let $\alpha \in \mathrm{ID}_{\mathrm{f}}(\mathcal{A})$, where α has the form

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \cdots \times \mathcal{A}_n . \qquad (\alpha)$$

For $i \in I$, let u_i be the greatest element of $\mathcal{A}_i$. Put

$$G_i = \bigcup_{n\in\mathbb{N}} [-nu_i, nu_i] .$$

Then G_i is an ℓ-subgroup of G; moreover, we have

$$G = (\mathrm{int})G_1 \times \cdots \times G_n . \qquad (\beta)$$

The mapping $\alpha \mapsto \beta$ is a bijection between $\mathrm{ID}_{\mathrm{f}}(\mathcal{A})$ and $\mathrm{ID}(G)$.

For the notion of pseudo MV-algebra, cf. Georgescu and Iorgulescu [5], [6] and Rachůnek [13] (in [13], the term "generalized MV-algebra" was applied).

2. Preliminaries

An element u of a partially ordered group G is a *strong unit* of G if for each $g \in G$ there exists a positive integer n such that $g \leqq nu$. A partially ordered group with a fixed strong unit is called *unital*.

A partial algebra $\mathcal{A} = (A; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants is called a *pseudo effect algebra* if for all $a, b, c \in A$ the following conditions are satisfied (cf. [2]):

(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$;

(ii) there is exactly one $d \in A$ and exactly one $e \in A$ such that $a+d = e+a = 1$;

(iii) if $a+b$ exists, then there are elements $d, e \in A$ such that $a+b = d+a = b+e$;

(iv) if $1+a$ or $a+1$ exists, then $a = 0$.

We put $a \leqq b$ if and only if there exists $c \in A$ such that $a+c = b$. Then $\leqq$ is a relation of a partial order on A and $0 \leqq a \leqq 1$ for each $a \in A$. Also, $a \leqq b$ if and only if there exists $d \in A$ with $d + a = b$.

In what follows, we always consider $\mathcal{A}$ as a partially ordered algebraic structure with the partial order $\leqq$ defined as above; i.e., we have $\mathcal{A} = (A; +, 0, 1, \leqq)$.

For the further definitions and for the results formulated in the present section cf. [2], [3], [4].

The group operation in a partially ordered group will be written additively, the commutativity of this operation is not assumed.

Let G be a partially ordered group and let $0 < u \in G$. Let A be the interval $[0, u]$ of G. Consider the partial binary operation $+$ on A which is defined by restricting the group operation $+$ on the set A. Put

$$\Gamma(G, u) = (A; +, 0, u).$$

Then $\Gamma(G, u)$ is a pseudo effect algebra.

We will deal with the following condition for a pseudo effect algebra $\mathcal{A}$:

(RDP_1) For any $a_1, a_2, b_1, b_2 \in A$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in A$ such that

(i) $d_1 + d_2 = a_1$, $d_2 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_3 + d_4 = b_2$;

(ii) for each $d_2', d_3' \in A$ with $d_2' \leqq d_2$, $d_3' \leqq d_3$ we have $d_2' + d_3' = d_3' + d_2'$.

THEOREM 2.1. (Cf. [3].) *Let $\mathcal{A}$ be a pseudo effect algebra satisfying the condition* (RDP_1). *Then there exists a partially ordered group G with a strong unit u such that $\mathcal{A}$ is isomorphic to $\Gamma(G, u)$. Moreover, the unital partially ordered group (G, u) is determined uniquely, up to isomorphisms.*

Therefore, when dealing with a pseudo effect algebra satisfying (RDP_1) we can assume without loss of generality, that $A = \Gamma(G, u)$ for some unital partially ordered group (G, u).

THEOREM 2.2. (Cf. [4].) *Let $\mathcal{A}$ be as in* 2.1. *Then $\mathcal{A}$ satisfies the following conditions:*

(RDP_0) *for every $a, b_1, b_2 \in A$ with $a \leqq b_1 + b_2$ there are $d_1, d_2 \in A$ such that $d_1 \leqq b_1$, $d_2 \leqq b_2$ and $a = d_1 + d_2$;*

(RIP) *for any $a_1, a_2, b_1, b_2 \in A$ with $a_1, a_2 \leqq b_1, b_2$ there is $c \in A$ such that $a_1, a_2 \leqq c \leqq b_1, b_2$.*

3. Direct product decompositions

In this section there is introduced the notion of internal direct product decomposition of a pseudo effect algebra.

Let I be a nonempty set of indices and for each $i \in I$ let $\mathcal{A}_i = (A_i; +, 0_i, 1_i)$ be a pseudo effect algebra. Let S be the set of all indexed systems $x = (x_i)_{i\in I}$ where $x_i \in A_i$ for each $i \in I$. Assume that $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$ are elements of S. If $a_i + b_i$ exists for each $i \in I$, then we put $a + b = (a_i + b_i)_{i\in I}$; otherwise, $a + b$ is not defined in S. Further, we put

$$1 = (1_i)_{i\in I}, \qquad 0 = (0_i)_{i\in I}.$$

It is clear that the algebraic structure $(S; +, 0, 1)$ is a pseudo effect algebra; we denote

$$(S; +, 0, 1) = \prod_{i\in I} \mathcal{A}_i$$

and we call this algebraic structure a *direct product of the system* $(\mathcal{A}_i)_{i\in I}$. If $I = \{1, 2, \dots, n\}$, then we write also $\mathcal{A}_1 \times \dots \times \mathcal{A}_n$.

Let $j \in I$. Further, let S_j be the set of all elements $a = (a_i)_{i\in I}$ of S such that $a_i = 0$ whenever $i \neq j$. We denote by 1^j the element of S_j such that $(1^j)_j = 1_j$.

If $a, b \in S_j$ and if $a + b$ exists in $(S; +, 0, 1)$, then clearly $a + b$ belongs to S_j. Hence the algebraic structure

$$\overline{\mathcal{A}}_j = (S_j, +, 0, 1^j)$$

is a pseudo effect algebra.

For $a_j \in A_j$ let $\overline{a^j}$ be the element of S_j such that $\left(\overline{a^j}\right)_j = a_j$. Then the mapping $\varphi_j \colon S_j \to A_j$ defined by $\varphi_j\left(\overline{a^j}\right) = a_j$ is an isomorphism of $\overline{\mathcal{A}}_j$ onto $\mathcal{A}_j$.

Let $\mathcal{A}$ be a pseudo effect algebra and let

$$\varphi \colon \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i \tag{1}$$

be an isomorphism. Then we say that φ *determines a direct product decomposition of* $\mathcal{A}$. For $a \in A$ and $i \in I$ we put $\big(\varphi(a)\big)_i = a_i$ and we say that a_i is the *component* of a in $\mathcal{A}_i$.

Further, for each $a \in A$ we set $\overline{\varphi}(a) = \left(\overline{a^i}\right)_{i\in I}$. In view of the upper mentioned properties of φ_j $(j \in I)$ we conclude that the mapping

$$\overline{\varphi} : \mathcal{A} \to \prod_{i\in I} \overline{\mathcal{A}}_i \tag{2}$$

also determines a direct product decomposition of $\mathcal{A}$; this direct product decomposition is called *internal.*

Recall that for each $i \in I$, the underlying set of $\overline{\mathcal{A}}_i$ (i.e., the set S_i) is a subset of A. Hence internal direct product decompositions of $\mathcal{A}$ form a set; on the other hand, there is a proper class of direct product decompositions of $\mathcal{A}$.

In view of (1) and (2), to each direct product decomposition φ of $\mathcal{A}$ there corresponds an internal direct product decomposition $\overline{\varphi}$ of $\mathcal{A}$ such that, up to isomorphism, $\overline{\varphi}$ does not differ from φ.

If (2) is valid, then we write

$$\mathcal{A} = (\mathrm{int}) \prod_{i \in I} \overline{\mathcal{A}}_i . \tag{3}$$

In view of the above definitions, we conclude that if (1) is valid, then φ determines an internal direct product decomposition of $\mathcal{A}$ if and only if the following conditions are satisfied:

(i) for each $i \in I$, the underlying partially ordered set $\ell(\mathcal{A}_i)$ of $\mathcal{A}_i$ is an interval of $\ell(\mathcal{A})$;

(ii) if $i \in I$ and $x \in \ell(\mathcal{A}_i)$, then $x_i = x$ and $x_j = 0$ for each $j \in I$, $j \neq i$.

In the same way we introduce the notion of internal direct product decomposition of a partially ordered group G, of the partially ordered semigroup G^+ or of a lattice L possessing the least element 0. We omit the detailed definitions.

If (3) holds, then the pseudo effect algebras $\overline{\mathcal{A}}_i$ are called internal direct factors of $\mathcal{A}$.

Assume that

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \mathcal{A}_2$$

and that this internal direct decomposition is determined by an isomorphism φ. Let $a \in A$ and $\varphi(a) = (a_1, a_2)$. Then

$$\varphi(a_1) = (a_1, 0), \qquad \varphi(a_2) = (0, a_2).$$

In the direct product $\mathcal{A}_1 \times \mathcal{A}_2$ we have

$$(a_1, a_2) = (a_1, 0) + (a_2, 0).$$

Hence in view of the isomorphism φ, the relation $a = a_1 + a_2$ is valid in $\mathcal{A}$. By induction we obtain:

Assume that

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \cdots \times \mathcal{A}_n ;$$

for $a \in A$ and $i \in \{1, 2, \ldots, n\}$ let a_i be the component of a in $\mathcal{A}_i$. Then $a = a_1 + \cdots + a_n$.

Let L be a directed partially ordered set with the least element 0. If $x, y, p, q \in L$ such that p is the infimum of $\{x, y\}$, then we write $x \wedge y = p$; the meaning of $x \vee y = q$ is analogous.

Lemma 3.1. *Assume that* $L = (\mathrm{int})L_1 \times L_2$.

(i) *If* $x \in L_1$ *and* $y \in L_2$, *then* $x \wedge y = 0$.

(ii) *Let* $z \in L$ *and let* z_i *be the component of* z *in* L_i $(i = 1, 2)$. *Then* $z_1 \vee z_2 = z$.

P r o o f . Assume that the internal direct product decomposition under consideration is determined by an isomorphism φ.

Let $x \in L_1$ and $y \in L_2$. Then $\varphi(x) = (x, 0)$, $\varphi(y) = (0, y)$. If $z \in L$, $z \leqq x$, $z \leqq y$, $\varphi(z) = (z_1, z_2)$, we obtain $\varphi(z) = (0, 0) = 0$. Thus $x \wedge y = 0$.

Let $z \in L$, $\varphi(z) = (z_1, z_2)$. Since $\varphi(z_1) = (z_1, 0)$, $\varphi(z_2) = (0, z_2)$, we get $z_1, z_2 \leqq z$. Let $t \in L$, $z_1 \leqq t$ and $z_2 \leqq t$, $\varphi(t) = (t_1, t_2)$. We get $z_1 \leqq t_1$ and $z_2 \leqq t_2$, yielding $z \leqq t$. Hence $z_1 \vee z_2 = z$. □

4. Relations between internal direct product decompositions of $\mathcal{A}$ and $\ell(\mathcal{A})$

In this section we assume that $\mathcal{A}$ is a pseudo effect algebra belonging to the class $\mathcal{D}$. Hence, without loss of generality we can suppose that $\mathcal{A} = \Gamma(G, u)$ for some unital partially ordered group (G, u). It suffices to deal with the case $G \neq \{0\}$, i.e., $u > 0$.

A directed group is called a *Riesz group* if it satisfies the condition analogous to the condition (RIP) from 2.2.

Lemma 4.1. *The partially ordered group* G *is a Riesz group.*

P r o o f . Since G possesses a strong unit it is directed. From $\mathcal{A} \in \mathcal{D}$ and from 2.2 it follows that $\mathcal{A}$ satisfies the condition (RIP). Therefore in view of [4] we conclude that G is a Riesz group. □

Let $u_1 \in A$, where A is the underlying set of $\mathcal{A}$. We denote by G_1 the convex subgroup of G generated by u_1. Hence $G_1 = \bigcup_{n \in \mathbb{N}} [-nu_1, nu_1]$. The element u_1 is a strong unit of G_1.

Put $A_1 = [0, u_1]$. For $x, y \in A_1$ consider $x + y$ to be defined in A_1 if $x + y \in A_1$. Then $\mathcal{A}_1 = (A_1; +, 0, u_1)$ is a pseudo effect algebra and we have

$$\mathcal{A}_1 = \Gamma(G_1, u_1).$$

Thus $\mathcal{A}_1 \in \mathcal{D}$. We call $\mathcal{A}_1$ an *interval subalgebra of* $\mathcal{A}$ (generated by u_1).

Lemma 4.2. *Let* $\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \mathcal{A}_2$. *Put* $u_i = u(\mathcal{A}_i)$ $(i = 1, 2)$. *Then* $\mathcal{A}_i$ *is an interval subalgebra of* $\mathcal{A}$ *generated by the element* u_i.

P r o o f . This is an immediate consequence of the definition of the internal direct product decomposition and of the relation $\mathcal{A} \in \mathcal{D}$. □

Also, since the partial order in $\mathcal{A}$ is defined by means of the partial operation $+$, we get:

LEMMA 4.3. *Assume that $\mathcal{A} = (\text{int})\mathcal{A}_1 \times \mathcal{A}_2$. Let $\ell(\mathcal{A}_i)$ be the underlying partially ordered set of $\mathcal{A}_i$ $(i = 1, 2)$. Then $\ell(\mathcal{A}) = (\text{int})\ell(\mathcal{A}_1) \times \ell(\mathcal{A}_2)$ and for $z \in A$, $i \in \{1, 2\}$ we have $z(\mathcal{A}_i) = z(\ell(\mathcal{A}_i))$.*

From 4.3 we obtain by induction:

LEMMA 4.3.1. *Assume that $\mathcal{A} = (\text{int})\mathcal{A}_1 \times \dots \times \mathcal{A}_n$. Then $\ell(\mathcal{A}) = (\text{int})(\ell(\mathcal{A}_1) \times \dots \times \ell(\mathcal{A}_n))$. For each $z \in A$ and $i \in \{1, 2, \dots, n\}$, $z(\mathcal{A}_i) = z(\ell(\mathcal{A}_i))$.*

LEMMA 4.4. *Assume that $\ell(\mathcal{A}) = (\text{int})L_1 \times L_2$. Put $u_i = u(L_i)$ $(i = 1, 2)$ and let $\mathcal{A}_i$ be the interval subalgebra of $\mathcal{A}$ generated by u_i. Then*

(i) *$\ell(\mathcal{A}_i) = L_i$ $(i = 1, 2)$;*
(ii) *$\mathcal{A} = (\text{int})\mathcal{A}_1 \times \mathcal{A}_2$;*
(iii) *for each $z \in A$, $z(\mathcal{A}_i) = z(L_i)$ $(i = 1, 2)$.*

Proof.

a) Let $z \in [0, u_1]$. Put $z_1 = z(L_1)$, $z_2 = z(L_2)$. In view of 3.1(ii), $z = z_1 \vee z_2$. Since $z_1 \in L_1$ and $z_2 \in L_2$, from 3.1(i) we obtain $z_2 = z \wedge z_2 = 0$, thus $z = z_1$. Hence $[0, u_1] \subseteq L_1$. Conversely, let $z \in L_1$. Then $z(L_1) = z$. In view of $z \leqq u$ we get $z(L_1) \leqq u(L_1) = u_1$, thus $z \leqq u_1$ and $L_1 \subseteq [0, u_1]$. Therefore $\ell(\mathcal{A}_1) = L_1$. Similarly, $\ell(\mathcal{A}_2) = L_2$.

b) Let $a \in L_1$, $b \in L_2$, $a_1 \in A$, $a_1 \leqq a + b$. We prove that $a_1 \in L_1$. In fact, there exist $a_2, b_2 \in A$ such that $a_2 \leqq a$, $b_2 \leqq b$ and $a_1 = a_2 + b_2$. Then $0 = a_1 \wedge b_2 = b_2$, whence $a_1 \leqq a$. Thus according to a) we have $a_1 \in L_1$.

c) Let $a \in L_1$, $b \in L_2$. Then $a \vee b = a + b$. We verify this assertion as follows. There exists $z \in A$ with $z(L_1) = a$, $z(L_2) = b$. Hence in view of 3.1, $z = a \vee b$; thus $a \vee b$ exists in L. We obviously have $a \vee b \leqq a + b$. Thus there exist $a_1, b_1 \in A$ such that $a_1 \leqq a$, $b_1 \leqq b$ and $a \vee b = a_1 + b_1$. In view of a) we obtain $a_1 \in L_1$, $a_2 \in L_2$. Further, $a \leqq a_1 + b_1$. According to b), $a \leqq a_1$. Thus $a = a_1$. Analogously, $b = b_1$. Therefore $a \vee b = a + b$.

d) Let a and b be as in c). From c) we infer $a + b = b + a$.

e) Let $a, a_1 \in L_1$ and $b, b_1 \in L_2$. Assume that $a + b = a_1 + b_1$. Then $a \leqq a_1 + b_1$, whence b) yields $a \leqq a_1$. Similarly we obtain $a_1 \leqq a$, hence $a_1 = a$. Analogously, $b_1 = b$. In view of 3.1 we conclude that each element $z \in A$ can be uniquely expressed in the form $z = a + b$ with $a \in L_1$, $b \in L_2$.

f) Let $a, a' \in L_1$ and suppose that $a + a'$ exists in A. Then $a + a' \in L_1$.

In fact, in view of e), $a + a'$ can be written in the form $a + a' = a_1 + b$ with $a_1 \in L_1$ and $b \in L_2$. Hence $b \leqq a + a'$. Thus there are $b_1, b_2 \in A$ such that $b = b_1 + b_2$, $b_1 \leqq a$ and $b_2 \leqq a'$. Further, $b_1 = b_1 \wedge a = 0$ and similarly $b_2 = 0$. Thus $b = 0$ and $a + a' = a_1 \in L_1$.

Analogously, if $b, b' \in L_2$ and if $b + b'$ exists in A, then $b + b' \in L_2$.

g) Let $z, z' \in A$. First suppose that $z + z'$ exists in A. We express z and z' as in e); we get $z = a + b$, $z' = a' + b'$. Put $z + z' = t$. Then in view of d),

$$z + z' = a + b + a' + b' = (a + a') + (b + b').$$

According to f), $a + a' \in L_1$ and $b + b' \in L_2$. Hence we have (cf. e))

$$\begin{aligned} z(L_1) &= a, & z'(L_1) &= a', & t(L_1) &= a + a', \\ z(L_2) &= b, & z'(L_2) &= b', & t(L_2) &= b + b'. \end{aligned}$$

Therefore $t(L_1) = z(L_1) + z'(L_1)$, $t(L_2) = z(L_2) + z'(L_2)$.

Secondly, suppose that $z + z'$ does not exist in A. We show that either $a + a'$ or $b + b'$ does not exist in A. By way of contradiction, suppose that both these elements exist in A. In view of f), we have $a + a' \in L_1$ and $b + b' \in L_2$. Thus $(a + a') \vee (b + b')$ exists in A and according to b),

$$(a + a') \vee (b + b') = (a + a') + (b + b').$$

Since $a' + b = b + a'$, we obtain $(a + a') \vee (b + b') = z + z'$. Hence $z + z' \in A$, which is a contradiction.

h) In view of g) we conclude that the mapping $\varphi\colon A \to L_1 \times L_2$ determines an isomorphism of $\mathcal{A}$ onto $\mathcal{A}_1 \times \mathcal{A}_2$.

Further, if $z \in L_1$, then $z_1 = z$ and $z_2 = 0$; similarly, if $z \in L_2$, then $z_2 = z$ and $z_1 = 0$. (We denote by z_i the component of z in L_i, $i = 1, 2$.) Therefore we obtain

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \mathcal{A}_2.$$

In view of the definition of φ, for each $z \in A$ we have $z(\mathcal{A}_i) = z(L_i)$ for $i = 1, 2$. $\square$

From 4.4 we obtain by a straightforward induction:

THEOREM 4.5. *Assume that $\mathcal{A}$ is a pseudo effect algebra belonging to the class $\mathcal{D}$. Let $\ell(\mathcal{A}) = (\mathrm{int})L_1 \times \cdots \times L_n$. Put $u_i = u(L_i)$ and let $\mathcal{A}_i$ be the interval subalgebra of $\mathcal{A}$ generated by u_i $(i = 1, 2, \dots, n)$. Then*

(i) *$\ell(\mathcal{A}_i) = L_i$ for $i \in \{1, 2, \dots, n\}$;*
(ii) *$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$;*
(iii) *for each $z \in A$ and $i \in \{1, 2, \dots, n\}$, $z(\mathcal{A}_i) = z(L_i)$.*

Let $\mathcal{A} \in \mathcal{D}$. In view of 4.1 and 4.5, there is a one-to-one correspondence between finite internal direct product decompositions of $\mathcal{A}$ and those of $\ell(\mathcal{A})$.

Now we will apply to above results for investigating internal direct decompositions which can be infinite. We need some auxiliary results.

Assume that an internal direct decomposition

$$\mathcal{A} = (\mathrm{int}) \prod_{i \in I} \mathcal{A}_i \tag{1}$$

is defined by a mapping φ. For $a \in A$ we put $\varphi(a) = (a_i)_{i \in I}$.

Let I_1 and I_2 be nonempty subsets of I such that $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = I$. Given $a \in A$, let a^1 and a^2 be elements of A such that

$$(a^1)_i = \begin{cases} a_i & \text{if } i \in I_1, \\ 0 & \text{otherwise,} \end{cases} \qquad (a^2)_i = \begin{cases} a_i & \text{if } i \in I_2, \\ 0 & \text{otherwise.} \end{cases}$$

We denote

$$A_1 = \{a^1 : a \in A\}, \qquad A_2 = \{a^2 : a \in A\}.$$

Then u^1 is the greatest element of A_1 and u^2 is the greatest element of A_2. Hence there exists an interval subalgebra $\mathcal{A}^1$ with $\ell(\mathcal{A}^1) = A_1$. The meaning of $\mathcal{A}^2$ is analogous.

For each $a \in A$ we put $\varphi^*(a) = (a^1, a^2)$. By a simple argument we can verify:

LEMMA 4.6. *The mapping φ^* determines an internal direct product decomposition*

$$\mathcal{A} = (\mathrm{int})\mathcal{A}^1 \times \mathcal{A}^2 .$$

For $j \in \{1,2\}$, the partial mapping $\varphi^j = \varphi\big|_{I_j}$ determines an internal direct product decomposition

$$\mathcal{A}^j = (\mathrm{int}) \prod_{k \in I_j} \mathcal{A}_k .$$

LEMMA 4.7. *Let φ be as above and let* (1) *be valid. Then φ determines, at the same time, an internal direct product decomposition*

$$\ell(\mathcal{A}) = (\mathrm{int}) \prod_{i \in I} \ell(\mathcal{A}_i) .$$

Proof. It suffices to apply the same argument as in 4.3.1. □

Further, consider the case when instead of (1) we have the relation

$$\ell(\mathcal{A}) = (\mathrm{int}) \prod_{i \in I} L_i \tag{2}$$

which is defined by a mapping φ with $\varphi(a) = (a_i)_{i \in I}$ for $a \in A$.

Under an analogous notation as above we obtain that A_1 and A_2 are sublattices of $\ell(\mathcal{A})$. Similarly as in 4.6 we have:

LEMMA 4.8. *The mapping φ^* determines an internal direct product decomposition*

$$\ell(\mathcal{A}) = (\mathrm{int})A_1 \times A_2\,;$$

for $j \in \{1,2\}$, the partial mapping $\varphi^j = \varphi\big|_{I_j}$ determines an internal direct product decomposition

$$A_j = (\mathrm{int}) \prod_{k \in I_j} L_k\,.$$

Let i be a fixed element of I. Put $I_1 = \{i\}$, $I_2 = I \setminus I_1$. Suppose that $I_2 \neq \emptyset$. Hence $A_1 = L_i$.

LEMMA 4.9. *Under the assumptions as above, there are internal direct factors $\mathcal{A}_i$ and $\mathcal{A}'_i$ of $\mathcal{A}$ such that*

(i) $\ell(\mathcal{A}_i) = L_i$, $\ell(\mathcal{A}'_i) = A_2$*;*

(ii) $\mathcal{A} = (\mathrm{int})\mathcal{A}_i \times \mathcal{A}'_i$*;*

(iii) *for each $a \in \mathcal{A}$, $a(\mathcal{A}_i) = a_i$ and*

$$\big(a(\mathcal{A}'_i)\big)_j = \begin{cases} 0 & \text{for } j = i\,, \\ a_j & \text{if } j \in I\,,\ j \neq i\,. \end{cases}$$

Proof. This is a consequence of 4.8 and 4.4. □

LEMMA 4.10. *Let* (2) *be valid and let φ be as above. Let $a \in A$. Then $a = \bigvee_{i \in I} a_i$.*

Proof. In view of 4.9 and 3.1 we have

$$a = a(\mathcal{A}_i) \vee a(\mathcal{A}'_i)\,.$$

Since $a(\mathcal{A}_i) = a_i$, $a_i \leqq a$.

Let $t \in A$ and suppose that $a_i \leqq t$ for each $i \in I$. Hence $(a_i)_i \leqq t_i$. In view of (2) we have $(a_i)_i = a_i$, thus $a_i \leqq t_i$ for each $i \in I$. Therefore $a \leqq t$. This yields $a = \bigvee_{i \in I} a_i$. □

LEMMA 4.10.1. *Assume that* (2) *is valid. Let $y \in A$. For each $i \in I$, let $a^i \in L_i$ and let $\bigvee_{i \in I} a^i = y$. Then $y_i = a^i$ for each $i \in I$.*

Proof. In view of 4.10 we have $y = \bigvee_{i \in I} y_i$. Further, according to (2), $a^i \wedge y_{i(1)} = 0$ whenever i and $i(1)$ are distinct elements of I. Hence for each $i \in I$,

$$y_i = y_i \wedge y = y_i \wedge \bigvee_{j \in I} a^j = \bigvee_{j \in I} (y_i \wedge a^j) = y_i \wedge a^i\,,$$

thus $y_i \leqq a^i$. Analogously we obtain $a^i \leqq y_i$. Therefore $y_i = a^i$. □

LEMMA 4.11. *Under the assumptions as in* 4.9, *let* $i(1) \in I$, $i(1) \neq i$ *and* $0 < a \in A_i$, $0 < b \in A_{i(1)}$. *Then* $a \vee b = a + b$.

Proof. From $b \in A_{i(1)}$ it follows $b \in A_2$. Then from 4.9(ii) and from 4.4 c) we obtain $a \vee b = a + b$. □

LEMMA 4.12. *We apply the notation as above. Let* $x, y \in A$. *Then the following conditions are equivalent:*

(i) $x + y$ *exists in* $\mathcal{A}$;
(ii) *for each* $i \in I$, $x_i + y_i$ *exists in* $\mathcal{A}_i$.

Proof. First we remark that if $i \in I$, then in view of 2.9(ii) and of 4.4 f), $x_i + y_i$ exists in $\mathcal{A}$ if and only if it exists in $\mathcal{A}_i$.

Assume that (i) holds. In view of 4.10, $x = \bigvee_{i \in I} x_i$ and $y = \bigvee_{i \in I} y_i$. Let $i \in I$. Hence $x_i \leqq x$ and $y_i \leqq y$. Thus $x_i + y_i \leqq x + y$. Then $x_i + y_i$ exists in $\mathcal{A}$. Therefore $x_i + y_i$ exists in $\mathcal{A}_i$.

Now let (ii) be valid. In G we have

$$x + y = \bigvee_{i \in I} x_i + \bigvee_{j \in I} y_j = \bigvee_{i \in I} \bigvee_{j \in j} (x_i + y_j).$$

If $i \neq j$, then according to 4.11, $x_i + y_j = x_i \vee y_j$, whence

$$x_i + y_j \leqq (x_i + y_i) \vee (x_j + y_j).$$

Therefore

$$\bigvee_{i \in I} \bigvee_{j \in J} (x_i + y_j) = \bigvee_{i \in I} (x_i + y_i).$$

In view of the assumption, $x_i + y_i \in A$ for each $i \in I$. Thus $x_i + y_i \leqq u$ for each $i \in I$ and then $\bigvee_{i \in I} (x_i + y_i) \leqq u$. Hence $\bigvee_{i \in I} (x_i + y_i)$ belongs to A. Therefore (i) is valid. □

LEMMA 4.13. *Let* $x, y \in A$ *and suppose that* $x+y$ *exists in* $\mathcal{A}$. *Then* $(x+y)_i = x_i + y_i$ *for each* $i \in I$.

Proof. The assertion follows from 4.12 and 4.10.1. □

LEMMA 4.14. *Let* (2) *be valid. Then* φ *determines an internal direct product decomposition*

$$\mathcal{A} = (\mathrm{int}) \prod_{i \in I} \mathcal{A}_i.$$

Proof. This is a consequence of 4.12 and 4.13. □

THEOREM 4.15. *Let $\mathcal{A}$ be a pseudo effect algebra belonging to the class $\mathcal{D}$. Let $\{\mathcal{A}_i\}_{i\in I}$ be a system of interval subalgebras of $\mathcal{A}$. Put $L_i = \ell(\mathcal{A}_i)$ for each $i \in I$. Assume that*

$$\varphi\colon A \to \prod_{i\in I} L_i$$

is a bijection. Then the following conditions are equivalent:

(i) *φ determines an internal direct product decomposition*

$$\mathcal{A} = (\mathrm{int}) \prod_{i\in I} \mathcal{A}_i\,;$$

(ii) *φ determines an internal direct product decomposition*

$$\ell(\mathcal{A}) = (\mathrm{int}) \prod_{i\in I} \ell(\mathcal{A}_i)\,.$$

P r o o f. It suffices to apply 4.7 and 4.14. □

This generalizes some results of [10] and [11]. From the result of H a s h i m o t o [7] it follows that any two internal direct product decompositions of the partially ordered set $\ell(\mathcal{A})$ have a common refinement; from this and from 4.15 it can be deduced that any two internal direct product decompositions of a pseudo effect algebra belonging to $\mathcal{D}$ have a common refinement.

5. The pseudo effect algebra $\Gamma(G, 2u)$

Again, we suppose that $\mathcal{A} \in \mathcal{D}$ and that $\mathcal{A} = \Gamma(G,u)$. In this section we investigate the pseudo effect algebra $\mathcal{A}_0 = \Gamma(G,2u)$.

LEMMA 5.1. *We have $\mathcal{A}_0 \in \mathcal{D}$.*

P r o o f. This is a consequence of the relation $\mathcal{A} \in \mathcal{D}$ and of [4; Theorem 2.3]. □

Assume that

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \mathcal{A}_2\,. \tag{1}$$

We will apply the notation as in the previous section. Hence for $z \in \mathcal{A}$ we put $z_i = z(\mathcal{A}_i)$ $(i = 1, 2)$. Thus we have

$$2u = (u_1 + u_2) + (u_1 + u_2) = 2u_1 + 2u_2\,,$$

since $u_2 \wedge u_1 = 0$.

Lemma 5.2. $2u_1 \wedge 2u_2 = 0$.

Proof. By way of contradiction, assume that there exists $0 < t \in G$ with $t \leqq 2u_1$ and $t \leqq 2u_2$. Hence there are $t_1, t_2 \in G$ with $0 \leqq t_i \leqq u_1$ for $i \in \{1,2\}$ such that $t = t_1 + t_2$. Without loss of generality we can suppose that $t_1 > 0$. We have $t_1 \leqq 2u_2$, hence $t_1 = t_{11} + t_{12}$, where $t_{1i} \in G$, $0 \leqq t_{1i} \leqq u_2$ for $i = 1,2$. Again, without loss of generality we can suppose that $t_{11} > 0$. Then $t_{11} \leqq u_1$, $t_{11} \leqq u_2$, which is a contradiction. □

We denote $A_0 = [0, 2u]$.

Lemma 5.3. *Let $z \in A_0$. Then z can be expressed uniquely in the form $z = t_1 + t_2$ with $t_1 \in [0, 2u_1]$, $t_2 \in [0, 2u_2]$.*

Proof. Since $z \leqq 2u$, there are $p, q \in A$ such that $z = p + q$. According to (1) there are $p_1, q_1 \in A_1$, $p_2, q_2 \in A_2$ such that $p = p_1 + p_2$ and $q = q_1 + q_2$. In view of $p_2 \wedge q_1 = 0$ we have $p_2 + q_1 = q_1 + p_2$, whence $z = p_1 + p_2 + q_1 + q_2$. Put $t_1 = p_1 + p_2$, $t_2 = q_1 + q_2$. Then $z = t_1 + t_2$ and $t_1 \in [0, 2u_1]$, $t_2 \in [0, 2u_2]$.

Let $t_1' \in [0, 2u_1]$, $t_2' \in [0, 2u_2]$, $z = t_1' + t_2'$. In view of 5.2 we have $t_1 \wedge t_2 = 0$ and $t_1' \wedge t_2' = 0$. From this we obtain by a simple calculation that the relations $t_1' = t_1$ and $t_2' = t_2$ are valid. □

Put $\mathcal{A}_i^0 = \Gamma(G, 2u_i)$ $(i = 1,2)$. Consider the mapping

$$\varphi_0 \colon \mathcal{A}_0 \to \mathcal{A}_1^0 \times \mathcal{A}_2^0$$

defined by $\varphi_0(z) = (t_1, t_2)$, where t_1 and t_2 are as in 5.3.

Lemma 5.4. *Let $t_1, t_1' \in [0, 2u_1]$. Assume that $t_1 + t_1'$ exists in $\mathcal{A}_0$. Then $t_1 + t_1' \in [0, 2u_1]$. An analogous result holds for $t_2, t_2' \in [0, 2u_2]$.*

Proof. In view of 5.3 there are $p \in [0, 2u_1]$ and $q \in [0, 2u_2]$ with $t_1 + t_1' = p + q$. We have to verify that $q = 0$. By way of contradiction, suppose that $q > 0$. Then $q = r + s$ such that $0 \leqq r \leqq t_1$, $0 \leqq s \leqq t_1'$ and either $r > 0$ or $s > 0$. Let, e.g., $r > 0$. We obtain $r \leqq 2u_1$, $r \leqq 2u_2$. In view of 5.2 we arrived at a contradiction. For $t_2, t_2' \in [0, 2u_2]$, the situation is analogous. □

Lemma 5.5. *Let $z, z' \in A_0$, $\varphi_0(z) = (t_1, t_2)$, $\varphi_0(z') = (t_1', t_2')$. Assume that $z + z'$ exists. Then $t_1 + t_1'$ exists in $\mathcal{A}_1^0$, $t_2 + t_2'$ exists in $\mathcal{A}_2^0$ and*

$$\varphi_0(z + z') = (t_1 + t_1', t_2 + t_2').$$

Proof. We have

$$z = t_1 + t_2, \qquad z' = t_1' + t_2'.$$

Further, $z + z' \leqq 2u = 2u_1 + 2u_2$. Then $t_1 + t_1' \leqq 2u_1 + 2u_2$. Hence there are $v_1, v_2 \in A_0$ such that $t_1 + t_1' = v_1 + v_2$, $v_1 \leqq 2u_1$, $v_2 \leqq 2u_2$. Then $v_2 \leqq t_1 + t_1'$, thus in view of 5.4, $v_2 \leqq 2u_1$. This yields $v_2 \leqq 2u_1 \wedge 2u_2 = 0$ (cf. 5.2), therefore $v_2 = 0$. Thus $t_1 + t_1' \leqq 2u_1$; hence $t_1 + t_1'$ exists in $\mathcal{A}_1^0$. Analogously, $t_2 + t_2'$ exists in $\mathcal{A}_2^0$. Further, since $t_2 \wedge t_1' = 0$ we get $t_2 + t_1' = t_1' + t_2$, whence

$$z + z' = (t_1 + t_1') + (t_2 + t_2') \leqq 2u_1 + 2u_2 = 2u\,.$$

In view of 5.3 and 5.4 we obtain

$$\varphi(z + z') = (t_1 + t_1', t_2 + t_2')\,.$$

□

LEMMA 5.6. *We apply the notation as in* 5.5. *Assume that* $t_1 + t_1'$ *and* $t_2 + t_2'$ *exist in* $\mathcal{A}_0$. *Then* $z + z'$ *exists in* $\mathcal{A}_0$.

P r o o f. We have $t_1, t_1' \in [0, 2u_1]$. Hence in view of 5.4, $t_1 + t_1' \in [0, 2u_1]$. Analogously, $t_2 + t_2' \in [0, 2u_2]$. Then $(t_1 + t_1') + (t_2 + t_2') \leqq 2u_1 + 2u_2 = 2u$. Therefore $(t_1 + t_1') + (t_2 + t_2')$ exists in $\mathcal{A}_0$. Since $t_1' + t_2 = t_2 + t_1'$, we obtain that $z + z'$ exists in $\mathcal{A}_0$. □

From 5.3–5.6 we conclude:

LEMMA 5.7. *The mapping* φ_0 *determines a direct product decomposition of* $\mathcal{A}_0$.

Recall that $\mathcal{A}_1^0$ and $\mathcal{A}_2^0$ are interval subalgebras of $\mathcal{A}_0$. Further, from the definition of φ_0 it follows that if $z \in [0, 2u_1]$, then $\varphi_0(z) = (z, 0)$; similarly, if $z \in [0, 2u_2]$, then $\varphi_0(z) = (0, z)$. Hence we have:

PROPOSITION 5.8. *The mapping* φ_0 *determines an internal direct product decomposition of* $\mathcal{A}_0$; *i.e.,* $\mathcal{A}_0 = (\mathrm{int})\mathcal{A}_1^0 \times \mathcal{A}_2^0$.

In view of definition of φ_0 we also obtain that for each $z \in A$,

$$z(\mathcal{A}_1) = z(\mathcal{A}_1^0)\,, \qquad z(\mathcal{A}_2) = z(\mathcal{A}_2^0)\,.$$

6. Internal direct product decompositions of G

We apply the assumptions and the notation as in the previous section with the exception that instead of $\mathcal{A}_0$ we write $\mathcal{A}^2$; similarly, instead of $\mathcal{A}_1^0$ and $\mathcal{A}_2^0$ we now write $\mathcal{A}_1^2$ or $\mathcal{A}_2^2$, respectively.

Let $n \in \mathbb{N}$ and $m = 2^n$. Put $\mathcal{A}^m = \Gamma(G, mu)$. Further, let $\mathcal{A}_1^m$ and $\mathcal{A}_2^m$ be interval subalgebras of $\mathcal{A}^m$ generated by mu_1 or mu_2, respectively.

By applying 5.8 and the induction we obtain:

PROPOSITION 6.1. *Let m be as above. Then $\mathcal{A}^m = (\mathrm{int})\mathcal{A}_1^m \times \mathcal{A}_2^m$. If $n > 1$ and $m_1 = 2^{n-1}$, then for $z \in [0, m_1 u]$ we have*

$$z(\mathcal{A}_i^m) = z(\mathcal{A}_i^{m_1}) \qquad (i = 1, 2).$$

We denote

$$A_1^* = \bigcup_{n \in \mathbb{N}} [0, 2^n u_1], \qquad A_2^* = \bigcup_{n \in \mathbb{N}} [0, 2^n u_2].$$

LEMMA 6.2. *Both A_1^* and A_2^* are closed with respect to the operation $+$.*

Proof. Let $z, z' \in A_1^*$. Since u is a strong unit of G, there exists $n \in \mathbb{N}$ such that $z + z' \leqq 2^n u$. Put $m = 2^n$ and consider the direct product decomposition of $\mathcal{A}^m$ from 6.1. We have $z(\mathcal{A}_1^m) = z$, $z'(\mathcal{A}_1^m) = z'$, thus $(z + z')(\mathcal{A}_1^m) = z + z'$. Therefore $z + z' \in [0, m u_1] \subseteq A_1^*$. Analogously we verify the assertion concerning A_2^*. □

LEMMA 6.2.1. *For each $z \in G^+$ there exist uniquely determined elements $z_1 \in A_1^*$ and $z_2 \in A_2^*$ such that $z = z_1 + z_2$.*

Proof. There is $n \in \mathbb{N}$ such that for $m = 2^n$ we have $z \leqq mu$. Put $z_1 = z(\mathcal{A}_1^m)$, $z_2 = z(\mathcal{A}_2^m)$. In view of 6.1, $z = z_1 + z_2$. If $z_1' \in A_1^*$, $z_2' \in A_2^*$ and $z = z_1' + z_2'$, then $z_1' + z_2' \leqq mu$. This yields $z_1' \leqq m u_1$ and $z_2' \leqq m u_2$, whence $z_2'(\mathcal{A}_1^m) = 0$ and

$$z_1 = z(\mathcal{A}_1^m) = z_1'(\mathcal{A}_1^m) + z_2'(\mathcal{A}_1^m) = z_1'(\mathcal{A}_1^m) = z_1'.$$

Analogously, $z_2 = z_2'$. □

Under the notation as in 6.2 we put $z_1^* = z_1$ and $z_2^* = z_2$.

LEMMA 6.3. *Let $z, t \in G^+$. Then*

$$(z + t)_i^* = z_i^* + t_i^* \qquad (i = 1, 2).$$

Proof. There is $n \in \mathbb{N}$ such that for $m = 2^n$, both z and t belong to the interval $[0, mu]$; then we have

$$(z + t)(\mathcal{A}_1^m) = z(\mathcal{A}_1^m) + t(\mathcal{A}_1^m).$$

According to the definitions of z_1^*, t_1^* and $(z + t)_1^*$ we obtain

$$(z + t)_1^* = z_1^* + t_1^*,$$

and analogously for $(z + t)_2^*$. □

LEMMA 6.4. *Let $z, t \in G^+$. Then $z \leqq t$ if and only if $z_1^* \leqq t_1^*$ and $z_2^* \leqq t_2^*$.*

P r o o f . Let m be as in the proof of 6.3. The assertion of the lemma is an immediate consequence of the validity of the corresponding assertion concerning the components of z and t in $\mathcal{A}_1^m$ and $\mathcal{A}_2^m$. □

For $z \in G^+$ we denote $\varphi^*(z) = (z_1^*, z_2^*)$. In view of 6.1–6.4 we have:

LEMMA 6.5. *The mapping φ^* determines a direct product decomposition of the partially ordered semigroup G^+ with the direct factors $\mathcal{A}_1^* = (A_1^*; +, \leqq)$ and $\mathcal{A}_2^* = (A_2^*; +, \leqq)$.*

Also, if $z \in A_1^*$, then $\varphi^*(z) = (z, 0)$; similarly if $z \in A_2^*$, then $\varphi^*(z) = (0, z)$. From this and from 6.5 we get:

LEMMA 6.6. $G^+ = (\mathrm{int})\mathcal{A}_1^* \times \mathcal{A}_2^*$.

Now, let us deal with the situation when instead of considering a two factor internal direct product decomposition of $\mathcal{A}$ we consider a relation of the form

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \cdots \times \mathcal{A}_n . \tag{α}$$

For each $i \in \{1, 2, \dots, n\}$ we construct A_i^* and $\mathcal{A}_i^*$ analogously as we did above for A_1^* and $\mathcal{A}_1^*$. By using induction, from 6.6 we obtain

PROPOSITION 6.7. *Assume that (α) is valid. For $i \in \{1, 2, \dots, n\}$ let $\mathcal{A}_1^*$ be as above. Then*

$$G^+ = (\mathrm{int})\mathcal{A}_1^* \times \cdots \times \mathcal{A}_n^*. \tag{α_1}$$

Consider the relation (α_1). For each $i \in \{1, 2, \dots, n\}$ let B_i be the set of all $g \in G$ such that there exists $a_i \in A_i^*$ with $-a_i \leqq g \leqq a_i$. The set B_i is partially ordered by the relation $\leqq$ induced from G. It is easy to verify that B_i is closed with respect to the operation $+$. Moreover, according to a result of S h i m b i r e v a [14] (cf. also the author [9]), the partially ordered structure $\mathcal{B}_i = (B_i; +, \leqq)$ is an internal direct factor of G and we have:

PROPOSITION 6.8. *Under the assumptions as above,*

$$G = (\mathrm{int})\mathcal{B}_1 \times \cdots \times \mathcal{B}_n. \tag{β}$$

Let $\mathrm{ID_f}(\mathcal{A})$ and $\mathrm{ID}(G)$ be as in Section 1. Put

$$\psi_1(\alpha) = \beta .$$

Hence ψ_1 is a mapping of $\mathrm{ID_f}(\mathcal{A})$ into $\mathrm{ID}(G)$.

Now assume that

$$G = (\mathrm{int})\mathcal{B}_1' \times \cdots \times \mathcal{B}_n' \tag{β_1}$$

is any internal direct product decomposition of G. For $i \in \{1,2,\dots,n\}$ let B_i' be the underlying set of $\mathcal{B}_i'$. Suppose that (β_1) is defined by a mapping $\varphi' \colon G \to B_1' \times \dots \times B_n'$. For $z \in G$ we denote

$$\varphi'(z) = (z_1', \dots, z_n').$$

Since (β_1) is internal, whenever $i \in \{1,2,\dots,n\}$ and $z \in B_i'$, then $z_i' = z$ and $z_j' = 0$ for $j \in \{1,2,\dots,n\}$, $j \neq i$.

Under our notation, $\varphi'(u) = (u_1', \dots, u_n')$. Put $A_i' = [0, u_i']$ and let A be as above, i.e., $A = [0,u]$. It is easy to verify that for $z \in G$ we have $z \in A$ if and only if $z_i' \in [0, u_i']$ for $i \in 1,2,\dots,n$. Hence the partial mapping

$$\varphi_1 = \varphi'\big|_A \colon A \to \prod_{i \in I} [0, u_i']$$

is a bijection. We denote by $\mathcal{A}_i'$ the interval subalgebra of $\mathcal{A}$ generated by the element u_i'.

The idea of the proof of the following assertion (+) is the same as in the proof of 4.12.

(+) *Let $z, t \in A$. Then $z + t$ is defined in $\mathcal{A}$ if and only if, for each $i \in \{1,2,\dots,n\}$, $z_i' + t_i'$ is defined in $\mathcal{A}_i'$.*

From the above facts we conclude that the relation

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1' \times \dots \times \mathcal{A}_n' \qquad (\alpha_1)$$

is valid.

Let us denote by $\mathrm{ID_f}(G)$ the set of all finite internal direct product decompositions of G. We put

$$\psi_2(\beta_1) = \alpha_1 .$$

Thus ψ_2 is a mapping of $\mathrm{ID_f}(G)$ into $\mathrm{ID_f}(\mathcal{A})$.

Our aim is to verify that $\mathrm{ID_f}(G) = \mathrm{ID}(G)$ and that $\psi_2 = \psi_1^{-1}$. This will be performed in the following section.

7. Components in an internal direct factor

Let $\mathcal{A}$ and G be as above. Again, let

$$\mathcal{A} = (\mathrm{int})\mathcal{A}_1 \times \dots \times \mathcal{A}_n \qquad (\alpha)$$

be an internal direct product decomposition of $\mathcal{A}$. For $i \in \{1,2,\dots,n\}$, the underlying set of $\mathcal{A}_i$ is denoted by A_i. Suppose that (α) is determined by a mapping φ; for $z \in A$ we have $\varphi(z) = (z_1, z_2, \dots, z_n)$. Then A_i is the interval $[0, u_i]$.

PROPOSITION 7.1. *For each $i \in \{1,2,\dots,n\}$ and each $z \in A$ we have $z_i = z \wedge u_i$.*

Proof. Let $i \in I$. We have $z = z_1 + \dots + z_n$. Further, $z_i \in A_i$, whence $z_i \leqq u_i$.

Let $t \in A$, $t \leqq z$ and $t \leqq u_i$. Then $t_i \leqq z_i$. Next, $t \in A_i$ and hence $t_j = 0$ for each $j \in \{1,2,\dots,n\}$ with $j \neq i$. Thus $t \leqq z$. Therefore $z_i = z \wedge u_i$. □

We conclude that the mapping φ is uniquely determined by the system $\{u_1,\dots,u_n\}$.

Let $\beta = \psi_1(\alpha)$ be as in Section 6. Then for each $i \in I$ we have $u(\mathcal{A}_i^*) = u_i$ and hence $u(\mathcal{B}_i) = u_i$. Further, let $\alpha_1 = \psi_2(\beta)$. From the definition of ψ_2 we then obtain $u_i' = u_i$, hence $\mathcal{A}_i' = \mathcal{A}_i$ and in view of 4.1, α_1 is equal to α. Thus $\psi_2(\psi_1(\alpha)) = \alpha$. This yields:

LEMMA 7.2. *The mapping ψ is a monomorphism.*

Now let us assume that the internal direct product decomposition β_1 is valid and let us apply the corresponding notation for β_1 as above.

PROPOSITION 7.3. *Let $z \in G^+$ and $i \in \{1,2,\dots,n\}$. Let n_0 be the first positive integer with $z \leqq n_0 u$. Then $z_i' = z \wedge n_0 u_i'$.*

Proof. We have $z = z_1' + \dots + z_n'$ and $z_1',\dots,z_n' \in G^+$. Hence $z_i' \leqq z$. From $z \leqq n_0 u$ we obtain $z_i' \leqq n_0 u_i'$.

Let $p \in G$, $p \leqq z$ and $p \leqq n_0 u_i'$. Then $p_i' \leqq z_i'$. Let $j \in \{1,2,\dots,n\}$, $j \neq i$. Then $(u_i')_j = 0$, thus $(n_0 u_i')_j = 0$. Also, $p_j' \leqq (n_0 u_i')_j$, hence $p_j' \leqq 0$. Therefore $p_j' \leqq z$, whence $p_j' \leqq z_j'$. This yields $p \leqq z'$. Summarizing, we verified that the relation $z_i' = z \wedge n_0 u_i'$ is valid. □

We remark that in 7.3 we can apply any $n_1 > n_0$ instead of n_0. From the fact that G is directed it follows

(∗) for each $z \in G$ there exist $x, y \in G^+$ with $z = x - y$.

In view of 7.3 and (∗), the mapping φ' yielding the internal direct product decomposition β_1 is uniquely determined by the system $\{u_1',\dots,u_n'\}$.

From the construction of α_1 in Section 6 it follows that for each $i \in I$, $u(\mathcal{A}_i') = u_i'$. This yields that in $\psi_1(\alpha_1)$ we have again the direct factors $\mathcal{A}_1',\dots,\mathcal{A}_n'$. Therefore $\psi_1(\alpha_1) = \beta_1$. Hence $\psi_1(\psi_2(\beta_1)) = \beta_1$. From this we obtain that the mapping ψ_1 is surjective. Thus, in view of 7.2, we have:

PROPOSITION 7.4. *The mapping ψ_1 is a bijection of $\mathrm{ID_f}(\mathcal{A})$ onto $\mathrm{ID_f}(G)$ and $\psi_2 = \varphi_1^{-1}$.*

PROPOSITION 7.5. *Let (G, u) be a unital partially ordered group. Then each its direct product decomposition with nonzero direct factors is finite.*

Proof. It suffices to apply the same argument as in the proof of [10; Proposition 2.2] (we correct a misprint in this proof: instead of $nu^0_{i(1)}$ it should be $nu^0_{i(n)}$). □

Therefore, in 7.4, $\mathrm{ID_f}(G)$ can be replaced by $\mathrm{ID}(G)$. Thus we have proved the assertion concerning the internal direct product decompositions α and β which was formulated in Section 1.

REFERENCES

[1] CIGNOLI, R.—D'OTTAVIANO, M. I.—MUNDICI, D.: *Algebraic Foundations of Many-Valued Reasoning.* Trends Log. Stud. Log. Libr. 7, Kluwer Academic Publishers, Dordrecht, 2000.

[2] DVUREČENSKIJ, A.—VETTERLEIN, T.: *Pseudoeffect algebras I. Basic properties*, Internat. J. Theoret. Phys. **40** (2001), 685–701.

[3] DVUREČENSKIJ, A.: *Pseudoeffect algebras II. Group representations*, Internat. J. Theoret. Phys. **40** (2001), 703–726.

[4] DVUREČENSKIJ, A.—VETTERLEIN, T.: *Infinitary lattice and Riesz properties of pseudoeffect algebras and po-groups*, J. Aust. Math. Soc. **75** (2003), 295–311.

[5] GEORGESCU, G.—IORGULESCU, A.: *Pseudo MV-algebras: a noncommutative extension of MV-algebras.* In: The Proceedings of the Fourth International Symposium on Economic Informatics, Bucharest, 6–9 May, Romania, 1999, pp. 961–968.

[6] GEORGESCU, G.—IORGULESCU, A.: *Pseudo MV-algebras*, Mult.-Valued Log. **6** (2001), 95 135.

[7] HASHIMOTO, J.: *On direct product decompositions of partially ordered sets*, Ann. of Math. (2) **54** (1951), 315–318.

[8] JAKUBÍK, J.: *Direct product decompositions of partially ordered groups*, Czechoslovak Math. J. **10** (1960), 231–243. (Russian)

[9] JAKUBÍK, J.: *Direct product decompositions of partially ordered groups II*, Czechoslovak Math. J. **11** (1961), 490–515. (Russian)

[10] JAKUBÍK, J.: *Direct product decompositions of MV-algebras*, Czechoslovak Math. J. **44** (1994), 725 739.

[11] JAKUBÍK, J.: *Direct product decompositions of pseudo MV-algebras*, Arch. Math. (Brno) **37** (2001), 131–142.

[12] KUROSH, A. G.: *Group Theory* (3rd ed.), Nauka, Moskva, 1967. (Russian)

[13] RACHŮNEK, J.: *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (2002), 255–273.

[14] SHIMBIREVA, E. P.: *To the theory of partially ordered groups*, Mat. Sb. **20** (1947), 145–178. (Russian)

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