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Dedicated to Prof. RNDr. Andrej Pázman DrSc. at the occasion of his 65th birthday

PARTIAL OPTIMUM ESTIMATOR IN TWO STAGE REGRESSION MODEL WITH CONSTRAINTS AND A PROBLEM OF EQUIVALENCE

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(*Communicated by Gejza Wimmer*)

ABSTRACT. Two stage regression model with special structure of the covariance matrix can lead to the problem how to estimate a given function of the second stage parameter in order that the influence of the uncertainty in estimation of the first stage parameter is minimized. The aim of the paper is to give a procedure for estimation, to make some numerical study of such a situation and to focus an attention on an equivalence of different procedure of estimation.

Introduction

The following problem occur, e.g. in geodesy (cf. the section Numerical example). Let an m-dimensional vector parameter Θ be estimated in the first stage and for its estimator $\hat{\Theta}$ be valid $\hat{\Theta} \sim_m (\Theta, \mathbf{V})$, i.e. the estimator $\hat{\Theta}$ has the mean value equal to Θ and its covariance matrix is $\mathbf{V}$. The estimate of the parameter Θ is at our disposal.

In connection to cited references the following notation will be used.

The model of the second stage measurement is

$$\boldsymbol{Y} \sim_n \left[(\mathbf{D}, \mathbf{X}_2)\begin{pmatrix}\Theta \\ \beta\end{pmatrix}, \Sigma_{2,2}\right],$$

where $\mathbf{D}$ and $\mathbf{X}_2$ are given matrices, the rank of the $n \times k$ matrix $\mathbf{X}_2$ is $\mathrm{r}(\mathbf{X}_2) = k < n$ and $\Sigma_{2,2}$ is a given positive definite (p.d.) covariance matrix

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of $\boldsymbol{Y}$. Thus the two stage model can be written in the form (cf. also [1]).

$$\begin{pmatrix} \hat{\Theta} \\ \boldsymbol{Y} \end{pmatrix} \sim_{m+n} \left[\begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta \\ \beta \end{pmatrix}, \begin{pmatrix} \mathbf{V}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2} \end{pmatrix} \right]. \tag{1}$$

The parametric space of the two stage model can be either $\mathbb{R}^{k+m}$ ($k+m$-dimensional Euclidean space), or

$$\underline{\Theta} = \left\{ \begin{pmatrix} \Theta \\ \beta \end{pmatrix} : \ \mathbf{C}\Theta + \mathbf{B}\beta + \boldsymbol{a} = \boldsymbol{O} \right\}, \tag{2}$$

(constraints I) or

$$3\underline{\Theta} = \left\{ \begin{pmatrix} \Theta \\ \beta \\ \gamma \end{pmatrix} : \ \mathbf{C}^*\Theta + \mathbf{B}^*\beta + \mathbf{G}\gamma + \boldsymbol{a}^* = \boldsymbol{O} \right\}$$

(constraints II). Here $\mathbf{B}$ and $\mathbf{C}$ are given $q \times k$ and $q \times m$ matrices and the rank of the matrix $\mathbf{B}$ is $r(\mathbf{B}) = q < k$. When the constraints II are under consideration, then $\mathbf{G}$ is the $q \times l$ matrix, $r(\mathbf{G}) = l < q$ and $r(\mathbf{B}^*, \mathbf{G}) = q < k + l$.

In the case of the constraints I, the aim is to obtain such an unbiased estimator $\tilde{\beta}$ which satisfies the constraints $\mathbf{C}\hat{\Theta} + \mathbf{B}\tilde{\beta} + \boldsymbol{a} = \boldsymbol{O}$. (Sometimes in practice the estimator $\hat{\Theta}$ is documented at some state record office and cannot be changed on the basis of the second stage measurement.) The estimator of β will be called $\mathbf{H}$-optimum estimator (see Definition 2.1) if it minimizes the value $\operatorname{Tr}[\mathbf{H} \operatorname{Var}(\tilde{\beta})]$ for a given positive semidefinite $k \times k$ matrix $\mathbf{H}$.

In the case of the constraints II, the aim is to obtain such an unbiased estimator $\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix}$ which satisfies the constraints $\mathbf{C}^*\hat{\Theta} + \mathbf{B}^*\tilde{\beta} + \mathbf{G}\tilde{\gamma} + \boldsymbol{a} = \boldsymbol{O}$. The estimator of $\begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ will be called $\mathbf{H}^*$-optimum estimator (cf. Definition 4.1) if it minimizes the value $\operatorname{Tr}\left[\mathbf{H}^* \operatorname{Var}\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix}\right]$ for a given positive semidefinite $(k+l) \times (k+l)$ matrix $\mathbf{H}^*$.

The aim of the paper is to give a procedure for a calculation of the $\mathbf{H}$- or $\mathbf{H}^*$-optimum estimators and to analyze, whether there exist equivalent estimators based on a standard reparametrization of the model which lead to the model without constraints.

This paper is intended as a homage to A. Pázman, who is outstanding expert in nonlinear statistics and thus in nonlinear models, which frequently occur in practice. However the ability of authors is not yet sufficient to solve such a problem in nonlinear version and therefore the linear approach was used only. A way to solution will need methods developed in stimulating book [8], but till now the problems are only a challenge for future.

1. Some auxiliary statements for constraints I

Lemma 1.1. *In the model* (1)&(2) *the* BLUE (*best linear unbiased estimator*) *of the parameter* $\binom{\Theta}{\beta}$ *is*

$$\begin{pmatrix}\hat{\hat{\Theta}}\\ \hat{\hat{\beta}}\end{pmatrix} =$$

$$= \left(\begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} \end{pmatrix} - \left[\begin{pmatrix} \mathbf{I}, & \mathbf{D}' \\ \mathbf{O}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \begin{pmatrix} \mathbf{C}' \\ \mathbf{B}' \end{pmatrix} \times \right.$$

$$\left. \times \left\{ (\mathbf{C}, \mathbf{B}) \left[\begin{pmatrix} \mathbf{I}, & \mathbf{D}' \\ \mathbf{O}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \begin{pmatrix} \mathbf{C}' \\ \mathbf{B}' \end{pmatrix} \right\}^{-1} (\mathbf{C}, \mathbf{B}) \right) \times$$

$$\times \left[\begin{pmatrix} \mathbf{I}, & \mathbf{D}' \\ \mathbf{O}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \begin{pmatrix} \mathbf{I}, & \mathbf{D}' \\ \mathbf{O}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} \hat{\Theta} \\ \mathbf{Y} \end{pmatrix}$$

$$- \left[\begin{pmatrix} \mathbf{I}, & \mathbf{D}' \\ \mathbf{O}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \begin{pmatrix} \mathbf{C}' \\ \mathbf{B}' \end{pmatrix} \times$$

$$\times \left\{ (\mathbf{C}, \mathbf{B}) \left[\begin{pmatrix} \mathbf{I}, & \mathbf{D}' \\ \mathbf{O}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \begin{pmatrix} \mathbf{C}' \\ \mathbf{B}' \end{pmatrix} \right\}^{-1} \boldsymbol{a}.$$

P r o o f. The statement is a direct consequence of the formula for the estimator in the model with constraints I. In detail, cf., e.g. [3] and [4]. □

The model (1) can be reparametrized in the following way. Without loss of generality and with respect to our assumption the matrix $\mathbf{B}$ can be expressed as $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2)$, where $\mathbf{B}_2$ is regular. Thus

$$\beta_2 = -\mathbf{B}_2^{-1}(\mathbf{B}_1\beta_1 + \mathbf{C}\Theta + \boldsymbol{a}),$$

where $\beta = (\beta_1', \beta_2')'$. If $\mathbf{X}_2\beta = \mathbf{X}_{2,1}\beta_1 + \mathbf{X}_{2,2}\beta_2$, then the model (1) can be rewritten in the form

$$\begin{pmatrix} \hat{\Theta} \\ \mathbf{Y} \end{pmatrix} \sim_{m+n}$$

$$\sim_{m+n} \left[\begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{C}, & \mathbf{X}_{2,1} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{B}_1 \end{pmatrix} \begin{pmatrix} \Theta \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \mathbf{V}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2} \end{pmatrix} \right]. \tag{4}$$

LEMMA 1.2. *In the model* (4) *the* BLUE *of the parameter* $(\boldsymbol{\Theta}', \boldsymbol{\beta}')'$ *is*

$$\begin{pmatrix} \hat{\hat{\Theta}} \\ \hat{\hat{\beta}}_1 \end{pmatrix} =$$

$$= \left[\begin{pmatrix} \mathbf{I}, & \mathbf{D}' - \mathbf{C}'(\mathbf{B}_2^{-1})'\mathbf{X}_{2,2}' \\ \mathbf{O}, & \mathbf{X}_{2,1}' - \mathbf{B}_1'(\mathbf{B}_2^{-1})'\mathbf{X}_{2,2}' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \times \right.$$

$$\left. \times \begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{D} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{C}, & \mathbf{X}_{2,1} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{B}_1 \end{pmatrix} \right]^{-1} \times$$

$$\times \begin{pmatrix} \mathbf{I}, & \mathbf{D}' - \mathbf{C}'(\mathbf{B}_2^{-1})'\mathbf{X}_{2,2}' \\ \mathbf{O}, & \mathbf{X}_{2,1}' - \mathbf{B}_1'(\mathbf{B}_2^{-1})'\mathbf{X}_{2,2}' \end{pmatrix} \begin{pmatrix} \mathbf{V}^{-1}, & \mathbf{O} \\ \mathbf{O}, & \Sigma_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} \hat{\Theta} \\ \boldsymbol{Y} + \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\boldsymbol{a} \end{pmatrix},$$

$$\hat{\hat{\beta}}_2 = -\mathbf{B}_2^{-1}(\mathbf{B}_1\hat{\hat{\beta}}_1 + \mathbf{C}\hat{\hat{\Theta}} + \boldsymbol{a}).$$

P r o o f . The statement is a direct consequence of the formula for the estimator in the model without constraints I. In detail, cf., e.g. [3] and [4]. □

LEMMA 1.3. *The estimator* $\hat{\hat{\boldsymbol{\beta}}}_1$ *from Lemma* 1.2 *can be written also in the form*

$$\hat{\hat{\beta}}_1 = \left\{ (\mathbf{X}_{2,1} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{B}_1)' \left[\Sigma_{2,2} + (\mathbf{D} - \mathbf{X}_{2,1}\mathbf{B}_2^{-1}\mathbf{C})\mathbf{V}(\mathbf{D} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{C})' \right]^{-1} \times \right.$$
$$\left. \times (\mathbf{X}_{2,1} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{B}_1) \right\}^{-1} (\mathbf{X}_{2,1} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{B}_1)' \times$$
$$\times \left[\Sigma_{2,2} + (\mathbf{D} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{C})\mathbf{V}(\mathbf{D} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{C})' \right]^{-1} \times$$
$$\times \left[\boldsymbol{Y} + \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\boldsymbol{a} - (\mathbf{D} - \mathbf{X}_{2,2}\mathbf{B}_2^{-1}\mathbf{C})\hat{\Theta} \right].$$

P r o o f . It is a consequence of [1]. □

Remark 1.4. The estimators of the parameter $(\boldsymbol{\Theta}', \boldsymbol{\beta}')'$ from Lemma 1.1 and Lemma 1.2 are the same. The estimator of $\boldsymbol{\beta}$ from Lemma 1.3 is the same as the estimator from Lemma 1.1 and 1.2. They satisfy the constraints

$$\boldsymbol{a} + \mathbf{C}\hat{\hat{\Theta}} + \mathbf{B}\hat{\hat{\beta}} = \boldsymbol{O}.$$

Lemmas 1.1, 1.2 and 1.3 give equivalent estimation algorithms.

Equivalent algorithms are frequently utilized in practice because of numerical check of the either tedious numerical calculation or a calculation of a great importance; e.g. when the state levelling network in late Czechoslovakia was created such a check was utilized many times.

In many cases the constraints

$$\boldsymbol{a} + \mathbf{C}\hat{\Theta} + \mathbf{B}\tilde{\beta} = \boldsymbol{O},$$

where $\tilde{\boldsymbol{\beta}}$ is an unbiased estimator, must be satisfied, since the estimate $\hat{\Theta}$ must not be changed into $\hat{\hat{\Theta}}$. In such a case the Lemmas 1.1, 1.2 and 1.3 are of no use (this situation occurred also during calculation of the state levelling network in Czechoslovakia).

For a solution of such a case the two following lemmas will help us.

Lemma 1.5. *The class $\mathcal{U}_\beta$ of all unbiased estimators $\tilde{\boldsymbol{\beta}}$ of the parameter $\boldsymbol{\beta}$ based on the vectors $\hat{\Theta}$ and $\boldsymbol{Y} - \mathbf{D}\hat{\Theta}$ in the model $\boldsymbol{Y} - \mathbf{D}\hat{\Theta} \sim_n (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_{2,2} + \mathbf{DVD}')$, $\boldsymbol{a} + \mathbf{C}\Theta + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{O}$, is*

$$\mathcal{U}_\beta = \Big\{\big[\mathbf{X}_2^- + \mathbf{Z}(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{EBX}_2^-\big](\boldsymbol{Y} - \mathbf{D}\hat{\Theta}) + \mathbf{EC}\hat{\Theta} + \mathbf{E}\boldsymbol{a} : \mathbf{Z} \text{ an arbitrary } k \times n \text{ matrix}, \mathbf{E} \text{ an arbitrary } k \times q \text{ matrix}\Big\},$$

where $\mathbf{X}_2^-$ is an arbitrary however fixed g-inverse (cf. in more detail in [9]) of the matrix $\mathbf{X}_2$.

Proof. Cf. [3; Lemma 8.3.3] and [2]. □

Lemma 1.6. *The class of all linear unbiased estimators $\tilde{\boldsymbol{\beta}}$, based on the vectors $\hat{\Theta}$ and $\boldsymbol{Y} - \mathbf{D}\hat{\Theta}$, satisfying the condition $\mathbf{C}\hat{\Theta} + \mathbf{B}\tilde{\boldsymbol{\beta}} + \boldsymbol{a} = \boldsymbol{O}$ is*

$$\begin{aligned}\tilde{\mathcal{U}}_\beta = \Big\{&(\mathbf{I} - \mathbf{B}^-\mathbf{B})\big[\mathbf{X}_2^- + \mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{BX}_2^-\big](\boldsymbol{Y} - \mathbf{D}\hat{\Theta}) \\ &+ \big[-\mathbf{B}^- + (\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{W}_2\big]\mathbf{C}\hat{\Theta} + (\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{W}_2\boldsymbol{a} - \mathbf{B}^-\boldsymbol{a} : \\ &\mathbf{W}_1 \text{ an arbitrary } k \times n \text{ matrix}, \ \mathbf{W}_2 \text{ an arbitrary } k \times q \text{ matrix}\Big\}.\end{aligned}$$

Here $\mathbf{X}_2^-$ and $\mathbf{B}^-$ are arbitrary, however, fixed g-inverse matrices.

The covariance matrix of the estimator $\tilde{\boldsymbol{\beta}} \in \tilde{\mathcal{U}}_\beta$ is

$$\begin{aligned}\operatorname{Var}(\tilde{\boldsymbol{\beta}}) &= \operatorname{Var}\big[\mathbf{M}_1(\boldsymbol{Y} - \mathbf{D}\hat{\Theta})\big] + \operatorname{Var}\big[\mathbf{M}_2\hat{\Theta}\big] + \operatorname{cov}\big[\mathbf{M}_1(\boldsymbol{Y} - \mathbf{D}\hat{\Theta}), \mathbf{M}_2\hat{\Theta}\big] \\ &\quad + \operatorname{cov}\big[\mathbf{M}_2\hat{\Theta}, \mathbf{M}_1(\boldsymbol{Y} - \mathbf{D}\hat{\Theta})\big] \\ &= \mathbf{M}_1(\boldsymbol{\Sigma}_{2,2} + \mathbf{DVD}')\mathbf{M}_1' + \mathbf{M}_2\mathbf{VM}_2' - \mathbf{M}_1\mathbf{DVM}_2' - \mathbf{M}_2\mathbf{VD}'\mathbf{M}_1',\end{aligned}$$

where

$$\begin{aligned}\mathbf{M}_1 &= (\mathbf{I} - \mathbf{B}^-\mathbf{B})\big[\mathbf{X}_2^- + \mathbf{W}_1(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) + \mathbf{W}_2\mathbf{BX}_2^-\big], \\ \mathbf{M}_2 &= \big[-\mathbf{B}^- + (\mathbf{I} - \mathbf{B}_2^-\mathbf{B})\mathbf{W}_2\big]\mathbf{C}.\end{aligned}$$

In this class, there does not exist the joint efficient linear unbiased estimator of the parameter $\boldsymbol{\beta}$.

Proof. Cf. [3; Lemma 8.3.4 and further pp.] and [2]. □

Lemma 1.7. *The estimator*

$$\bar{\boldsymbol{\beta}} =$$
$$= \Big(\mathbf{I} - \big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{B}'\{\mathbf{B}\big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{B}'\}^{-1}\mathbf{B}\Big)\times$$
$$\times \big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}(\boldsymbol{Y}-\mathbf{D}\hat{\boldsymbol{\Theta}})$$
$$- \big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{B}'\Big\{\mathbf{B}\big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{B}'\Big\}^{-1}(\mathbf{C}\hat{\boldsymbol{\Theta}}+\boldsymbol{a})$$

is an element of the class $\tilde{\mathcal{U}}_{\beta}$.

Proof. The estimator $\bar{\boldsymbol{\beta}}$ is a standard BLUE in the model

$$\boldsymbol{Y}-\mathbf{D}\hat{\boldsymbol{\Theta}} \sim_n (\mathbf{X}_2\boldsymbol{\beta}, \Sigma_{2,2}+\mathbf{DVD}')$$

with constraints $\boldsymbol{a}+\mathbf{C}\hat{\boldsymbol{\Theta}}+\mathbf{B}\boldsymbol{\beta}=\boldsymbol{O}$. In the constraints the vector $\hat{\boldsymbol{\Theta}}$ is considered as a nonrandom vector. If in Lemma 1.6 we choose

$$\mathbf{W}_1 = \mathbf{O},$$
$$\mathbf{W}_2 = \mathbf{B}^- = \big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{B}'\Big\{\mathbf{B}\big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{B}\Big\}^{-1},$$
$$\mathbf{X}_2^- = \big[\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'(\Sigma_{2,2}+\mathbf{DVD}')^{-1},$$

we also obtain the considered estimator. □

Lemma 1.8. *The estimator*

$$\bar{\boldsymbol{\beta}} =$$
$$= \Big\{\mathbf{I} - (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\big]^{-1}\mathbf{B}\Big\}(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1}(\boldsymbol{Y}-\mathbf{D}\hat{\boldsymbol{\Theta}})$$
$$-(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\big]^{-1}(\mathbf{C}\hat{\boldsymbol{\Theta}}+\boldsymbol{a})$$

is a standard BLUE *in the model* $\boldsymbol{Y}-\mathbf{D}\boldsymbol{\Theta} \sim_n (\mathbf{X}_2\boldsymbol{\beta}, \Sigma_{2,2})$ *and it is an element of the class* $\tilde{\mathcal{U}}_{\beta}$.

Proof. If in Lemma 1.6 we choose

$$\mathbf{W}_1 = \mathbf{O},$$
$$\mathbf{W}_2 = \mathbf{B}^- = (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}\big]^{-1},$$
$$\mathbf{X}_2^- = (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1},$$

we obtain the considered estimator. □

Remark 1.9. The covariance matrix of the estimator from Lemma 1.8 is

$$\operatorname{Var}(\bar{\boldsymbol{\beta}}) = \operatorname{Var}\big[\mathbf{L}_1(\boldsymbol{Y}-\mathbf{D}\hat{\boldsymbol{\Theta}})\big] + \operatorname{Var}\big[\mathbf{L}_2(\mathbf{C}\hat{\boldsymbol{\Theta}}+\boldsymbol{a})\big] + \operatorname{cov}\big[\mathbf{L}_1(\boldsymbol{Y}-\mathbf{D}\hat{\boldsymbol{\Theta}}), \mathbf{L}_2(\mathbf{C}\hat{\boldsymbol{\Theta}}+\boldsymbol{a})\big]$$
$$+ \operatorname{cov}\big[\mathbf{L}_2(\mathbf{C}\hat{\boldsymbol{\Theta}}+\boldsymbol{a}), \mathbf{L}_1(\boldsymbol{Y}-\mathbf{D}\hat{\boldsymbol{\Theta}})\big]$$
$$= \mathbf{L}_1(\Sigma_{2,2}+\mathbf{DVD}')\mathbf{L}_1' + \mathbf{L}_2\mathbf{CVC}'\mathbf{L}_2' - \mathbf{L}_1\mathbf{DVD}'\mathbf{L}_2' - \mathbf{L}_2\mathbf{CVD}'\mathbf{L}_1',$$

where

$$\mathbf{L}_1 = \left\{\mathbf{I} - (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\left[\mathbf{B}(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\right]^{-1}\mathbf{B}\right\}\times$$
$$\times (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1},$$
$$\mathbf{L}_2 = -(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\left[\mathbf{B}(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\right]^{-1}.$$

Since in the class $\tilde{\mathcal{U}}_\beta$ there does not exist the joint efficient estimator, the only possibility is to find an unbiased estimator which minimizes the variance of an estimator of some linear functional of the parameter $\boldsymbol{\beta}$. This problem is solved in the next section.

2. H-optimum estimator in the case of constraints I

Definition 2.1. Let $\mathbf{H}$ be a given $k \times k$ positive semidefinite matrix. The estimator $\tilde{\boldsymbol{\beta}}$ from $\tilde{\mathcal{U}}_\beta$ is said to be $\mathbf{H}$-*optimum* if it minimizes the value $\operatorname{Tr}\left[\mathbf{H}\operatorname{Var}(\tilde{\boldsymbol{\beta}})\right]$.

Lemma 2.2. *Let $\tilde{\boldsymbol{\beta}}$ be an $\mathbf{H}$-optimum estimator from $\tilde{\mathcal{U}}_\beta$. Then the matrices $\mathbf{W}_1$ and $\mathbf{W}_2$ from Lemma 1.6 are solution of the equation*

$$\mathbf{U}_1(\mathbf{W}_1,\mathbf{W}_2)\begin{pmatrix}\mathbf{V}_1, & \mathbf{T}_1\\ \mathbf{V}_2, & \mathbf{T}_2\end{pmatrix} = (\mathbf{P}_1,\mathbf{P}_2),$$

where

$$\begin{aligned}
\mathbf{U}_1 &= \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}(\mathbf{I} - \mathbf{B}^-\mathbf{B}),\\
\mathbf{V}_1 &= (\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)(\Sigma_{2,2} + \mathbf{DVD}')\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right],\\
\mathbf{V}_2 &= \mathbf{BX}_2^-(\Sigma_{2,2} + \mathbf{DVD}')\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right] - \mathbf{CVD}'\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right],\\
\mathbf{P}_1 &= -\left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}(\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{X}_2^-(\Sigma_{2,2} + \mathbf{DVD}')\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right]\\
&\quad - \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{HB}^-\mathbf{CVD}'\left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right],\\
\mathbf{T}_1 &= \left[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'\right]\left[(\Sigma_{2,2} + \mathbf{DVD}')(\mathbf{X}_2^-)'\mathbf{B}' - \mathbf{DVC}'\right],\\
\mathbf{T}_2 &= \mathbf{BX}_2^-(\Sigma_{2,2} + \mathbf{DVD}')(\mathbf{X}_2^-)'\mathbf{B}' + \mathbf{CVC}' - \mathbf{CVD}'(\mathbf{X}_2^-)'\mathbf{B}' - \mathbf{BX}_2^-\mathbf{DVC}',\\
\mathbf{P}_2 &= -\left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}(\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{X}_2^-(\Sigma_{2,2} + \mathbf{DVD}')(\mathbf{X}_2^-)'\mathbf{B}'\\
&\quad + \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{HB}^-\mathbf{CVC}' - \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{HB}^-\mathbf{CVD}'(\mathbf{X}_2^-)'\mathbf{B}'\\
&\quad + \left[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)'\right]\mathbf{H}(\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{X}_2^-\mathbf{DVC}'.
\end{aligned}$$

Proof. Cf. [3; Theorem 8.3.10] and [2]. □

3. Some auxiliary statements for constraints II

Let in the model (1) & (3) the parameter $\boldsymbol{\Theta}$ be known. For the sake of simplicity the second stage of the model (1), i.e.

$$\boldsymbol{Y} - \mathbf{D}\boldsymbol{\Theta} \sim_n (\mathbf{X}_2\boldsymbol{\beta}, \Sigma_{2,2}), \qquad \boldsymbol{a}^* + \mathbf{C}^*\boldsymbol{\Theta} + \mathbf{B}^*\boldsymbol{\beta} + \mathbf{G}\boldsymbol{\gamma} = \boldsymbol{O}, \tag{5}$$

is under consideration.

Lemma 3.1. *The* BLUE *of the parameter* $(\boldsymbol{\beta}', \boldsymbol{\gamma}')'$ *in the model* (5) *is*

$$\begin{aligned}
\hat{\hat{\boldsymbol{\beta}}} = \Big(\mathbf{I} &- (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\Big\{\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1} \\
&- \big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{G}\Big\{\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\times \\
&\times (\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\Big\}\mathbf{B}^*\Big)\times \\
&\times (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1}(\boldsymbol{Y} - \mathbf{D}\boldsymbol{\Theta}) \\
&- (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\Big\{\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1} \\
&- \big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{G}\Big\{\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' \\
&+ \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\Big\}(\boldsymbol{a}^* + \mathbf{C}^*\boldsymbol{\Theta}),
\end{aligned}$$

and

$$\begin{aligned}
\hat{\hat{\boldsymbol{\gamma}}} = -&\Big\{\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}'\times \\
&\times \big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\times \\
&\times \big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1}\boldsymbol{Y} + \boldsymbol{a}^* + \mathbf{C}^*\boldsymbol{\Theta}\big].
\end{aligned}$$

Their covariance matrices and cross covariance matrix are

$$\begin{aligned}
\operatorname{Var}(\hat{\hat{\boldsymbol{\beta}}}) = &(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1} - (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\times \\
&\times (\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1} + (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\times \\
&\times (\mathbf{B}^*)'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\times \\
&\times \mathbf{G}\Big\{\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}'\times \\
&\times \big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1},
\end{aligned}$$

$$\begin{aligned}
\operatorname{cov}(\hat{\hat{\boldsymbol{\beta}}}, \hat{\hat{\boldsymbol{\gamma}}}) = &-(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\times \\
&\times \mathbf{G}\Big\{\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{G}\Big\}^{-1},
\end{aligned}$$

$$\mathrm{Var}(\hat{\hat{\gamma}}) = \left\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\right\}^{-1} - \mathbf{I}.$$

P r o o f . When the least squares method is used, then the standard procedure is (where $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers)

$$\Phi(\boldsymbol{\beta},\boldsymbol{\gamma}) = (\boldsymbol{Y} - \mathbf{X}_2\boldsymbol{\beta})'\Sigma_{2,2}^{-1}(\boldsymbol{Y} - \mathbf{X}_2\boldsymbol{\beta}) - 2\boldsymbol{\lambda}'(\mathbf{B}^*\boldsymbol{\beta} + \mathbf{G}\boldsymbol{\gamma} + \boldsymbol{a} + \mathbf{C}^*\boldsymbol{\Theta}),$$

$$\frac{\partial\Phi(\boldsymbol{\beta},\boldsymbol{\gamma})}{\partial\boldsymbol{\beta}} = \boldsymbol{O} \quad \& \quad \frac{\partial\Phi(\boldsymbol{\beta},\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}} = \boldsymbol{O}$$

$$\Longrightarrow \hat{\hat{\boldsymbol{\beta}}} = \hat{\boldsymbol{\beta}} + (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\boldsymbol{\lambda}, \quad \hat{\boldsymbol{\beta}} = (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1}(\boldsymbol{Y} - \mathbf{D}\boldsymbol{\Theta}),$$

$$\begin{pmatrix} \mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)', & \mathbf{G} \\ \mathbf{G}', & \mathbf{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \hat{\hat{\boldsymbol{\gamma}}} \end{pmatrix} = \begin{pmatrix} -(\boldsymbol{a}^* + \mathbf{C}^*\boldsymbol{\Theta} + \mathbf{B}^*\hat{\boldsymbol{\beta}}) \\ \mathbf{O} \end{pmatrix}.$$

With respect to Pandora-Box theorem (cf. [9]) we have

$$\begin{pmatrix} \mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)', & \mathbf{G} \\ \mathbf{G}', & \mathbf{O} \end{pmatrix}^{-} = \begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix},$$

where

$$\boxed{1} = \left[\mathbf{M}_{\mathbf{G}}\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\mathbf{M}_{\mathbf{G}}\right]^{+},$$

$$\boxed{2} = (\mathbf{G}')^{-}_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)']},$$

$$\boxed{3} = \left[(\mathbf{G}')^{-}_{m[\mathbf{B}^*(\mathbf{X}'\Sigma_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)']}\right]',$$

$$\boxed{4} = -\left[(\mathbf{G}')^{-}_{m[\mathbf{B}^*(\mathbf{X}'\Sigma_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)']}\right]'(\mathbf{B}^*)'(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\times$$
$$\times\, \mathbf{B}^*(\mathbf{G}')^{-}_{m[\mathbf{B}^*(\mathbf{X}'\Sigma_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)']}.$$

When the equalities

$$\left[\mathbf{M}_{\mathbf{G}}\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\mathbf{M}_{\mathbf{G}}\right]^{+}$$
$$= \left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} - \left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\times$$
$$\times\, \mathbf{G}\left\{\mathbf{G}\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}'\right\}^{-1}\times$$
$$\times\, \mathbf{G}\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1},$$

$$(\mathbf{G}')^{-}_{m[\mathbf{B}^*(\mathbf{X}'\Sigma_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)']}$$
$$= \left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\left\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\right\}^{-1}$$

and

$$-\left[(\mathbf{G}')^-_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)']}\right]'\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'(\mathbf{G}')^-_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)']}$$
$$= \left\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\right\}^{-1} - \mathbf{I}$$

are used, then we can easily finish the proof. □

Remark 3.2. The least squares estimator of the parameter β and γ obtained under the condition $\mathbf{V} = \mathbf{O}$ ($\Longrightarrow \mathrm{Var}(\hat{\Theta}) = \mathbf{O}$) is called the *standard estimator* if in this estimator the vector Θ is substituted by $\hat{\Theta}$. Thus if Θ in Lemma 3.1 is substituted by $\hat{\Theta}$, the standard estimator is obtained. Its covariance matrix is given by the following relationships.

$$\begin{aligned}\mathrm{Var}(\hat{\beta}) &= \mathrm{Var}\left[\mathbf{N}_1(\boldsymbol{Y} - \mathbf{D}\hat{\Theta})\right] + \mathrm{Var}\left[\mathbf{N}_2(\mathbf{C}^*\hat{\Theta} + \boldsymbol{a})\right] \\ &\quad + \mathrm{cov}\left[\mathbf{N}_1(\boldsymbol{Y} - \mathbf{D}\hat{\Theta}), \mathbf{N}_2(\mathbf{C}^*\hat{\Theta} + \boldsymbol{a})\right] + \mathrm{cov}\left[\mathbf{N}_2(\mathbf{C}^*\hat{\Theta} + \boldsymbol{a}), \mathbf{N}_1(\boldsymbol{Y} - \mathbf{D}\hat{\Theta})\right] \\ &= \mathbf{N}_1(\Sigma_{2,2} + \mathbf{DVD}')\mathbf{N}_1' + \mathbf{N}_2\mathbf{C}^*\mathbf{V}(\mathbf{C}^*)'\mathbf{N}_2' - \mathbf{N}_1\mathbf{DV}(\mathbf{C}^*)'\mathbf{N}_2' - \mathbf{N}_2\mathbf{C}^*\mathbf{VD}'\mathbf{N}_1',\end{aligned}$$

where

$$\begin{aligned}\mathbf{N}_1 = \big(\mathbf{I} - (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big)\Big\{&\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} \\ &- \left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} \times \\ &\times \mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}(\mathbf{B}^*)'(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1}\Big\},\end{aligned}$$

$$\begin{aligned}\mathbf{N}_2 = -(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\Big\{&\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} \\ &- \left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G} \times \\ &\times \left\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\right\}^{-1}\mathbf{G}' \times \\ &\times \left[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\Big\}.\end{aligned}$$

THEOREM 3.3. *In the model*

$$\boldsymbol{Y} - \mathbf{D}\hat{\Theta} \sim_n (\mathbf{X}_2\beta, \Sigma_{2,2} + \mathbf{DVD}'), \qquad \boldsymbol{a}^* + \mathbf{C}^*\Theta + \mathbf{B}^*\beta + \mathbf{G}\gamma = \boldsymbol{O},$$

the class of all unbiased linear estimators of $\begin{pmatrix}\beta\\ \gamma\end{pmatrix}$ *based on the vectors* $\hat{\Theta}$ *and* $\boldsymbol{Y} - \mathbf{D}\hat{\Theta}$ *is*

$$\mathcal{U}_{\beta,\gamma} = \left\{\begin{pmatrix}\tilde{\beta}\\ \tilde{\gamma}\end{pmatrix} : \begin{pmatrix}\tilde{\beta}\\ \tilde{\gamma}\end{pmatrix} = \begin{pmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{pmatrix} + \begin{pmatrix}\mathbf{K}_1, \mathbf{K}_2\\ \mathbf{K}_3, \mathbf{K}_4\end{pmatrix}\begin{pmatrix}\hat{\Theta}\\ \boldsymbol{Y} - \mathbf{D}\hat{\Theta}\end{pmatrix}\right\},$$

where

$$\begin{pmatrix} \boldsymbol{k}_1 \\ \boldsymbol{k}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{O} \\ -\mathbf{G}^- \end{pmatrix} \boldsymbol{a}^* + \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_3 \end{pmatrix} (\mathbf{I} - \mathbf{G}\mathbf{G}^-)\boldsymbol{a}^*,$$

$$\begin{pmatrix} \mathbf{K}_1, & \mathbf{K}_2 \\ \mathbf{K}_3, & \mathbf{K}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{O}, & \mathbf{X}_2^- \\ -\mathbf{G}^-\mathbf{C}^*, & -\mathbf{G}^-\mathbf{B}^*\mathbf{X}_2^- \end{pmatrix}$$

$$+ \begin{pmatrix} \mathbf{Z}_1, & \mathbf{Z}_2 \\ \mathbf{Z}_3, & \mathbf{Z}_4 \end{pmatrix} \begin{pmatrix} (\mathbf{I} - \mathbf{G}\mathbf{G}^-)\mathbf{C}^*, & (\mathbf{I} - \mathbf{G}\mathbf{G}^-)\mathbf{B}^*\mathbf{X}_2^- \\ \mathbf{O}, & \mathbf{I} - \mathbf{X}_2\mathbf{X}_2^- \end{pmatrix},$$

where $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$ *are arbitrary matrices with suitable dimensions.*

The covariance matrix of the estimator $\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix} \in \mathcal{U}_{\beta,\gamma}$ *is*

$$\operatorname{Var}\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_1, & \mathbf{K}_2 \\ \mathbf{K}_3, & \mathbf{K}_4 \end{pmatrix} \begin{pmatrix} \mathbf{V}, & -\mathbf{V}\mathbf{D}' \\ -\mathbf{D}\mathbf{V}, & \Sigma_{2,2} + \mathbf{D}\mathbf{V}\mathbf{D}' \end{pmatrix} \begin{pmatrix} \mathbf{K}_1, & \mathbf{K}_2 \\ \mathbf{K}_3, & \mathbf{K}_4 \end{pmatrix}'.$$

P r o o f . The linear unbiased estimator of the parameter $\begin{pmatrix} \beta \\ \gamma \end{pmatrix}$

$$\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \boldsymbol{k}_1 \\ \boldsymbol{k}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{K}_1, & \mathbf{K}_2 \\ \mathbf{K}_3, & \mathbf{K}_4 \end{pmatrix} \begin{pmatrix} \hat{\Theta} \\ \boldsymbol{Y} - \mathbf{D}\hat{\Theta} \end{pmatrix},$$

which is based on the vector $\begin{pmatrix} \hat{\Theta} \\ \boldsymbol{Y} - \mathbf{D}\hat{\Theta} \end{pmatrix}$, must satisfy the relationship

$$E\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \boldsymbol{k}_1 \\ \boldsymbol{k}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{K}_1, & \mathbf{K}_2 \\ \mathbf{K}_3, & \mathbf{K}_4 \end{pmatrix} \begin{pmatrix} \Theta \\ \mathbf{X}_2\beta \end{pmatrix},$$

what is equivalent with the equality

$$\begin{pmatrix} \boldsymbol{k}_1 \\ \boldsymbol{k}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{K}_1\Theta + (\mathbf{K}_2\mathbf{X}_2 - \mathbf{I})\beta \\ \mathbf{K}_3\Theta + \mathbf{K}_4\mathbf{X}_2\beta - \mathbf{I}\gamma \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{O} \end{pmatrix}.$$

Regarding the structure of the parametric space, it must be valid that

$$\begin{aligned} &\boldsymbol{k}_1 = \mathbf{U}_1\boldsymbol{a}^*, \quad && \mathbf{K}_1 = \mathbf{U}_1\mathbf{C}^*, \quad && \mathbf{K}_2\mathbf{X}_2 - \mathbf{I} = \mathbf{U}_1\mathbf{B}^*, \quad && \mathbf{O} = \mathbf{U}_1\mathbf{G}, \\ &\boldsymbol{k}_2 = \mathbf{U}_2\boldsymbol{a}^*, && \mathbf{K}_3 = \mathbf{U}_2\mathbf{C}^*, && \mathbf{K}_4\mathbf{X}_2 = \mathbf{U}_2\mathbf{B}^*, && -\mathbf{I} = \mathbf{U}_2\mathbf{G} \end{aligned}$$

for some matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ of the types $k_1 \times q$ and $k_2 \times q$, respectively.

The equation $\mathbf{O} = \mathbf{U}_1\mathbf{G}$ implies

$$\mathbf{U}_1 = \mathbf{Z}_1(\mathbf{I} - \mathbf{G}\mathbf{G}_0^-),$$

where the $k_1 \times q$ matrix $\mathbf{Z}_1$ is arbitrary. From the equation $\mathbf{U}_2\mathbf{G} = -\mathbf{I}$ we obtain analogously

$$\mathbf{U}_2 = -\mathbf{G}_0^- + \mathbf{Z}_3(\mathbf{I} - \mathbf{G}\mathbf{G}^-)$$

(it is to be remarked that assumption on regularity implies $\mathbf{G}_0^-\mathbf{G} = \mathbf{I}$). Further

$$\mathbf{K}_2 = (\mathbf{I} + \mathbf{U}_1\mathbf{B}^*)\mathbf{X}_2^- + \mathbf{Z}_2(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-),$$
$$\mathbf{K}_4 = \mathbf{U}_2\mathbf{B}^*\mathbf{X}_2^- + \mathbf{Z}_4(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-).$$

Now the proof can be easily finished. □

LEMMA 3.4. *Let in Lemma* 3.1, Θ *be substituted by* $\hat{\Theta}$. *Then such estimator* (*it is usually used in practice*) *belong to the class* $\mathcal{U}_{\beta,\gamma}$.

P r o o f . In Theorem 3.3 we choose

$$\mathbf{Z}_1 = -(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big]^{-1}, \qquad \mathbf{Z}_3 = \mathbf{O},$$
$$\mathbf{Z}_2 = \mathbf{O}, \qquad \mathbf{Z}_4 = \mathbf{O},$$

and

$$\mathbf{G}_0^- = \Big\{\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}'\big[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big]^{-1}$$
$$= \Big[(\mathbf{G}')^-_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)']}\Big]',$$

$$\mathbf{X}_2^- = (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1}.$$

Thus we obtain

$$\begin{pmatrix} \boldsymbol{k}_1 \\ \boldsymbol{k}_2 \end{pmatrix} = \begin{pmatrix} -(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big[\mathbf{M}_G\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big]^+ \\ -\Big[(\mathbf{G}')^-_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)']}\Big]' \end{pmatrix} \boldsymbol{a}^*$$

and

$$\mathbf{K}_1 = -(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big[\mathbf{M}_\mathbf{G}\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big]^+\mathbf{C}^*,$$
$$\mathbf{K}_2 = \mathbf{I} - (\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big[\mathbf{M}_\mathbf{G}\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)'\big]^+ \times$$
$$\times \mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_{2,2}^{-1},$$
$$\mathbf{K}_3 = -\Big[(\mathbf{G}')^-_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)']}\Big]'\mathbf{C}^*,$$
$$\mathbf{K}_4 = -\Big[(\mathbf{G}')^-_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)']}\Big]'\mathbf{B}^*\mathbf{X}_2^-,$$

what proves the statement. □

Remark 3.5. Estimators from the class $\mathcal{U}_{\beta,\gamma}$ need not satisfy the constraints

$$\boldsymbol{a}^* + \mathbf{C}^*\hat{\Theta} + \mathbf{B}^*\tilde{\beta} + \mathbf{G}\tilde{\gamma} = \boldsymbol{O}.$$

If the estimator from Theorem 3.2 $\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix}$ is substituted into expression $\boldsymbol{a}^* + \mathbf{C}^*\hat{\Theta} + (\mathbf{B}^*, \mathbf{G})\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix}$, we obtain

$$(\mathbf{I} + \mathbf{B}^*\mathbf{Z}_1 + \mathbf{G}\mathbf{Z}_3)(\mathbf{I} - \mathbf{G}\mathbf{G}^-)\Big\{\boldsymbol{a}^* + \mathbf{C}^*\hat{\Theta} + \mathbf{B}^*\mathbf{X}_2^-(\boldsymbol{Y} - \mathbf{D}\hat{\Theta})$$
$$-\mathbf{G}\Big[(\mathbf{G}')^-_{m[\mathbf{B}^*(\mathbf{X}_2'\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}(\mathbf{B}^*)']}\Big]'(\mathbf{B}^*\tilde{\beta} + \boldsymbol{a}^* + \mathbf{C}^*\hat{\Theta})\Big\}$$
$$-(\mathbf{B}^*\mathbf{Z}_2 + \mathbf{G}\mathbf{Z}_4)(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)(\boldsymbol{Y} - \mathbf{D}\hat{\Theta}).$$

It is obvious that it cannot be in general a zero vector.

THEOREM 3.6. *The class* $\tilde{\mathcal{U}}_{\beta,\gamma}$ *of all linear unbiased estimators which in addition satisfy the constraints*

$$\boldsymbol{a}^* + \mathbf{C}^*\hat{\Theta} + \mathbf{B}^*\tilde{\beta} + \mathbf{G}\tilde{\gamma} = \boldsymbol{O}$$

is given by such a choice of the matrices $\mathbf{Z}_1, \dots, \mathbf{Z}_4$ *in Theorem* 3.3 *which satisfy the following equation*

$$\begin{pmatrix} \mathbf{Z}_1, & \mathbf{Z}_2 \\ \mathbf{Z}_3, & \mathbf{Z}_4 \end{pmatrix} =$$

$$= (\mathbf{B}^*, \mathbf{G})^- [-(\mathbf{I} - \mathbf{G}\mathbf{G}^-), \mathbf{O}] \begin{pmatrix} \mathbf{I} - \mathbf{G}\mathbf{G}^-, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} - \mathbf{X}_2\mathbf{X}_2^- \end{pmatrix}$$

$$+ \begin{pmatrix} \mathbf{W}_1, & \mathbf{W}_2 \\ \mathbf{W}_3, & \mathbf{W}_4 \end{pmatrix} - (\mathbf{B}^*, \mathbf{G})^-(\mathbf{B}^*, \mathbf{G}) \begin{pmatrix} \mathbf{W}_1, & \mathbf{W}_2 \\ \mathbf{W}_3, & \mathbf{W}_4 \end{pmatrix} \begin{pmatrix} \mathbf{I} - \mathbf{G}\mathbf{G}^-, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} - \mathbf{X}_2\mathbf{X}_2^- \end{pmatrix},$$

where the matrices $\mathbf{W}_1, \dots, \mathbf{W}_4$ *are arbitrary.*

P r o o f . If the estimator (Theorem 3.3)

$$\begin{pmatrix} \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_1(\mathbf{I} - \mathbf{G}\mathbf{G}^-) \\ -\mathbf{G}^- + \mathbf{Z}_3(\mathbf{I} - \mathbf{G}\mathbf{G}^-) \end{pmatrix} \boldsymbol{a} + \left[\begin{pmatrix} \mathbf{O}, & \mathbf{X}_2^- \\ -\mathbf{G}^-\mathbf{C}^*, & -\mathbf{G}^-\mathbf{B}^*\mathbf{X}_2^- \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \mathbf{Z}_1, & \mathbf{Z}_2 \\ \mathbf{Z}_3, & \mathbf{Z}_4 \end{pmatrix} \begin{pmatrix} (\mathbf{I} - \mathbf{G}\mathbf{G}^-)\mathbf{C}^*, & (\mathbf{I} - \mathbf{G}\mathbf{G}^-)\mathbf{B}^*\mathbf{X}_2^- \\ \mathbf{O}, & \mathbf{I} - \mathbf{X}_2\mathbf{X}_2^- \end{pmatrix} \right] \begin{pmatrix} \hat{\Theta} \\ \boldsymbol{Y} - \mathbf{D}\hat{\Theta} \end{pmatrix}$$

satisfies the constraints $\boldsymbol{a}^* + \mathbf{C}^*\hat{\Theta} + \mathbf{B}^*\tilde{\beta} + \mathbf{G}\tilde{\gamma} = \boldsymbol{O}$, then

$$(\mathbf{I} + \mathbf{B}^*\mathbf{Z}_1 + \mathbf{G}\mathbf{Z}_3)(\mathbf{I} - \mathbf{G}\mathbf{G}^-)[\boldsymbol{a}^* + \mathbf{C}^*\hat{\Theta} + \mathbf{B}^*\mathbf{X}_2^-(\boldsymbol{Y} - \mathbf{D}\hat{\Theta})]$$
$$+ (\mathbf{B}^*\mathbf{Z}_2 + \mathbf{G}\mathbf{Z}_4)(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)(\boldsymbol{Y} - \mathbf{D}\hat{\Theta}) = \boldsymbol{O}$$

must be valid, what is equivalent with the following relationships

$$(\mathbf{I} + \mathbf{B}^*\mathbf{Z}_1 + \mathbf{G}\mathbf{Z}_3)(\mathbf{I} - \mathbf{G}\mathbf{G}^-) = \mathbf{O} \quad \& \quad (\mathbf{B}^*\mathbf{Z}_2 + \mathbf{G}\mathbf{Z}_4)(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-) = \mathbf{O}.$$

The matrices $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$, which satisfy the last two relationships, are solutions of the equations

$$(\mathbf{B}^*, \mathbf{G}) \begin{pmatrix} \mathbf{Z}_1, & \mathbf{Z}_2 \\ \mathbf{Z}_3, & \mathbf{Z}_4 \end{pmatrix} \begin{pmatrix} \mathbf{I} - \mathbf{G}\mathbf{G}^-, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} - \mathbf{X}_2\mathbf{X}_2^- \end{pmatrix} = [-(\mathbf{I} - \mathbf{G}\mathbf{G}^-), \mathbf{O}]. \qquad (6)$$

Since

$$(\mathbf{B}^*, \mathbf{G})(\mathbf{B}^*, \mathbf{G})^- [-(\mathbf{I} - \mathbf{G}\mathbf{G}^-), \mathbf{O}] \begin{pmatrix} \mathbf{I} - \mathbf{G}\mathbf{G}^-, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} - \mathbf{X}_2\mathbf{X}_2^- \end{pmatrix}$$
$$= (\mathbf{B}^*, \mathbf{G})(\mathbf{B}, \mathbf{G})^- [-(\mathbf{I} - \mathbf{G}\mathbf{G}^-), \mathbf{O}]$$
$$= [-(\mathbf{I} - \mathbf{G}\mathbf{G}^-), \mathbf{O}],$$

the considered equation is solvable (it is to be remarked that $(\mathbf{B}, \mathbf{G})(\mathbf{B}, \mathbf{G})^- = \mathbf{I}$). The general solution of the equation is given in the statement of the theorem (the equalities $(\mathbf{I} - \mathbf{G}\mathbf{G}^-)^- = \mathbf{I} - \mathbf{G}\mathbf{G}^-$ a $(\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)^- = \mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-$ are utilized here). $\square$

4. $\mathbf{H}^*$-optimum estimator for constraints II

DEFINITION 4.1. Let $\mathbf{H}^*$ be a given $(k+l)\times(k+l)$ positive semidefinite matrix. The estimator $\begin{pmatrix}\tilde{\beta}\\ \tilde{\gamma}\end{pmatrix}$ from $\tilde{\mathcal{U}}_{\beta,\gamma}$ is said to be $\mathbf{H}^*$-*optimum* if it minimizes the value $\operatorname{Tr}\left[\mathbf{H}^* \operatorname{Var}\begin{pmatrix}\tilde{\beta}\\ \tilde{\gamma}\end{pmatrix}\right]$.

THEOREM 4.2. *An estimator $\begin{pmatrix}\tilde{\beta}\\ \tilde{\gamma}\end{pmatrix}$ is $\mathbf{H}^*$-optimum if the matrices $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$ (Theorem 3.6) are solution of the equation*

$$\begin{aligned}\left\{\mathbf{I}-(\mathbf{B}^*,\mathbf{G})'[(\mathbf{B}^*,\mathbf{G})^-]'\right\}\mathbf{H}^*\left[\mathbf{I}-(\mathbf{B}^*,\mathbf{G})^-(\mathbf{B}^*,\mathbf{G})\right]\mathbf{WSTS}'\\ = -\left\{\mathbf{I}-(\mathbf{B}^*,\mathbf{G})'[(\mathbf{B}^*,\mathbf{G})^-]'\right\}\mathbf{H}^*(\mathbf{RTS}'+\mathbf{ASTS}'),\end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathbf{A} &= (\mathbf{B}^*,\mathbf{G})^-\left[-(\mathbf{I}-\mathbf{GG}^-),\mathbf{O}\right]\begin{pmatrix}\mathbf{I}-\mathbf{GG}^-, & \mathbf{O}\\ \mathbf{O}, & \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-\end{pmatrix},\\ \mathbf{W} &= \begin{pmatrix}\mathbf{W}_1, & \mathbf{W}_2\\ \mathbf{W}_3, & \mathbf{W}_4\end{pmatrix},\\ \mathbf{R} &= \begin{pmatrix}\mathbf{O}, & \mathbf{X}_2^-\\ -\mathbf{G}^-\mathbf{C}^*, & -\mathbf{G}^-\mathbf{B}^*\mathbf{X}_2^-\end{pmatrix},\\ \mathbf{S} &= \begin{pmatrix}(\mathbf{I}-\mathbf{GG}^-)\mathbf{C}^*, & (\mathbf{I}-\mathbf{GG}^-)\mathbf{B}^*\mathbf{X}_2^-\\ \mathbf{O}, & \mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-\end{pmatrix},\\ \mathbf{T} &= \operatorname{Var}\begin{pmatrix}\hat{\Theta}\\ Y-\mathbf{D}\hat{\Theta}\end{pmatrix} = \begin{pmatrix}\mathbf{V}, & -\mathbf{VD}'\\ -\mathbf{DV}, & \Sigma_{2,2}+\mathbf{DVD}'\end{pmatrix}.\end{aligned} \tag{8}$$

Proof. With respect to the notation (8)

$$\begin{aligned}\frac{1}{2}\frac{\partial \operatorname{Tr}\left[\mathbf{H}^*\operatorname{Var}\begin{pmatrix}\tilde{\beta}\\ \tilde{\gamma}\end{pmatrix}\right]}{\partial\mathbf{W}} &= \left\{\mathbf{I}-(\mathbf{B}^*,\mathbf{G})'[(\mathbf{B}^*,\mathbf{G})^-]'\right\}\mathbf{H}^*\left[\mathbf{I}-(\mathbf{B}^*,\mathbf{G})^-(\mathbf{B}^*,\mathbf{G})\right]\mathbf{WSTS}'\\ &\quad+\left\{\mathbf{I}-(\mathbf{B}^*,\mathbf{G})'[(\mathbf{B}^*,\mathbf{G})^-]'\right\}\mathbf{H}^*(\mathbf{RTS}'+\mathbf{ASTS}')\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2 \operatorname{Tr}\left[\mathbf{H}^*\operatorname{Var}\begin{pmatrix}\tilde{\beta}\\ \tilde{\gamma}\end{pmatrix}\right]}{\partial[\operatorname{vec}(\mathbf{W})]\,\partial[\operatorname{vec}(\mathbf{W})]'}\\ &\qquad= (\mathbf{STS}')\otimes\left\{\mathbf{I}-(\mathbf{B}^*,\mathbf{G})'[(\mathbf{B}^*,\mathbf{G})^-]'\right\}\mathbf{H}^*\left[\mathbf{I}-(\mathbf{B}^*,\mathbf{G})^-(\mathbf{B}^*,\mathbf{G})\right],\end{aligned}$$

what is p.s.d. Hessian and thus the statement is proved. □

Remark 4.3. Since the matrices $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$ of the $\mathbf{H}^*$-optimum estimator are functions of the matrix $\mathbf{H}^*$, the joint efficient estimator does not exist in the class $\tilde{\mathcal{U}}_{\beta,\gamma}$.

5. Problem of equivalence

Equivalent algorithms of estimation are useful in practice especially in large and tedious numerical calculations, when numerical correctness is of a great importance. Thus also in the case of $\mathbf{H}$-optimum and $\mathbf{H}^*$-optimum estimators, respectively, such a problem is actual.

It seems to be natural to use the procedure before Lemma 1.2 and except the calculation according Lemma 1.7 to use the model without constraints and to use Lemma 1.2. However, regarding Lemma 1.6 we know that in $\tilde{\mathcal{U}}_\beta$ there does not exist the joint efficient linear unbiased estimator and at the same time, Lemma 1.2 told us that there exists the BLUE of the parameter $\boldsymbol{\beta}$.

An analogous consideration can be made for the case of constraints II. We obtain a similar conclusion. Thus we can formulate the following rule.

Remark 5.1. In the two stage model ($\hat{\boldsymbol{\Theta}}$ must not be changed) either with constraints (2) or with constraints (3) the $\mathbf{H}$-optimum and $\mathbf{H}^*$-optimum estimators respectively, cannot be calculated from the reparametrized model.

6. Numerical example

EXAMPLE 6.1. Let the heights (given in *meters*) of four points A, P_1, P_2 and B be given by the scheme $A \to \Theta_1$, $P_1 \to \Theta_1 + \beta_1$, $P_2 \to \Theta_1 + \beta_1 + \beta_2$, $B \to \Theta_1 + \beta_1 + \beta_2 + \beta_3 = \Theta_2$. In the first stage the heights Θ_1 and Θ_2 are estimated, i.e.

$$\begin{pmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, 0.04^2 \mathbf{I}_{2,2} \right]$$

(the value $\sigma^2 = 0.04^2$ is given in *meters*^{2}). The second stage of measurement is given by the model

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim \left[\begin{pmatrix} 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, 0.01^2 \mathbf{I}_{3,3} \right]. \tag{9}$$

The constraint is

$$\beta_1 + \beta_2 + \beta_3 + \Theta_1 - \Theta_2 = 0.$$

Thus

$$\mathbf{X}_2 = \mathbf{I}_{2,2}, \qquad \mathbf{D} = \mathbf{O}, \qquad \mathbf{B} = (1,1,1), \qquad \mathbf{C} = (1,-1), \qquad \boldsymbol{a} = \boldsymbol{O}.$$

Let an estimate of $\boldsymbol{\Theta}$ be $\begin{pmatrix} 0.97 \\ 3.02 \end{pmatrix}$ and a realization of $\boldsymbol{Y}$ be $\begin{pmatrix} 1.010 \\ 0.590 \\ 0.390 \end{pmatrix}$.

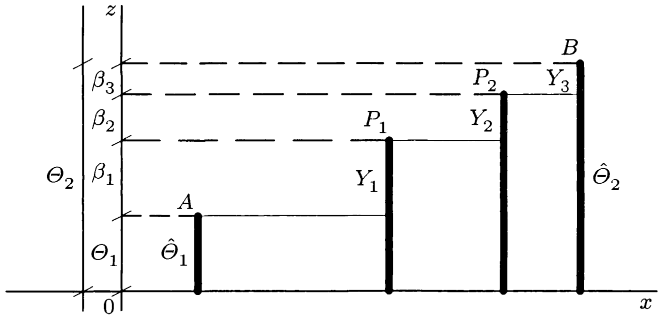

FIGURE 1. Model with two inner points with constraints of type I.

Then

$$\begin{pmatrix} \hat{\hat{\Theta}}_1 \\ \hat{\hat{\Theta}}_2 \\ \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \\ \hat{\hat{\beta}}_3 \end{pmatrix} = \begin{pmatrix} 0.99743 \\ 2.99257 \\ 1.01171 \\ 0.59171 \\ 0.39171 \end{pmatrix} \qquad \text{(Lemma 1.2)},$$

$$\begin{pmatrix} \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \end{pmatrix} = \begin{pmatrix} 1.01171 \\ 0.59171 \end{pmatrix} \qquad \text{(Lemma 1.3)},$$

and

$$\boldsymbol{a} + \mathbf{C}\hat{\boldsymbol{\Theta}} + \mathbf{B}\hat{\hat{\boldsymbol{\beta}}} = -0.0054857 \neq 0\,.$$

Further

$$\begin{pmatrix} \bar{\beta}_1 \\ \bar{\beta}_2 \\ \bar{\beta}_3 \end{pmatrix} = \begin{pmatrix} 1.03000 \\ 0.61000 \\ 0.41000 \end{pmatrix} \quad \text{(Lemma 1.7 and Lemma 1.8)} \qquad \mathbf{D} = \mathbf{O}\,,$$

$$\begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \end{pmatrix} = \begin{pmatrix} 1.03914 \\ 0.59171 \\ 0.41914 \end{pmatrix} \quad \text{(Lemma 2.2)} \qquad \mathbf{H} = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 0 \end{pmatrix}.$$

Here $\mathrm{Var}(\bar{\beta}_2) = 0.000422 > 0.000097 = \mathrm{Var}(\tilde{\beta}_2)$ (cf. Remark 1.9 and Lemma 1.6).

EXAMPLE 6.2. Let the horizontal coordinates of four points A, P_1, P_2 and B from Example 6.1 be situated on the axis x with coordinates (in *meters*) $A \to x = 1.00$, $P_1 \to x = 3.70$, $P_2 \to x = 4.50$, $B \to x = 4.95$ and they creates a parabola $z = \gamma_1 + \gamma_2 x + \gamma_3 x^2$ in the plane x, z. Then the model of the second stage is (9), however, the constraints are

$$\begin{pmatrix} 1, & 0 \\ -1, & 1 \\ 0, & 0 \\ 0, & 0 \\ 0, & 0 \end{pmatrix} \boldsymbol{\Theta} + \begin{pmatrix} 0, & 0, & 0 \\ -1, & -1, & -1 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} -1, & -1, & -1 \\ 0, & 0, & 0 \\ 0, & -2.7, & -(3.7^2 - 1) \\ 0, & -0.8, & -(4.5^2 - 3.7^2) \\ 0, & -0.45, & -(4.95^2 - 4.5^2) \end{pmatrix} \boldsymbol{\gamma} = \boldsymbol{O}.$$

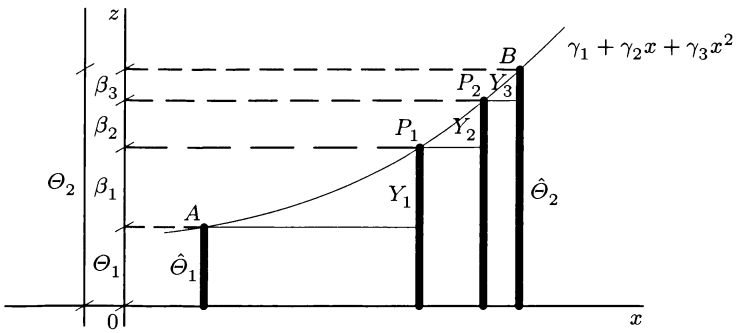

FIGURE 2. Model with two inner points with constraints of type II.

Let an estimate of $\boldsymbol{\Theta}$ and a realization of $\boldsymbol{Y}$ be the same as in Example 6.1. Then

$$\begin{pmatrix} \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \\ \hat{\hat{\beta}}_3 \\ \hat{\hat{\gamma}}_1 \\ \hat{\hat{\gamma}}_2 \\ \hat{\hat{\gamma}}_3 \end{pmatrix} = \begin{pmatrix} 1.030142 \\ 0.613123 \\ 0.406735 \\ 0.873687 \\ -0.048923 \\ 0.085235 \end{pmatrix} \quad \text{(Lemma 3.1, where } \boldsymbol{\Theta} \text{ is substituted by } \hat{\boldsymbol{\Theta}}\text{)}.$$

Estimator for the matrix $\mathbf{H}^* = \begin{pmatrix} 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \end{pmatrix}$ then we obtain from Theorems 3.3, 3.6 and 4.2

$$\begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \\ \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ \tilde{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1.070077 \\ 0.591824 \\ 0.388099 \\ 0.936756 \\ -0.064886 \\ 0.098130 \end{pmatrix}.$$

Here $\mathrm{Var}(\hat{\hat{\beta}}_2) = 0.0005006 > 0.0000595 = \mathrm{Var}(\tilde{\beta}_2)$ (Lemma 3.1, where Θ is substituted by $\hat{\Theta}$, Remark 3.2 ($\mathbf{V} \neq \mathbf{O}$) and Theorem 3.3 and Lemma 3.4).

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