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GRÖBNER BASES AND THE IMMERSION OF REAL FLAG MANIFOLDS IN EUCLIDEAN SPACE

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ABSTRACT. The purpose of this note is to try to answer some questions about the immersion problem of real flag manifolds in Euclidean spaces.

1. Introduction

Let M^m, N^n be compact connected differentiable manifolds without boundary and let $f\colon M \to N$ be a differentiable mapping. If $T(M)$, $T(N)$ are their corresponding tangent bundles, f induces a bundle homomorphism $df\colon T(M) \to T(N)$. The mapping f is called an *immersion* if $df(x)$ is a monomorphism for each x in the manifold. In this case $n \geq m$. We call the integer number $n - m$ the *codimension* of the immersion.

The natural question then arises:

Given manifolds M and N, is there an immersion of M in N?

The problem we are concerned with in our work is to find, for each real flag manifold F, the least integer $i(F)$ such that F immerses in $\mathbb{R}^{i(F)}$.

Hirsch in [12] located the essential element related to each manifold that gives us the minimum codimension, namely, the minimal rank of a normal bundle of an immersion. He established the following result.

THEOREM 1.1. *M^m can be immersed in $\mathbb{R}^{m+k}$ if and only if there exists a bundle ξ of dimension k such that $TM \oplus \xi$ is isomorphic to $(m+k)\varepsilon$, the trivial bundle of dimension $m+k$ over M.*

So, we need to find the least k among the ξ^k's that satisfy $TF \oplus \xi \equiv (m+k)\varepsilon$.

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As the vanishing of some of the Stiefel-Whitney characteristic classes is a necessary condition for the immersion, we address our problem, now a problem of vector bundles, by dealing with these classes. The idea involved is quite standard in that the appropriate Stiefel-Whitney class of the normal bundle is nonzero. In this sense, they provide us with results of obstructions to the differentiable manifolds immersions.

2. Preliminaries (Background of classical results)

Let G be a compact Lie group of dimension n and H be a closed subgroup of dimension h.

Consider the H-principal bundle $G \to G/H$; the adjoint representation of G restricted to H, $Ad|_H\colon H \to O(n)$; the adjoint representation of H, $Ad_H\colon H \to O(h)$; and the isotropic representation of H, $\iota\colon H \to O(n-h)$. All the above give us the decomposition $Ad|_H = Ad_H \oplus \iota$. Here $O(n)$ is the orthogonal group (see [13], [21]).

This decomposition and the mixing construction give us another decomposition:

$$(Ad|_H) = (Ad_H) \oplus (\iota)\,,$$

where $(Ad|_H)$, (Ad_H) and (ι) are corresponding vector bundles over G/H.

Classical results show us that $(Ad|_H)$ is trivial and that (ι) is isomorphic to $T(G/H)$ (see [5], [6], [10]).

With this information, we conclude, by Theorem 1.1, that (Ad_H) is a normal bundle of the homogeneous space G/H for some immersion of G/H in a Euclidean space.

The following commutative diagram contains all the information we need.

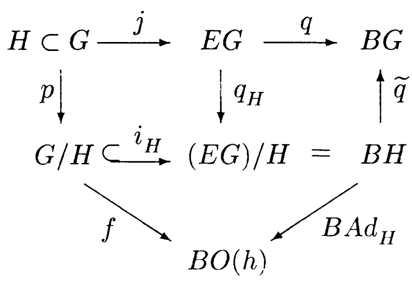

Diagram 2.1.

Here $EG \xrightarrow{q} BG$ is the G-principal bundle (known as the classifying bundle of G, and, in this sense, the universal bundle of G) given by the Milnor construction; q_H is the quotient map of the action of H on EG (since EG is contractible, q_H is also a classifying map of H); p is the quotient map of the bundle we are dealing with; i_H, j are inclusions; BAd_H is the map induced by the adjoint representation of H, $Ad_H \colon H \to O(h)$; and $\tilde{q}$ comes from the following fact:

$$\begin{array}{ccc} & H \subset EG \supset G & \\ & {\scriptstyle q_H}\swarrow \qquad \searrow{\scriptstyle q} & \\ G/H \subset BH & \xrightarrow[\tilde{q}]{\qquad} & BG \end{array}$$

Diagram 2.2.

As $H \subset G$, we have two bundles, namely, a H-principal bundle and a G-bundle with fibre G/H.

In addition, the composed map $f = BAd_H \circ i_H$ is the classifying map of the normal bundle over G/H.

3. The real flag manifolds

The background information below can be found in [2], [14], [15], [16], [24].

Definition 3.1. Let $n_1, \dots, n_s$ be positive integers numbers. A *real flag of type* $(n_1, \dots, n_s)$ is an s-tuple $(V_1, \dots, V_s)$, where each V_i is an n_i-subspace of the real Euclidean n-space $\mathbb{R}^n$ $(n = n_1 + \cdots + n_s)$ and the V_j's are mutually orthogonal. The space of all such flags, $F(n_1, \dots, n_s)$, may be identified with the homogeneous space $O(n)/\big(O(n_1) \times \cdots \times O(n_s)\big)$, where $O(n), O(n_1), \dots, O(n_s)$ are appropriate orthogonal groups.

With this identification, $F = F(n_1, \dots, n_s)$, becomes a compact connected differentiable manifold of dimension $\left(n^2 - \sum_{i=1}^{s} n_i^2\right)/2$. We call such a manifold a *real flag manifold*.

Thus, these manifolds F are spaces of type G/H where $G = O(n)$ is a compact Lie group and $H = O(n_1) \times \cdots \times O(n_s)$ is a closed subgroup of G. Then we can conclude that the normal bundle of F, $\eta(F)$, is given by (Ad_H).

Lam in [16] proves the following result.

THEOREM 3.2. *The real flag manifold F can be immersed in Euclidean space with codimension h, where $h = \dim H = \sum_{i=1}^{s} \binom{n_i}{2}$, provided this codimension is nonzero.*

Remark 3.3. The fact that the codimension is zero tells us that we are dealing with $F(1,\dots,1)$ and this flag has a trivial tangent bundle. So, it immerses in codimension 1 (see [16], [24]).

4. Some obstructions to the immersion of real flag manifolds

Let $D(n)$ be $O(1)\times\cdots\times O(1)$ (n-times). Then we have the inclusion map $j\colon D(n)\hookrightarrow H$ and $BD(n) = \mathbb{R}P^\infty\times\cdots\times\mathbb{R}P^\infty$ (n-times).

A part of Diagram 2.1 gives us the following diagram:

$$
\begin{array}{ccccc}
F(1,\dots,1) & \xhookrightarrow{\;i_{D(n)}\;} & BD(n) & \xrightarrow{\;BAd|_{D(n)}\;} & BD(h) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle Bj} & & \Big\downarrow{\scriptstyle \bar{q}} \\
F(n_1,\dots,n_s) & \xhookrightarrow{\;i\;} & BH & \xrightarrow{\;BAd\;} & BO(h)
\end{array}
$$

Diagram 4.1.

where π is the canonical quotient map; BAd is induced by the adjoint representation of H, $Ad\colon H\to O(h)$; $BAd|_{D(n)}$ is the induced of $Ad|_{D(n)}$; $\bar{q}$ is the map induced by the inclusion $D(h)\hookrightarrow O(h)$.

Taking the cohomology, with coefficients in $\mathbb{Z}_2$, we have:

$$
\begin{array}{ccccc}
H^*(BD(h)) & \xrightarrow{\;BAd|^*_{D(n)}\;} & H^*(BD(n)) & \xrightarrow{\;i^*_{D(n)}\;} & H^*(F(1,\dots,1)) \\
\Big\uparrow{\scriptstyle \bar{q}^*} & & \Big\uparrow{\scriptstyle (Bj)^*} & & \Big\uparrow{\scriptstyle \pi^*} \\
H^*(BO(h)) & \xrightarrow{\;BAd^*\;} & H^*(BH) & \xrightarrow{\;i^*\;} & H^*(F(n_1,\dots,n_s))
\end{array}
$$

Diagram 4.2.

where the vertical arrows are monomorphisms (see [4]).

Therefore, we can transform our study of BAd^* and i^* into the study of $BAd|^*_{D(n)}$ and $i^*_{D(n)}$.

The cohomology of $BD(n)$ with coefficients in $\mathbb{Z}_2$ is a polynomial ring in n variables $x_1, \ldots, x_n$, i.e., $H^*(BD(n)) \equiv \mathbb{Z}_2[x_1, \ldots, x_n]$, where $x_i \in H^1(BD(n))$.

On the other hand $\bar{q}$: $BD(n) \to BO(n)$ induces the monomorphism $\bar{q}^*$: $H^*(BO(n)) \to H^*(BD(n))$ on the invariant elements for the symmetric group in n letters (see [23]).

Then,

$$\begin{aligned} \bar{q}^* : \mathbb{Z}_2[W_1, \ldots, W_n] &\to \mathbb{Z}_2[x_1, \ldots, x_n] \\ W_k \in H^k(BO(n); \mathbb{Z}_2) &\mapsto \sigma_k(x_1, \ldots, x_n), \end{aligned}$$

where W_k is the kth Stiefel-Whitney class of the universal bundle of $O(n)$ and σ_k is the kth elementary symmetric polynomial in n variables $x_1, \ldots, x_n$ (see [3]).

The Lie algebra of $O(n)$ consists of matrices of the form:

$$\begin{pmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1n} \\ -b_{12} & 0 & b_{23} & \cdots & b_{2n} \\ -b_{13} & -b_{23} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{1n} & -b_{2n} & -b_{3n} & \cdots & 0 \end{pmatrix}.$$

So, the matrices of the Lie algebra of H are formed by skew-symmetric blocks of order n_i, $i = 1, \ldots, s$.

Furthermore, as $D(h) = O(1) \times \cdots \times O(1)$ (h-times), the number $h = \dim(H) = \sum \binom{n_i}{2}$ is the number of generators of $H^*(BD(h))$.

Indexing such generators with the same format as the Lie algebra of $O(n)$ we obtain:

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & t_{12} & \cdots\, t_{1n_1} \\ \cdots & \cdots & \cdots \\ -t_{1n_1} & -t_{2n_1} & \cdots\ 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & t_{n_1+1\, n_1+2} & \cdots\, t_{n_1+1\, n_1+n_2} \\ \cdots & \cdots & \cdots \\ -t_{n_1+1\, n_1+n_2} & -t_{n_1+2\, n_1+n_2} & \cdots\ 0 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{} \end{pmatrix}$$

and then our $BAd|^*_{D(n)}$ is given by:

$$\begin{array}{ccc} H^*(BD(h)) & \xrightarrow{BAd|^*_{D(n)}} & H^*(BD(n)) \\ ||| & & ||| \\ \mathbb{Z}_2[t_{12},\dots,t_{1n_1},t_{23},\dots,t_{2n_1},\dots,t_{n-1n}] & \longrightarrow & \mathbb{Z}_2[x_1,\dots,x_n] \\ & & t_{ij} \longmapsto BAd|^*_{D(n)}(t_{ij}) = x_i + x_j \end{array}$$

Diagram 4.3.

The explanation for the formula $t_{ij} \mapsto x_i + x_j$ in the above diagram is as follows. First notice that for any m, the representation $Ad\colon O(m) \to O\left(\binom{m}{2}\right)$, when restricted to the diagonals, is the map $Ad|_{D(m)}\colon D(m) \to D\left(\binom{m}{2}\right)$ described on the diagonal matrices (with $a_i = \pm 1$ along the diagonal) by $\operatorname{diag}(a_1,\dots,a_m) \mapsto \operatorname{diag}(a_i \cdot a_j)$, $1 \le i < j \le m$ (due to the fact that on the orthogonal group Ad is the same as the second exterior power representation). Now, for the flag manifold, taking m to be successively $n_1,\dots,n_s$ we find the same description for $Ad|_{D(n)}$ within each of the s blocks since $H = O(n_1) \times \cdots \times O(n_s)$. This same description applies to π_0 of these (finite discrete) spaces, which is the same as π_1 of the respective classifying spaces and again this is isomorphic to H_1 of these classifying spaces under the Hurewicz homomorphism. Dualizing this description to the $\mathbb{Z}_2$-cohomology gives the formula in the diagram.

The bundle $F(n_1,\dots,n_s) \overset{i}{\hookrightarrow} BH \overset{\tilde{q}}{\to} BO(n)$ induces $H^*(BO(n)) \overset{\tilde{q}^*}{\to} H^*(BH) \overset{i^*}{\to} H^*(F(n_1,\dots,n_s))$.

Moreover, i^* is an epimorphism with kernel generated by $\tilde{q}^*(W_1),\dots,\tilde{q}^*(W_n)$.

Using $F(1,\dots,1)$ and $D(n)$ in place of $F = F(n_1,\dots,n_s)$ and H, we obtain:

$$\begin{array}{ccccc} H^*(BO(n)) & \xrightarrow{\overline{q}^*} & H^*(BD(n)) & \xrightarrow{i^*_{D(n)}} & H^*(F(1,\dots,1)) \\ ||| & & ||| & & ||| \\ \mathbb{Z}_2[W_1,\dots,W_n] & & \mathbb{Z}_2[x_1,\dots,x_n] & & \mathbb{Z}_2[x_1,\dots,x_n]/_{\mathcal{I}_n} \end{array}$$

Diagram 4.4.

where the kernel of $i^*_{D(n)}$, i.e. $\mathcal{I}_n$, is $\langle \sigma_1,\dots,\sigma_n \rangle$, the ideal of $\mathbb{Z}_2[x_1,\dots,x_n]$ generated by the elementary symmetric polynomials $\sigma_1,\dots,\sigma_n$ in n variables $x_1,\dots,x_n$ (see [3], [15]).

As $BAd \circ i$ is the classifying map of $\eta(F)$, we can write $i^* \circ BAd^*(W_k) = w_k(\eta(F))$, where $w_k(\eta(F))$ is the kth Stiefel-Whitney class of $\eta(F)$.

Combining all this information, we finally get the commutative diagram:

$$\begin{array}{ccccc}
\mathbb{Z}_2[t_{12},\dots,t_{n-1n}] & \xrightarrow{BAd|^*_{D(n)}} & \mathbb{Z}_2[x_1,\dots,x_n] & \xrightarrow{i^*_{D(n)}} & \mathbb{Z}_2[x_1,\dots,x_n]/\mathcal{I}_n \\
\uparrow & & \uparrow & & \uparrow \pi^* \\
\mathbb{Z}_2[W_1,\dots,W_h] & \xrightarrow{BAd^*} & H^*(BH) & \xrightarrow{i^*} & H^*(F(1,\dots,1))
\end{array}$$

Diagram 4.5.

Here the kth Stiefel-Whitney class of the universal bundle of $O(h)$, i.e. W_k, is mapped on τ_k, the kth elementary symmetric polynomial in variables t_{ij} with $i < j$. So, we have:

THEOREM 4.1. *The kth Stiefel-Whitney class of the normal bundle of the real flag manifold F is zero if and only if the kth elementary symmetric polynomial $\tau_k(t_{12},\dots,t_{n-1n})$ evaluated on h elements $x_i + x_j$ of $\mathbb{Z}_2[x_1,\dots,x_n]$ belongs to the ideal $\mathcal{I}_n$.*

Proof. As π^* is a monomorphism, then $w_k(\eta(F)) = 0$ if and only if $\pi^*(w_k(\eta(F))) = 0$, i.e., if and only if $\pi^*(i^*(BAd^*(W_k))) = 0$.

On the other hand, from the commutativity of the Diagram 4.5 this is equivalent to $BAd|^*_{D(n)}(\tau_k) \in \mathcal{I}_n$.

Observing that $BAd|^*_{D(n)}(\tau_k) = \tau_k(x_i + x_j)$ we arrive at the result. □

With this Theorem the immersion problem leads to an algebraic problem. We need to know if the polynomial $\tau_k((x_i + x_j))$ belongs to $\mathcal{I}_n$.

The following is in [17].

PROPOSITION 4.2. *Let $\mathbb{Z}_2[\sigma_1,\dots,\sigma_n]$ be the subring of $\mathbb{Z}_2[x_1,\dots,x_n]$ generated by the elementary symmetric polynomials $\sigma_1,\dots,\sigma_n$ in variables $x_1,\dots,x_n$. Then $\mathbb{Z}_2[x_1,\dots,x_n]$ is a free $\mathbb{Z}_2[\sigma_1,\dots,\sigma_n]$-module with standard base given by the monomials $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, where $0 \le \alpha_i \le n - i$, $i = 1,\dots,n$.*

By Proposition 4.2, every p in $\mathbb{Z}_2[x_1, \ldots, x_n]$ can be written in a unique way as $p = \sum_{\alpha_1 \cdots \alpha_n} a_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $a_{\alpha_1 \cdots \alpha_n} = a_{\alpha_1 \cdots \alpha_n}(\sigma_1, \ldots . \sigma_n) \in \mathbb{Z}_2[\sigma_1, \ldots, \sigma_n]$. So, with the notation above, we have:

LEMMA 4.3. *If $p \in \mathcal{I}_n$, then no coefficients $a_{\alpha_1 \cdots \alpha_n}$ can be equal to $1 \in \mathbb{Z}_2$.*

Proof. If $p \in \mathcal{I}_n$, then $p = \sum_{j=1}^{n} p_j \sigma_j$, where $p_j \in \mathbb{Z}_2[x_1, \ldots, x_n]$, $j = 1, \ldots, n$. On the other hand, $p_j = \sum_{\alpha_1 \cdots \alpha_n} p^j_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $p^j_{\alpha_1 \cdots \alpha_n} \in \mathbb{Z}_2[\sigma_1, \ldots, \sigma_n]$. Then $p = \sum_{\alpha_1 \cdots \alpha_n} \left(\sum_{j=1}^{n} p^j_{\alpha_1 \cdots \alpha_n} \sigma_j \right) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and we can conclude that $a_{\alpha_1 \cdots \alpha_n} = \sum_{j=1}^{n} p^j_{\alpha_1 \cdots \alpha_n} \sigma_j$ cannot be constant and we have the result. □

THEOREM 4.4. *The real flag manifold $F(2, \ldots, 2)$ (s-times) can be immersed in Euclidean space with codimension s and this codimension is the best possible.*

Proof. As by Theorem 3.2, $F(2, \ldots, 2)$ can be immersed in Euclidean space with codimension s, we only have to show that $\tau_s = \tau_s(x_1 + x_2, \ldots \ldots, x_{2s-1} + x_{2s}) \notin \mathcal{I}_{2s}$. In $\mathbb{Z}_2[x_1, \ldots, x_n]$, $x_{2s} = \sigma_1 + x_1 + \cdots + x_{2s-1}$ and then $\tau_s = [(x_1 + x_2) \cdots (x_{2s-3} + x_{2s-2})](\sigma_1 + x_1 + \cdots + x_{2s-2}) = [(x_1 + x_2) \cdots \cdots (x_{2s-3} + x_{2s-2})]\sigma_1 + [(x_1 + x_2) \cdots (x_{2s-3} + x_{2s-2})](x_1 + \cdots + x_{2s-2})$. The first term in the last sum belongs to $\mathcal{I}_{2s}$ but the second term, by the Lemma 4.3, does not. □

Remark 4.5. This result was presented by Conde in [7] with a more elementary but much longer proof. The proof given here was communicated to us by Prof. Brasil T. Leme, whom we thank.

5. Gröbner bases and the algebraic problem

All the information in this section can be found in [1] and [8].

The first aspect to be observed is that the division algorithm gives us the solution for our algebraic problem in the case of one variable. Thus, the way to solve this problem is to generalize the division algorithm to several variables.

The key point is to fix a monomial ordering in the ring $\mathbb{Z}_2[x_1, \cdots, x_n]$.

To *divide a polynomial* p by a finite sequence of polynomials $g_1, \ldots . g_t$ means expressing p as $p = q_1 g_1 + \cdots + q_t g_t + r$ with quotients q_i $(i = 1, \ldots t)$ and remainder r in this ring. For this, we must be careful in characterizing r. At this point, we need the monomial ordering.

If after the division of p by $(g_1, \dots, g_t)$ we get $r = 0$, then $p \in \langle g_1, \dots, g_t \rangle$, the ideal of $\mathbb{Z}_2[x_1, \dots, x_n]$ generated by $g_1, \dots, g_t$.

Since r is not uniquely determined, the converse of the above mentioned is not true.

In order to overcome this inconvenience, we need to find a "good" set of generators for the ideal, where the condition $r = 0$ is equivalent to $\tau_k\big((x_i + x_j)\big) \in \mathcal{I}_n$.

The set of "good" generators for the ideal $\mathcal{I}_n$ is the Gröbner basis.

5.1. Gröbner Bases.

The set of "good" generators was introduced for the first time in the middle of the sixties by H. Hironaka and later by B. Buchberger in his PhD thesis. The name "Gröbner bases" was created by Buchberger in honour of W. Gröbner (1899 - 1980).

The idea involved is that once one chooses a monomial ordering, each $p \in \mathbb{Z}_2[x_1, \dots, x_n]$ has a unique leading term $\mathrm{lt}(p)$.

Every nonzero polynomial $p \in \mathbb{Z}_2[x_1, \dots, x_n]$ can be written as $p = X^{\alpha_1} + X^{\alpha_2} + \cdots + X^{\alpha_r}$, where for each $\alpha_i = (\alpha_{i_1}, \dots, \alpha_{i_n}) \in \mathbb{N}^n$ $(i = 1, \dots, r)$, $X^{\alpha_i} = x_1^{\alpha_{i_1}} \dots x_n^{\alpha_{i_n}}$ and $X^{\alpha_r} < \cdots < X^{\alpha_2} < X^{\alpha_1}$.

Under this condition, we put $\mathrm{lt}(p) = 1X^{\alpha_1}$ and we call it *leading term* (the term of highest degree) of p.

DEFINITION 5.1. Fix a monomial order. A finite subset $G = \{g_1, \dots, g_t\}$ of an ideal $\mathcal{I}$ is a *Gröbner basis* if $\big\langle \mathrm{lt}(g_1), \dots, \mathrm{lt}(g_t) \big\rangle = \big\langle \mathrm{lt}(\mathcal{I}) \big\rangle$ where $\mathrm{lt}(\mathcal{I})$ is the set of leading terms of elements of $\mathcal{I}$.

One can show that every nonzero ideal $\mathcal{I}$ of $\mathbb{Z}_2[x_1, \dots, x_n]$ has a Gröbner basis, which is a basis of $\mathcal{I}$.

In the seventies and eighties, Buchberger et al. devised an algorithm that calculates such bases. With this algorithm, $\tau_k\big((x_i + x_j)\big) \in \mathcal{I}_n$ if and only if the remainder r, under the division of this polynomial by a Gröbner basis G for our ideal, is zero. Under these conditions, we call r the *normal form* of τ_k related to G and we write $\mathrm{normalf}(\tau_k, G)$.

Bearing this and the Theorem 4.1 in mind, we developed **an** algorithm, using the software Maple V Release 4 (Waterloo Maple Inc., June 1996) which produced the following table (see [18]):

$n = n_1 + \cdots + n_s$	$F = F(n_1, \ldots, n_s)$	$\dim F$	codimension of immersion (LAM)	obstruction to immersion ($\overline{w}_i \neq 0$)
3	$F(1,2)$	2	1	$\overline{w}_1$
4	$F(1,1,2)$	5	1	$\overline{w}_1$
4	$F(2,2)$	4	2	$\overline{w}_2$
4	$F(1,3)$	3	3	$\overline{w}_i = 0$, $\forall i = 1, \ldots, 3$
5	$F(1,1,1,2)$	9	1	$\overline{w}_1$
5	$F(1,2,2)$	8	2	$\overline{w}_2$
5	$F(1,1,3)$	7	3	$\overline{w}_3$
5	$F(2,3)$	6	4	$\overline{w}_4$
5	$F(1,4)$	4	6	$\overline{w}_3$
6	$F(1,1,1,1,2)$	14	1	$\overline{w}_1$
6	$F(1,1,2,2)$	13	2	$\overline{w}_2$
6	$F(2,2,2)$	12	3	$\overline{w}_3$
6	$F(1,1,1,3)$	12	3	$\overline{w}_3$
6	$F(1,2,3)$	11	4	$\overline{w}_4$
6	$F(3,3)$	9	6	$\overline{w}_6$
6	$F(1,1,4)$	9	6	$\overline{w}_5$
6	$F(2,4)$	8	7	$\overline{w}_6$
6	$F(1,5)$	5	10	$\overline{w}_2$
7	$F(1,1,1,1,1,2)$	20	1	$\overline{w}_1$
7	$F(1,1,1,2,2)$	19	2	$\overline{w}_2$
7	$F(1,2,2,2)$	18	3	$\overline{w}_3$
7	$F(1,1,1,1,3)$	18	3	$\overline{w}_3$
7	$F(1,1,2,3)$	17	4	$\overline{w}_4$
7	$F(2,2,3)$	16	5	$\overline{w}_5$
7	$F(1,3,3)$	15	6	$\overline{w}_6$

$n = n_1 + \cdots + n_s$	$F = F(n_1, \ldots, n_s)$	$\dim F$	codimension of immersion (LAM)	obstruction to immersion ($\overline{w}_i \neq 0$)
7	$F(1,1,1,4)$	15	6	$\overline{w}_6$
7	$F(1,2,4)$	14	7	$\overline{w}_7$
7	$F(3,4)$	12	9	$\overline{w}_9$
7	$F(1,1,5)$	11	10	$\overline{w}_3$
7	$F(2,5)$	10	11	$\overline{w}_4$
7	$F(1,6)$	6	15	$\overline{w}_1$
8	$F(1,1,1,1,1,2)$	27	1	$\overline{w}_1$
8	$F(1,1,1,1,2,2)$	26	2	$\overline{w}_2$
8	$F(1,1,2,2,2)$	25	3	$\overline{w}_3$
8	$F(1,1,1,1,1,3)$	25	3	$\overline{w}_3$
8	$F(2,2,2,2)$	24	4	$\overline{w}_4$
8	$F(1,1,1,2,3)$	24	4	$\overline{w}_4$
8	$F(1,2,2,3)$	23	5	$\overline{w}_5$
8	$F(1,1,3,3)$	22	6	$\overline{w}_6$
8	$F(1,1,1,1,4)$	22	6	$\overline{w}_6$
8	$F(2,3,3)$	21	7	$\overline{w}_7$
8	$F(1,1,2,4)$	21	7	$\overline{w}_7$
8	$F(2,2,4)$	20	8	$\overline{w}_8$
8	$F(1,3,4)$	19	9	$\overline{w}_9$
8	$F(1,1,1,5)$	18	10	$\overline{w}_3$
8	$F(1,2,5)$	17	11	$\overline{w}_4$
8	$F(4,4)$	16	12	$\overline{w}_{12}$
8	$F(3,5)$	15	13	$\star$
8	$F(1,1,6)$	13	15	$\star$
8	$F(2,6)$	12	16	$\star$
8	$F(1,7)$	7	21	$\overline{w}_i = 0$, $\forall i = 1, \ldots, 7$

$n = n_1 + \cdots + n_s$	$F = F(n_1, \ldots, n_s)$	dim F	codimension of immersion (LAM)	obstruction to immersion ($\overline{w}_i \neq 0$)
9	$F(1,1,1,1,1,1,1,2)$	35	1	$\overline{w}_1$
9	$F(1,1,1,1,1,2,2)$	34	2	$\overline{w}_2$
9	$F(1,1,1,2,2,2)$	33	3	$\overline{w}_3$
9	$F(1,1,1,1,1,1,3)$	33	3	$\overline{w}_3$
9	$F(1,2,2,2,2)$	32	4	$\overline{w}_4$
9	$F(1,1,1,1,2,3)$	32	4	$\overline{w}_4$
9	$F(1,1,2,2,3)$	31	5	$\overline{w}_5$
9	$F(2,2,2,3)$	30	6	$\overline{w}_6$
9	$F(1,1,1,3,3)$	30	6	$\overline{w}_6$
9	$F(1,1,1,1,1,4)$	30	6	$\overline{w}_6$
9	$F(1,2,3,3)$	29	7	$\overline{w}_7$
9	$F(1,1,1,2,4)$	29	7	$\overline{w}_7$
9	$F(1,2,2,4)$	28	8	$\overline{w}_8$
9	$F(3,3,3)$	27	9	$\overline{w}_9$
9	$F(1,1,3,4)$	27	9	$\overline{w}_9$
9	$F(2,3,4)$	26	10	$\overline{w}_{10}$
9	$F(1,1,1,1,5)$	26	10	$\overline{w}_{10}$
9	$F(1,1,2,5)$	25	11	$\overline{w}_{11}$
9	$F(1,4,4)$	24	12	$\overline{w}_{12}$
9	$F(2,2,5)$	24	12	$\overline{w}_{12}$
9	$F(1,3,5)$	23	13	$\overline{w}_{13}$
9	$F(1,1,1,6)$	21	15	$\star$
9	$F(4,5)$	20	16	$\overline{w}_{16}$
9	$F(1,2,6)$	20	16	$\star$
9	$F(3,6)$	18	18	$\star$
9	$F(1,1,7)$	15	21	$\star$

$n = n_1 + \cdots + n_s$	$F = F(n_1, \ldots, n_s)$	$\dim F$	codimension of immersion (LAM)	obstruction to immersion ($\overline{w}_i \neq 0$)
9	$F(2,7)$	14	22	$\star$
9	$F(1,8)$	8	28	$\overline{w}_7 \neq 0$
10	$F(1,1,1,1,1,1,1,1,2)$	44	1	$\overline{w}_1$
10	$F(1,1,1,1,1,1,2,2)$	43	2	$\overline{w}_2$
10	$F(1,1,1,1,2,2,2)$	42	3	$\overline{w}_3$
10	$F(1,1,1,1,1,1,1,3)$	42	3	$\overline{w}_3$
10	$F(1,1,2,2,2,2)$	41	4	$\overline{w}_4$
10	$F(1,1,1,1,1,2,3)$	41	4	$\overline{w}_4$
10	$F(2,2,2,2,2)$	40	5	$\overline{w}_5$
10	$F(1,1,1,2,2,3)$	40	5	$\overline{w}_5$
10	$F(1,2,2,2,3)$	39	6	$\overline{w}_6$
10	$F(1,1,1,1,3,3)$	39	6	$\overline{w}_6$
10	$F(1,1,1,1,1,1,4)$	39	6	$\overline{w}_6$
10	$F(1,1,2,3,3)$	38	7	$\overline{w}_7$
10	$F(1,1,1,1,2,4)$	38	7	$\overline{w}_7$
10	$F(2,2,3,3)$	37	8	$\overline{w}_8$
10	$F(1,1,2,2,4)$	37	8	$\overline{w}_8$
10	$F(1,3,3,3)$	36	9	$\overline{w}_9$
10	$F(2,2,2,4)$	36	9	$\overline{w}_9$
10	$F(1,1,1,3,4)$	36	9	$\overline{w}_9$
10	$F(1,2,3,4)$	35	10	$\overline{w}_{10}$
10	$F(1,1,1,1,1,5)$	35	10	$\overline{w}_{10}$
10	$F(1,1,1,2,5)$	34	11	$\overline{w}_{11}$
10	$F(3,3,4)$	33	12	$\overline{w}_{12}$
10	$F(1,1,4,4)$	33	12	$\overline{w}_{12}$
10	$F(1,2,2,5)$	33	12	$\overline{w}_{12}$

$n = n_1 + \cdots + n_s$	$F = F(n_1, \ldots, n_s)$	$\dim F$	codimension of immersion (LAM)	obstruction to immersion ($\overline{w}_i \neq 0$)
10	$F(2,4,4)$	32	13	$\overline{w}_{13}$
10	$F(1,1,3,5)$	32	13	$\overline{w}_{13}$
10	$F(2,3,5)$	31	14	$\overline{w}_{14}$
10	$F(1,1,1,1,6)$	30	15	$\star$
10	$F(1,4,5)$	29	16	$\overline{w}_{16}$
10	$F(1,1,2,6)$	29	16	$\star$
10	$F(2,2,6)$	28	17	$\star$
10	$F(1,3,6)$	27	18	$\star$
10	$F(5,5)$	25	20	$\overline{w}_{20}$
10	$F(4,6)$	24	21	$\star$
10	$F(1,1,1,7)$	24	21	$\star$
10	$F(1,2,7)$	23	22	$\star$
10	$F(3,7)$	21	24	$\star$
10	$F(1,1,8)$	17	28	$\star$
10	$F(2,8)$	16	29	$\star$
10	$F(1,9)$	9	36	$\overline{w}_6$

Since $F(n_1, \ldots, n_s)$ is diffeomorphic to $F(n_{i_1}, \ldots, n_{i_s})$, where $\{i_1, \ldots, i_s\} = \{1, \ldots, s\}$, we assume, without loss of generality, that $n_1 \leq n_2 \leq \cdots \leq n_s$.

Observe that the Lam estimation tends to be good when the n_i are "small", but is useless when the n_i tend to be unbalanced with some of them relatively large. In these cases this codimension of immersion is much worse than $n - 1$, the codimension of immersion given by Whitney.

The $\star$ on the last column of the table indicates that the software Maple and the Gröbner package were unable, in their current scope, to check if all the Stiefel-Whitney classes $\overline{w}_k = w_k(\eta(F))$ are zero, or nonzero. In such cases, we obtained:

For $n = 8$:

— $F(3,5)$: $\overline{w}_1 = \overline{w}_2 = \overline{w}_3 = 0$, $\overline{w}_4 \neq 0$, $\overline{w}_5 = 0$, $\overline{w}_{12} = \overline{w}_{13} = 0$;
— $F(1,1,6)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 = \overline{w}_3 = \overline{w}_4 = 0$, $\overline{w}_{11} = \cdots = \overline{w}_{13} = 0$;
— $F(2,6)$: $\overline{w}_1 = 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 = \overline{w}_4 = 0$, $\overline{w}_{11} = \cdots = \overline{w}_{12} = 0$.

For $n = 9$:

— $F(1,1,1,6)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 = \overline{w}_3 = 0$, $\overline{w}_4 \neq 0$, $\overline{w}_{15} = 0$;
— $F(1,2,6)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 = 0$, $\overline{w}_4 \neq 0$, $\overline{w}_{16} = 0$;
— $F(3,6)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 \neq 0$, $\overline{w}_4 \neq 0$, $\overline{w}_{17} = \overline{w}_{18} = 0$;
— $F(1,1,7)$: $\overline{w}_1 = 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 \neq 0$, $\overline{w}_{12} = \cdots = \overline{w}_{15} = 0$;
— $F(2,7)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 = 0$, $\overline{w}_{12} = \cdots = \overline{w}_{14} = 0$.

For $n = 10$:

— $F(1,1,1,1,6)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 = \overline{w}_3 = 0$, $\overline{w}_4 \neq 0$, $\overline{w}_{15} = 0$;
— $F(1,1,2,6)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 = 0$, $\overline{w}_4 \neq 0$, $\overline{w}_{16} = 0$;
— $F(2,2,6)$: $\overline{w}_1 = 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 \neq 0$, $\overline{w}_4 \neq 0$, $\overline{w}_{17} = 0$;
— $F(1,3,6)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 \neq 0$, $\overline{w}_4 \neq 0$, $\overline{w}_{18} = 0$;
— $F(4,6)$: $\overline{w}_1 = \cdots = \overline{w}_3 = 0$, $\overline{w}_{21} = 0$;
— $F(1,1,1,7)$: $\overline{w}_1 = 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 \neq 0$, $\overline{w}_{21} = 0$;
— $F(1,2,7)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 \neq 0$, $\overline{w}_{22} = 0$;
— $F(3,7)$: $\overline{w}_1 = 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 = 0$, $\overline{w}_{19} = \cdots = \overline{w}_{21} = 0$;
— $F(1,1,8)$: $\overline{w}_1 \neq 0$, $\overline{w}_2 \neq 0$, $\overline{w}_3 \neq 0$, $\overline{w}_{16} = \cdots = \overline{w}_{17} = 0$;
— $F(2,8)$: $\overline{w}_1 = \cdots = \overline{w}_3 = 0$.

In the process we employed a Pentium 133 MHz processor (32 MB RAM).

In [22], Stong proved:

Theorem 5.2. *If* $w = \{n_1, \ldots, n_s\}$ *can be partitioned as* $w = w_1 \cup w_2 \cup w_3$, *where* $w_1, w_2 \neq \emptyset$, *and*

1) $\left| \sum_{n \in w_1} n - \sum_{n \in w_2} n \right| \leq 1$,
2) $m \in w_3 \implies m \leq \sum_{n \in w_1} n + \sum_{n \in w_2} n + 1$,

then Lam's result is best possible for $F(n_1, \ldots, n_s)$.

Remark 5.3. Using the same argument we can clearly extend this last Theorem. So, we stated it as follows:

If $w = \{n_1, \dots, n_s\}$ *can be partitioned as* $w = w_1 \cup w_2 \cup \cdots \cup w_r$, *where* $w_1, w_2 \neq \emptyset$, *and*

1) $\left| \sum_{n \in w_1} n - \sum_{n \in w_2} n \right| \leq 1$,
2) *for* $i \geq 3$, $m \in w_i \Longrightarrow m \leq \sum_{n \in w_1} n + \cdots + \sum_{n \in w_{i-1}} n + 1$,

then Lam's result is best possible for $F(n_1, \dots, n_s)$.

We shall call the partition of n that satisfies conditions 1 and 2 above as *Stong partition.*

Observing the table with $n \leq 10$, we put:

CONJECTURE 5.4. *If the hth Stiefel-Whitney class of a normal bundle of* F, $w_h(\eta(F)) = \overline{w}_h$, *is different from zero, where* $h = \dim H$, *then the partition* $\{n_1, \dots, n_s\}$ *of* n *is a Stong partition.*

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