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SOME AXIOMATIZATIONS OF B-ALGEBRAS

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ABSTRACT. Some systems of axioms defining a B-algebra are given with a proof of the independence of the axioms. In addition, we obtain a simplified axiomatization of commutative B-algebras.

1. Introduction

B-algebras have been introduced by J. Neggers and H. S. Kim in [4]. They defined a *B-algebra* as an algebra $(A, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

A1. $x * x = 0$,
A2. $x * 0 = x$,
A3. $(x * y) * z = x * (z * (0 * y))$.

We will denote by $\mathcal{B}$ the class of all B-algebras. In [1], J. R. Cho and H. S. Kim proved that every B-algebra is a quasigroup. M. Kondo and Y. B. Jun [3] showed that the class $\mathcal{B}$ is equivalent in one sense to the class of groups. In [2], Y. B. Jun, E. H. Roh and H. S. Kim introduced the notion of BH-algebras, which is a generalization of $BCH/BCI/BCK$-algebras. Moreover, $\mathcal{B}$ is a proper subclass of the class of BH-algebras (cf. [4; Lemma 2.9]). For another useful generalization of B-algebras see [6].

2. Some axiomatizations of B-algebras

THEOREM 2.1. *Let $(A, -, +, 0)$ be an algebra of type $(2, 2, 0)$ satisfying the following axioms:*

B1. $x - x = 0$,
B2. $x - 0 = x$,
B3. $(x - y) - z = x - (z + y)$,
B4. $x + y = x - (0 - y)$.

Then $(A, -, 0)$ is a B-algebra.

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Conversely, if $(A, -, 0) \in \mathcal{B}$*, and if we define* $x + y$ *by* $x * (0 * y)$*, then* $(A, -, +, 0)$ *obeys the equations* B1-B4.

P r o o f . Straightforward. □

In [6], J. N e g g e r s and H. S. K i m introduced the notion of β-algebras. They defined a *β-algebra* as an algebra $(A, -, +, 0)$ of type $(2, 2, 0)$ that obeys B2, B3, and the following axiom:

$$(0 - x) + x = 0 \,.$$

It is easy to verify that if an algebra $(A, -, +, 0)$ satisfies B1–B4, then it is a β-algebra.

THEOREM 2.2. *Let* $\mathbf{A} = (A, *, 0)$ *be an algebra of type* $(2, 0)$*. Then* $\mathbf{A} \in \mathcal{B}$ *if and only if* $\mathbf{A}$ *obeys the laws:*

C1. $x * x = 0$,
C2. $0 * (0 * x) = x$,
C3. $(x * z) * (y * z) = x * y$.

P r o o f . Suppose that $\mathbf{A}$ is a B-algebra. For each $x \in A$ we have $0 * (0 * x) = x$ (see [4; Lemma 2.9]). Consequently, C2 is valid in $\mathbf{A}$. By A3 we obtain

$$(x * z) * (y * z) = x * \left[(y * z) * (0 * z)\right] = x * \left[y * \left((0 * z) * (0 * z)\right)\right].$$

Hence applying A1 and A2 we get C3.

Conversely, assume that C1–C3 hold in $\mathbf{A}$. Then we have

$$x = 0 * (0 * x) = (x * x) * (0 * x) = x * 0 \,.$$

From this and from C3 we deduce that

$$(x * y) * (0 * y) = x \,. \tag{1}$$

Combining (1) with C3 we get

$$x * \left(z * (0 * y)\right) = \left[(x * y) * (0 * y)\right] * \left[z * (0 * y)\right] = (x * y) * z \,,$$

i.e., A3 holds. Therefore $\mathbf{A} \in \mathcal{B}$. □

LEMMA 2.3. *Let* $(A, *, 0)$ *be an algebra of type* $(2, 0)$ *obeying the following laws:*

D1. $x * x = 0$,
D2. $x * \left\{\left[(0 * y) * z\right] * \left[(0 * x) * z\right]\right\} = y$.

Then:

(i) $x * 0 = x$,
(ii) $0 * (0 * x) = x$,
(iii) $0 * x = 0 * y \implies x = y$,
(iv) $(x * y) * (0 * y) = x$,
(v) $x * y = 0 * (y * x)$.

Proof.

(i): To obtain (i), substitute x for y in D2 and then use D1.

(ii): Substituting $x = 0$, $y = x$, and $z = 0$, D2 becomes

$$0 * \Big\{ \big[(0 * x) * 0\big] * \big[(0 * 0) * 0\big] \Big\} = x .$$

Applying (i) we obtain (ii).

(iii) follows from (ii).

(iv): Let $a, b \in A$. Using D2 with $x = 0$, $y = 0 * a$, $z = b$ we have

$$0 * \Big\{ \big[\big(0 * (0 * a)\big) * b\big] * \big[(0 * 0) * b\big] \Big\} = 0 * a .$$

Hence applying (i) and (ii) we conclude that

$$0 * \big[(a * b) * (0 * b)\big] = 0 * a .$$

That $(a * b) * (0 * b) = a$ follows from (iii).

(v): Let $a, b \in A$. Substituting $x = a$, $y = 0 * (b * a)$, $z = 0$ in D2 we deduce that

$$a * \Big\{ \big[\big(0 * \big(0 * (b * a)\big)\big) * 0\big] * \big[(0 * a) * 0\big] \Big\} = 0 * (b * a) .$$

Then $a * \big[(b * a) * (0 * a)\big] = 0 * (b * a)$. By (iv), $a * b = 0 * (b * a)$, verifying (v). □

THEOREM 2.4. *An algebra* $\mathbf{A} = (A, *, 0)$ *of type* $(2, 0)$ *is a* B*-algebra if and only if the equations* D1 *and* D2 *are valid in* $\mathbf{A}$.

Proof. Let $\mathbf{A}$ satisfy D1 and D2. C1 holds in $\mathbf{A}$ by D1. From Lemma 2.3 (iii) we conclude that $\mathbf{A}$ obeys C2. If we let $x = a * c$, $y = a * b$ and $z = 0 * a$ in D2, then we have

$$(a * c) * \Big\{ \big[\big(0 * (a * b)\big) * (0 * a)\big] * \big[\big(0 * (a * c)\big) * (0 * a)\big] \Big\} = a * b .$$

By Lemma 2.3,

$$\big(0 * (a * b)\big) * (0 * a) = (b * a) * (0 * a) = b ,$$

and similarly, $\big(0 * (a * c)\big) * (0 * a) = c$. Consequently,

$$(a * c) * (b * c) = a * b .$$

This shows that $\mathbf{A}$ also satisfies C3. Then $\mathbf{A} \in \mathcal{B}$ by Theorem 2.2.

For the converse, suppose that $\mathbf{A}$ is a B-algebra. Obviously D1 is valid in $\mathbf{A}$. From Theorem 2.2 we see that C3 holds in $\mathbf{A}$, and therefore

$$\big[(0 * y) * z\big] * \big[(0 * x) * z\big] = (0 * y) * (0 * x) . \qquad (2)$$

It follows that

$$\begin{aligned} x * \Big\{\big[(0*y)*z\big] * \big[(0*x)*z\big]\Big\} &= x * \big[(0*y)*(0*x)\big] && \text{(by (2))} \\ &= (x*x)*(0*y) && \text{(by A3)} \\ &= 0*(0*y) && \text{(by A1)} \\ &= y && \text{(by C2),} \end{aligned}$$

proving D2. The proof is finished. □

Following J. Neggers and H. S. Kim [4] (see also [1]) we give:

DEFINITION 2.5. A B-algebra $(A,*,0)$ is said to be 0*-commutative* if $a*(0*b) = b*(0*a)$ for all $a,b \in A$.

In [1], J. R. Cho and H. S. Kim showed that a B-algebra $\mathbf{A} = (A,*,0)$ is 0-commutative if and only if the equation

C2′ $y*(y*x) = x$

holds in $\mathbf{A}$.

From this and from Theorem 2.2 we have:

COROLLARY 2.6. *An algebra* $(A,*,0)$ *of type* $(2,0)$ *is a* 0*-commutative* B*-algebra if and only if it obeys the laws* C1, C2′, *and* C3.

3. Proof of the independence of the axioms

The independence of the axioms A1, A2, and A3 was proved by J. Neggers and H. S. Kim in [4].

THEOREM 3.1. *The axioms* B1–B4 *are independent, i.e., none of them can be deduced from the others.*

Proof. We are going to give some examples of algebras in which only three of the axioms hold.

Let $A = \{0,1\}$. Define binary operations $\ominus$ and $\oplus$ on A as follows:

$$\begin{aligned} x \ominus y &= x && \text{for all } x,y \in A, \\ x \oplus y &= 0 && \text{for all } x,y \in A. \end{aligned}$$

Then $(A,\ominus,\ominus,0)$ fulfils the axioms B2–B4, but not B1, since $1 \ominus 1 = 1 \neq 0$. (Independence of B1.)

It is easily seen that $(A,\oplus,\oplus,0)$ satisfies B1, B3, and B4, but not B2 (independence of B2).

Now we define the binary operations $-$ and $+$ on A by the following table.

x	y	$x-y$	$x+y$
0	0	0	0
0	1	0	0
1	0	1	1
1	1	0	1

The equations B1, B2, and B4 are valid in $(A,-,+,0)$, but B3 does not hold because $(1-1)-0=0$, while $1-(0+1)=1$. (Independence of B3.)

Finally, let $\mathbf{A}=(A,-,+,0)$ be the algebra, where $-$ is given in the above table and $+$ is defined by

$$x+y=\begin{cases} 0 & \text{if } x=y=0\,, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

Obviously, B1–B3 hold in $\mathbf{A}$, while B4 does not (independence of B4). □

THEOREM 3.2. *The system of axioms* C1–C3 *is independent.*

P r o o f. Let $A=\{0,1\}$. We use the table below in order to define $*$, $\circledast$, and $\divideontimes$.

x	y	$x*y$	$x\circledast y$	$x\divideontimes y$
0	0	1	0	0
0	1	0	0	1
1	0	0	0	0
1	1	1	0	0

We can see that the algebra $(A,*,0)$ satisfies C2–C3, but not C1. The axioms C1 and C3 hold in $(A,\circledast,0)$, while C2 does not. It is evident that $(A,\divideontimes,0)$ obeys C1 and C2. The axiom C3 does not hold because $(0\divideontimes 1)\divideontimes(1\divideontimes 1)=0$, while $0\divideontimes 1=1$. □

Remark 3.3. It is easy to see that the axiom system D1–D2 of B-algebras is independent.

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