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ASYMPTOTIC AND OSCILLATORY BEHAVIOUR OF SOLUTIONS OF CERTAIN SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH FORCING TERM

SVATOSLAV STANĚK

ABSTRACT. Sufficient conditions are obtained for the oscillatory and asymptotic behaviour of solutions of the equation

$$\Big[a(t)\big(x'(h(t)) - p(t)g(x'(t))\big)\Big]' + f\big(t, x(\alpha_0(t)), x'(\alpha_1(t))\big) = e(t),$$

where $-1 < \lim\limits_{t\to\infty} p(t) < 1$ and $h(t) > t$.

1. Introduction

Consider the second order neutral delay differential equation ($\mathbb{R}_+ = \langle 0, \infty)$)

$$\Big[a(t)\big(x'(h(t)) - p(t)g(x'(t))\big)\Big]' + f\big(t, x(\alpha_0(t)), x'(\alpha_1(t))\big) = e(t), \qquad (1)$$

in which $a, p, e \in C^0(\mathbb{R}_+; \mathbb{R})$, $g \in C^0(\mathbb{R}; \mathbb{R})$, $f \in C^0(\mathbb{R}_+ \times \mathbb{R}^2; \mathbb{R})$, $h, \alpha_i \in C^0(\mathbb{R}_+; \mathbb{R}_+)$, $-1 < \lim\limits_{t\to\infty} p(t) =: \gamma < 1$, $h(t) > t$ on $\mathbb{R}_+$, $\lim\limits_{t\to\infty} \alpha_i(t) = \infty$ $(i = 0, 1)$.

By a solution x of (1) we mean a function $x \in C^1(\langle T_x, \infty); \mathbb{R})$ for some $T_x \in \mathbb{R}_+$ such that $a(t)\big(x'(h(t)) - p(t)g(x'(t))\big)$ is continuously differentiable on the interval $\langle T_x, \infty)$ and such that (1) is satisfied for all $t \geqq T_x$, $\alpha_i(t) \geqq T_x$, $(i = 0, 1)$.

As it is customary, a solution x of (1) is called oscillatory, if it has arbitrarily large zeros; otherwise it is called non-oscillatory.

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This paper was motivated by recent papers [3] and [5], where the authors give some criteria for the asymptotic and oscillatory behaviour of solutions of the delay differential equation

$$x''(t) + q(t)f\big(x(\sigma_1(t))\big)g\big(x'(\sigma_2(t))\big) = e(t)$$

and the neutral delay differential equation

$$\big[a(t)\big(x(t) - px(t-\tau)\big)'\big]' + q(t)f\big(x(t-\sigma)\big) = 0,$$

where $0 \leqq p < 1$ is a constant, respectively. The purpose of this paper is to present a new criterion for the oscillatory and asymptotic behaviour of solutions of (1), which extends results in [3].

We observe that the oscillatory and asymptotic behaviour of solutions for second order and higher order neutral and non-neutral delay differential equations has been studied in many papers, e.g. [1]-[14], [17]-[19].

2. Notation, lemmas

We denote by $h^{[n]}$ for any integer n ($\geqq 0$) the function defined inductively by $h^{[0]}(t) = t$ and $h^{[n]}(t) = h \circ h^{[n-1]}(t)$ for $n > 0$ and $t \in \mathbb{R}_+$. One can readily check that $\lim\limits_{n\to\infty} h^{[n]}(t) = \infty$ for all $t \in \mathbb{R}_+$ (see e.g. [15]) and for each $t_0 \in \mathbb{R}_+$,

$$\langle t_0, \infty) = \bigcup_{n=0}^{\infty} \big\langle h^{[n]}(t_0), h^{[n+1]}(t_0)\big\rangle .$$

We shall assume that the functions a, g, f, e satisfy some of the following assumptions:

(H_1) There exists $\lim\limits_{t\to\infty} a(t) =: A > 0$;

(H_2) $g(z) - z$ is bounded on $\mathbb{R}$ and $\dfrac{z_1 - z_2}{g(z_1) - g(z_2)} > |\gamma|$ for all $z_1, z_2 \in \mathbb{R}$, $z_1 \neq z_2$;

(H_3) $f(t,y,z)y \geqq 0$ for all $(t,y,z) \in \mathbb{R}_+ \times \mathbb{R}^2$, and $f(t,\cdot,z)$ is nondecreasing on $\mathbb{R}$ for each fixed $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$;

(H_4) $\int\limits_0^\infty e(s)\,\mathrm{d}s$ is convergent.

LEMMA 1. *Let $t_0 \in \mathbb{R}_+$ be a number, $z\colon \langle t_0, \infty) \to \mathbb{R}$ be a function such that $\lim\limits_{t\to\infty}\big(z(h(t)) - p(t)g(z(t))\big)$ ($=: b$) exists. If assumption (H_2) is fulfilled and z is locally bounded on $\langle t_0, \infty)$, then $\lim\limits_{t\to\infty} z(t)$ exists.*

P r o o f. Let assumption (H_2) be fulfilled and let z be locally bounded on $\langle t_0, \infty)$. First we will prove z is bounded on $\langle t_0, \infty)$. Setting $u(t) :=$

$z(h(t)) - p(t)g(z(t))$ and $r(t) := u(t) + p(t)(g(z(t)) - z(t))$ for $t \in \langle t_0, \infty)$, then $\lim_{t\to\infty} u(t) = b$ and since (cf. (H_2)) $g(z(t)) - z(t)$ is bounded on $\langle t_0, \infty)$, we have $|r(t)| \leqq L$ for $t \geqq t_0$ with a positive constant L. Let $|p(t)| \leqq \varepsilon$ for $t \geqq t_1$ ($\geqq t_0$), where $|\gamma| < \varepsilon < 1$. Using the equality

$$z(h^{[2]}(t)) = r(h(t)) + p(h(t))r(t) + p(h(t))p(t)z(t)$$

we deduce

$$\begin{aligned} z(h^{[2n]}(t)) &= r(h^{[2n-1]}(t)) + p(h^{[2n-1]}(t))r(h^{[2n-2]}(t)) \\ &\quad + \sum_{k=0}^{n-2}\Big(r(h^{[2k+1]}(t)) + p(h^{[2k+1]}(t))r(h^{[2k]}(t))\Big)\prod_{j=2k+2}^{2n-1} p(h^{[j]}(t)) \\ &\quad + z(t)\prod_{k=0}^{2n-1} p(h^{[k]}(t)) \qquad (2) \end{aligned}$$

for $t \geqq t_0$ and $n \in \mathbb{N}$, $n \geqq 2$. Hence

$$|z(h^{[2n]}(t))| \leqq (1+\varepsilon)L + (1+\varepsilon)L\sum_{k=0}^{n-2}\varepsilon^{2(n-k-1)} + m\varepsilon^{2n} \leqq \frac{L}{1-\varepsilon} + m$$

for $t \in \langle t_1, h^{[2]}(t_1))$, $n \geqq 2$, where $m = \sup\{|z(t)|;\ t_1 \leqq t \leqq h^{[2]}(t_1)\}$ and consequently, z is bounded on $\langle t_0, \infty)$.

If $\gamma = 0$, then $\lim_{t\to\infty} z(h(t)) = \lim_{t\to\infty}(u(t) + p(t)g(z(t))) = \lim_{t\to\infty} u(t) = b$ and $\lim_{t\to\infty} z(t)$ exists.

Let $\gamma \neq 0$ and let $\{t_n\}$ and $\{t'_n\}$ be sequences of points in $\langle t_0, \infty)$, $\lim_{n\to\infty} t_n = \infty = \lim_{n\to\infty} t'_n$ such that

$$\alpha := \limsup_{t\to\infty} z(t) = \lim_{n\to\infty} z(h^{[2]}(t_n)),$$
$$\beta := \liminf_{t\to\infty} z(t) = \lim_{n\to\infty} z(h^{[2]}(t'_n)).$$

Using the equality $z(h^{[2]}(t)) = u(h(t)) + p(h(t))g(u(t) + p(t)g(z(t)))$ and the fact that (cf. (H_2)) g is increasing on $\mathbb{R}$ (and then also $\gamma \cdot g(b + \gamma \cdot g(t))$ is increasing on $\mathbb{R}$) we obtain the following inequalities

$$\alpha \leqq b + \gamma g(b + \gamma g(\alpha)), \qquad \beta \geqq b + \gamma g(b + \gamma g(\beta)).$$

Then

$$\alpha - \beta \leqq \gamma\big(g\big(b + \gamma g(\alpha)\big) - g\big(b + \gamma g(\beta)\big)\big)$$

and if $\alpha \neq \beta$, $\dfrac{\alpha - \beta}{\gamma\big(g\big(b + \gamma g(\alpha)\big) - g\big(b + \gamma g(\beta)\big)\big)} \leqq 1$ which contradicts (cf. (H_2))

$$\frac{\alpha - \beta}{\gamma\big(g\big(b + \gamma g(\alpha)\big) - g\big(b + \gamma g(\beta)\big)\big)} = \frac{b + \gamma g(\alpha) - b - \gamma g(\beta)}{\gamma\big(g\big(b + \gamma g(\alpha)\big) - g\big(b + \gamma g(\beta)\big)\big)} \cdot \frac{\alpha - \beta}{\gamma\big(g(\alpha) - g(\beta)\big)} > 1 .$$

Whence $\alpha = \beta$ that is $\lim_{t\to\infty} z(t)$ exists.

Remark 1. From our Lemma 1 follow Lemma 1 in [16] and Lemma 1 in [20] (with $n = 1$).

LEMMA 2. *Assume* $t_0 \in \mathbb{R}_+$, $c\colon \langle t_0, \infty) \to \mathbb{R}$ *is a bounded function and* $z\colon \langle t_0, \infty) \to \mathbb{R}$ *is such a function that* $(u(t) :=)\ a(t)\big(z\big(h(t)\big) - p(t) g\big(z(t)\big)\big) + c(t)$ *is non-increasing on* $\langle t_0, \infty)$ *and* $\lim_{t\to\infty} u(t) = -\infty$. *If assumptions* (H_1), (H_2) *are fulfilled and* z *is locally bounded on* $\langle t_0, \infty)$, *then* $\lim_{t\to\infty} z(t) = -\infty$.

Proof. Let assumptions (H_1), (H_2) be fulfilled and let z be locally bounded on $\langle t_0, \infty)$. Assume $a(t) > 0$ for $t \geqq t_1$ ($\geqq t_0$) and for this t define r by $r(t) = \big(1/a(t)\big)\big(u(t) - b(t)\big)$, where $b(t) = c(t) - a(t)p(t)\big(g\big(z(t)\big) - z(t)\big)$. Then $z\big(h(t)\big) = r(t) + p(t)z(t)$, b is bounded on $\langle t_1, \infty)$, say $|b(t)| \leqq B$ for $t \geqq t_1$, and $\lim_{t\to\infty} r(t) = -\infty$. Choose numbers ε, t_2, $|\gamma| < \varepsilon < 1$, $t_2 \geqq t_1$ so that $|p(t)| \leqq \varepsilon$, $u(t) + B < 0$ and

$$\frac{(1 + 3\varepsilon)A}{2(1 + \varepsilon)} \leqq a(t) \leqq \frac{(3 + \varepsilon)A}{2(1 + \varepsilon)}$$

for $t \geqq t_2$. Then

$$\frac{2(1 + \varepsilon)}{(1 + 3\varepsilon)A}\big(u(t) - B\big) \leqq r(t) \leqq \frac{2(1 + \varepsilon)}{(3 + \varepsilon)A}\big(u(t) + B\big)$$

and

$$\begin{aligned} & r\big(h^{[2k+1]}(t)\big) + p\big(h^{[2k+1]}(t)\big) r\big(h^{[2k]}(t)\big) \\ & \leqq \frac{2(1 + \varepsilon)}{(3 + \varepsilon)A}\big(u\big(h^{[2k+1]}(t)\big) + B\big) - \varepsilon \frac{2(1 + \varepsilon)}{(1 + 3\varepsilon)A}\big(u\big(h^{[2k]}(t)\big) - B\big) \\ & \leqq \frac{1 - \varepsilon^2}{(3 + \varepsilon)A} u\big(h^{[2k+1]}(t)\big) + 2B/A \end{aligned} \tag{3}$$

for $t \geqq t_2$ and $k = 0,1,2,\dots$. Setting $m = \sup\{|z(t)|;\ t_2 \leqq t \leqq h^{[2]}(t_2)\}$ we have (cf. (2), (3))

$$z\big(h^{[2n]}(t)\big) \leqq \frac{1-\varepsilon^2}{(3+\varepsilon)A} u\big(h^{[2n-1]}(t)\big) + 2B/A + \sum_{k=0}^{n-2} (2B/A)\varepsilon^{2(n-k-1)} + m$$
$$\leqq \frac{1-\varepsilon^2}{(3+\varepsilon)A} u\big(h^{[2n-1]}(t)\big) + \big(2B/A(1-\varepsilon^2)\big) + m$$

for $t \in \langle t_2, h^{[2]}(t_2)\rangle$ and $n \geqq 2$. Consequently, $\lim\limits_{t\to\infty} z(t) = -\infty$.

3. Results

THEOREM 1. *Suppose (H_1)–(H_4) hold and for each $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$*

$$\operatorname{sign}\varepsilon \int_0^\infty f\big(s,\ \varepsilon\alpha_0(s),\ z\cdot\operatorname{sign}\varepsilon\big)\,\mathrm{d}s = \infty \tag{4}$$

uniformly on $\langle\, |\varepsilon|,\ 2|\varepsilon|\,\rangle$ with respect to z. Then every solution x of (1) is either oscillatory or $\lim\limits_{t\to\infty} x'(t) = 0$.

Proof. Let x be a non-oscillatory solution of (1), say $x(t) > 0$ for $t \geqq t_1$ $(\geqq 0)$ and let $\alpha_i(t) \geqq t_1$ for $t \geqq t_2$ $(\geqq t_1)$, $i = 0,1$. Then

$$f\big(t, x(\alpha_0(t)), x'(\alpha_1(t))\big) \geqq 0 \qquad \text{for} \quad t \geqq t_2,$$

hence

$$\Big[a(t)\big(x'(h(t)) - p(t)g(x'(t))\big)\Big]' - e(t) \leqq 0 \qquad \text{for} \quad t \geqq t_2$$

and

$$a(t)\big(x'(h(t)) - p(t)g(x'(t))\big) - \int_0^t e(s)\,\mathrm{d}s$$

is a non-increasing function on $\langle t_2, \infty)$. Consequently, either

$$\lim_{t\to\infty}\Big\{a(t)\big(x'(h(t)) - p(t)g(x'(t))\big) - \int_0^t e(s)\,\mathrm{d}s\Big\} = -\infty$$

or $\lim_{t\to\infty} a(t)\big(x'(h(t))-p(t)g(x'(t))\big)$ is finite. From Lemma 1 and Lemma 2 (with $z = x'$, $c(t) = -\int_0^t e(s)\,ds$) we infer either $\lim_{t\to\infty} x'(t) = -\infty$ which contradicts

$$x(t) > 0 \qquad \text{for} \quad t \geqq t_1 \tag{5}$$

or $\lim_{t\to\infty} x'(t)$ is finite, say d.

If $d < 0$, then $\lim_{t\to\infty} x(t) = -\infty$ which contradicts (5). Let $d > 0$. Then there exists a t_3 ($\geqq t_2$) so that

$$3d/4 \leqq x'(t) \leqq 5d/4 \qquad \text{for} \quad t \geqq t_3$$

and $x(t) \geqq x(t_3) + 3d(t - t_3)/4$ for $t \geqq t_3$. Hence $x(t) \geqq \varepsilon t$ for $t \geqq t_4$ ($\geqq t_3$) and $\varepsilon = 5d/8$, which implies $x(\alpha_0(t)) \geqq \varepsilon\alpha_0(t)$ for $t \geqq t_5$ ($\geqq t_4$), where t_5 is a number with $\alpha_i(t) \geqq t_4$ for $t \geqq t_5$ ($i = 0, 1$). Then

$$f\big(t, x(\alpha_0(t)), x'(\alpha_1(t))\big) \geqq f\big(t, \varepsilon\alpha_0(t), x'(\alpha_1(t))\big) \qquad \text{and}$$

$$\Big[a(t)\big(x'(h(t)) - p(t)g(x'(t))\big)\Big]' \leqq e(t) - f\big(t, \varepsilon\alpha_0(t), x'(\alpha_1(t))\big) \qquad \text{for} \quad t \geqq t_5 .$$

Since $\varepsilon \leqq x'(\alpha_1(t)) \leqq 2\varepsilon$ for $t \geqq t_5$ using assumption (4) we get

$$\lim_{t\to\infty} a(t)\big(x'(h(t)) - p(t)g(x'(t))\big) = -\infty ,$$

which contradicts

$$\lim_{t\to\infty} \big(x'(h(t)) - p(t)g(x'(t))\big) = d - \gamma g(d) \quad \text{and} \ (H_1).$$

For the case $x(t) < 0$ on a ray the proof is similar and therefore it is omitted.

Remark 2. Let $f(t,y,z) = q(t)k(y)m(z)$ for $(t,y,z) \in \mathbb{R}_+ \times \mathbb{R}^2$ with continuous functions q, k, m. If $q(t) \geqq 0$ on $\mathbb{R}_+$, $k(y)y \geqq 0$ for $y \in \mathbb{R}$, k is non-decreasing on $\mathbb{R}$, $m(z) > 0$ for $z \in \mathbb{R} - \{0\}$ and

$$\operatorname{sign}\varepsilon \int_0^\infty q(t)k\big(\varepsilon\alpha_0(t)\big)\,dt = \infty$$

for each $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$, then the statement of Theorem 1 holds.

R e m a r k 3. The result of Theorem 1 can be extended to the equation of the form

$$\Big[a(t)\big(x'(h(t)) - p(t)g(x'(t))\big)\Big]' + f\Big(t, x(t), x\big(\alpha_0(t)\big),\dots,x\big(\alpha_n(t)\big), x'(t), x'\big(\beta_0(t)\big),\dots, x'\big(\beta_m(t)\big)\Big) = e(t).$$

The following examples show if at least one of the assumptions (H_1)–(H_4), (4) and $-1 < \gamma < 1$ is violated then the conclusion of Theorem 1 is false.

E x a m p l e 1. Consider the neutral differential equation

$$\Big[e^{-t}\big(x'(t+1) + e^{-1}\,x'(t)\big)\Big]' + \frac{x(t)}{1 + x'^2(2t)} = \frac{e^t}{1 + e^{4t}}. \tag{6}$$

All assumptions of Theorem 1 are fulfilled except (H_1). Equation (6) has a solution $x(t) = e^t$.

E x a m p l e 2. Consider the neutral differential equation

$$\big(x'(t+1) - 2e^{1-2t}x'^3(t)\big)' + x(t+1) = 0. \tag{7}$$

The assumptions of Theorem 1 are fulfilled except (H_2). Equation (7) has a solution $x(t) = e^t$.

E x a m p l e 3. Consider the neutral differential equation

$$\Big[(3/4)\big(x'(t+2\pi) + (1/3)x'(t)\big)\Big]' + \frac{\big(x(t)-2\big)\big(1+x'^2(t)\big)}{1+\cos^2 t} = 0. \tag{8}$$

The assumptions of Theorem 1 are satisfied except (H_3). Equation (8) has a solution $x(t) = 2 - \sin t$.

E x a m p l e 4. The neutral differential equation

$$\Big[(2/3)\big(x'(t+2\pi) + (1/2)x'(t)\big)\Big]' + \frac{x(t)\big(1+x'^2(t)\big)}{(2-\sin t)(1+\cos^2 t)} = 1 + \sin t$$

fulfils all assumptions of Theorem 1 except (H_4) and admits a solution $x(t) = 2 - \sin t$.

E x a m p l e 5. Consider the differential equation

$$x''(t+1) + \frac{x(t+1)}{t^2(t+e^{-t})} = t^{-2} + e^{-t}. \tag{9}$$

All assumptions of Theorem 1 are fulfilled except (4). Equation (9) has a solution $x(t) = t - 1 + e^{1-t}$.

Example 6. Consider the neutral differential equation

$$\big(x'(t+\ln 3) - 4x'(t)\big)' + (1/e)x(t+1) = 0. \tag{10}$$

All assumptions of Theorem 1 are fulfilled except $-1 < \gamma < 1$. Equation (10) has a solution $x(t) = e^t$.

The following example shows that under the assumptions of Theorem 1 there exists an equation having a non-oscillatory solution x with $\lim_{t\to\infty} x'(t) = 0$ and $\lim_{t\to\infty} x(t) \neq 0$.

Example 7. The neutral differential equation

$$x''(t+\ln 2) - (1/2)x''(t) + (1/t^2)x(t^2) = t^{-2}(1 + e^{-t^2})$$

admits a solution $x(t) = 1 + e^{-t}$.

Our results can be extended to the neutral differential equation of the form

$$\Big[a(t)\big(x'\big(h^{[2]}(t)\big) + (\alpha+\beta)x'\big(h(t)\big) + \alpha\beta x'(t)\big)\Big]' + f\big(t, x\big(\alpha_0(t)\big), x'\big(\alpha_1(t)\big)\big) = e(t), \tag{11}$$

where a, h, α_0, α_1, f, e are as in equation (1) and $\alpha, \beta \in \mathbb{R}$.

By a solution (11) we mean a C^1-function x on an interval $\langle T_x, \infty)$ ($T_x \geqq 0$), $a(t)\big(x'\big(h^{[2]}(t)\big) + (\alpha+\beta)x'\big(h(t)\big) + \alpha\beta x'(t)\big)$ is continuously differentiable on $\langle T_x, \infty)$ and (11) is satisfied for all $t \geqq T_x$, $\alpha_i(t) \geqq T_x$ ($i = 0, 1$).

THEOREM 2. *Let assumptions (H_1), (H_3), (H_4), (4) and $|\beta| < 1$ be satisfied. If*

$$-1 < \alpha \leqq 0 \tag{12}$$

or

$$0 < \alpha < 1, \quad h \in C^2(\mathbb{R}_+), \quad h''(t) \geqq 0 \quad \text{on} \quad \mathbb{R}_+ \quad \text{and} \quad \liminf_{t\to\infty} h'(t) > 0, \tag{13}$$

then every solution x of (11) is either oscillatory or $\lim_{t\to\infty} x'(t) = 0$.

Proof. Let x be a non-oscillatory solution of (11), say $x(t) < 0$ for $t \geqq t_1$ ($\geqq 0$) and let $\alpha_i(t) \geqq t_1$, for $t \geqq t_2$ ($\geqq t_1$), $i = 0, 1$. Then

$$f\big(t, x\big(\alpha_0(t)\big), x'\big(\alpha_1(t)\big)\big) \leqq 0 \qquad \text{on} \quad \langle t_2, \infty)$$

and setting $r(t) := x'(h(t)) + \alpha x'(t)$ for $t \geqq t_1$ we have

$$\Big[a(t)\big(r(h(t)) + \beta r(t)\big)\Big]' - e(t) \geqq 0 \qquad \text{for} \quad t \geqq t_2 .$$

Therefore $u(t) := a(t)\big(r(h(t)) + \beta r(t)\big) - \int\limits_0^t e(s)\,\mathrm{d}s$ is a non-decreasing function on $\langle t_2, \infty)$, and consequently either $\lim\limits_{t\to\infty} u(t) = \infty$ and then by Lemma 2 $\lim\limits_{t\to\infty} r(t) = \infty$ or $\lim\limits_{t\to\infty} u(t)$ is finite and then by Lemma 1 $\lim\limits_{t\to\infty} r(t) =: c$ is finite too.

Let $\lim\limits_{t\to\infty} r(t) = \lim\limits_{t\to\infty}\big(x'(h(t)) + \alpha x'(t)\big) = \infty$. If $-1 < \alpha \leqq 0$, we have

$$x'\big(h^{[n]}(t)\big) = r\big(h^{[n-1]}(t)\big) + \sum_{k=0}^{n-2} r\big(h^{[k]}(t)\big)|\alpha|^{n-k-1} + x'(t)|\alpha|^n$$

for $t \geqq t_1$ and $n \geqq 2$, hence $\lim\limits_{t\to\infty} x'(t) = \infty$ which contradicts

$$x(t) < 0 \qquad \text{for} \quad t \geqq t_1 . \tag{14}$$

If assumption (13) is satisfied and $h'(t) > 0$ for $t \geqq t_2$, then

$$\int_{t_2}^{t} r(s)\,\mathrm{d}s = \int_{t_2}^{t} x'\big(h(s)\big)\,\mathrm{d}s + \alpha\big(x(t) - x(t_2)\big) \leqq \int_{t_2}^{t} x'\big(h(s)\big)\,\mathrm{d}s - \alpha x(t_2)$$

and therefore

$$\int_{t_2}^{\infty} x'\big(h(s)\big)\,\mathrm{d}s = \infty ,$$

which contradicts

$$\int_{t_2}^{t} x'\big(h(s)\big)\,\mathrm{d}s = \frac{1}{h'(t)} x\big(h(t)\big) - \frac{1}{h'(t_2)} x\big(h(t_2)\big) + \int_{t_2}^{t} \frac{h''(s)x\big(h(s)\big)}{h'^2(s)}\,\mathrm{d}s$$
$$\leqq -\frac{x\big(h(t_2)\big)}{h'(t_2)} \qquad \text{for} \quad t \geqq t_2 .$$

Let $\lim\limits_{t\to\infty} r(t) = \lim\limits_{t\to\infty}\big(x'(h(t)) + \alpha x'(t)\big) = c$. By Lemma 1 there exists $\lim\limits_{t\to\infty} x'(t) =: d$. Due to (14), $d \leqq 0$. If $d < 0$, then there exists a t_3 ($\geqq t_2$) so that

$$(5/4)d \leqq x'(t) \leqq (3/4)d \qquad \text{for} \quad t \geqq t_3$$

and $x(t) \leqq x(t_3) + (3/4)d(t - t_3)$ for $t \geqq t_3$. Hence $x(t) \leqq \varepsilon t$ for $\varepsilon = (5/8)d$ and $t \geqq t_4$ ($\geqq t_3$). If t_5 ($\geqq t_4$) is such a number that $\alpha_i(t) \geqq t_4$ on $\langle t_5, \infty)$ for $i = 0, 1$, then

$$f\big(t, x(\alpha_0(t)),\, x'(\alpha_1(t))\big) \leqq f\big(t, \varepsilon\alpha_0(t),\, x'(\alpha_1(t))\big)$$

and

$$\Big[a(t)\big(x'(h^{[2]}(t)) + (\alpha+\beta)x'(h(t)) + \alpha\beta x'(t)\big)\Big]' \geqq e(t) - f\big(t, \varepsilon\alpha_0(t),\, x'(\alpha_1(t))\big)$$

for $t \geqq t_5$. Since $\varepsilon \geqq x'(t) \geqq 2\varepsilon$ for $t \geqq t_5$, using assumption (4) we have

$$\lim_{t\to\infty} a(t)\Big[x'\big(h^{[2]}(t)\big) + (\alpha+\beta)x'\big(h(t)\big) + \alpha\beta x'(t)\Big] = \infty,$$

which contradicts (H_1) and

$$\lim_{t\to\infty}\Big[x'\big(h^{[2]}(t)\big) + (\alpha+\beta)x'\big(h(t)\big) + \alpha\beta x'(t)\Big] = d(1+\alpha+\beta+\alpha\beta).$$

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