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ON A SINGULAR SET FOR THE RESTRICTION OF THE CHARACTERISTIC MAP TO A LINEAR SUBSPACE OF $\mathcal{M}_n$

Marcin Skrzyński

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ABSTRACT. We give some basic characterizations of the set of matrices $\mathbf{A} \in \mathcal{M}_n$ (over an algebraically closed field $\mathbb{F}$ of characteristic zero) such that the image $\chi_{\mathbf{A}}(\mathcal{L})$ is not dense in $\mathbb{F}^n$, where $\mathcal{L} \subseteq \mathcal{M}_n$ is a fixed linear subspace and $\chi_{\mathbf{A}} \colon \mathcal{M}_n \to \mathbb{F}^n$ is the characteristic map associated with $\mathbf{A}$. We study the case of $n \in \{2,3\}$ in a more detailed way.

Preliminaries and introduction

Throughout the note we work over an algebraically closed field $\mathbb{F}$ of characteristic zero. We write $\mathbb{F}^*$ instead of $\mathbb{F} \setminus \{0\}$. We mean by $\mathcal{M}_{m\times n}$ the set of all $m \times n$-matrices whose entries are elements of $\mathbb{F}$. We write $\mathcal{M}_n$ instead of $\mathcal{M}_{n\times n}$. We denote by $\mathbf{0}$ and $\mathbf{I}$ the zero matrix and the unit matrix belonging to $\mathcal{M}_n$. We define $\mathcal{GL}_n$ to be the full linear group of size n over $\mathbb{F}$, i.e.

$$\mathcal{GL}_n := \{\mathbf{U} \in \mathcal{M}_n : \ \mathbf{U} \text{ is invertible}\}.$$

Furthermore, we put $\mathfrak{sl}_n = \{\mathbf{A} \in \mathcal{M}_n : \ \operatorname{tr}(\mathbf{A}) = 0\}$ and $\mathcal{T}_n = \{\mathbf{A} \in \mathcal{M}_n : \mathbf{A} \text{ is upper triangular}\}$. We refer to the linear subspaces of $\mathcal{M}_n$ of codimension one as (linear) hyperplanes. For an $\mathbf{A} \in \mathcal{M}_n$ we define $\operatorname{Diag}(\mathbf{A}) \in \mathcal{M}_n$ to be the diagonal matrix whose diagonal entries coincide with those of $\mathbf{A}$.

A map $\Psi \colon \mathcal{M}_n \to Y$, where Y is an arbitrary set, is $\mathcal{GL}_n$*-invariant* if $\Psi(\mathbf{U}^{-1}\mathbf{A}\mathbf{U}) = \Psi(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{M}_n$ and for all $\mathbf{U} \in \mathcal{GL}_n$. A set $\mathcal{E} \subseteq \mathcal{M}_n$ is $\mathcal{GL}_n$-invariant if $\mathbf{U}^{-1}\mathcal{E}\mathbf{U} := \{\mathbf{U}^{-1}\mathbf{A}\mathbf{U} : \ \mathbf{A} \in \mathcal{E}\} \subseteq \mathcal{E}$ for all $\mathbf{U} \in \mathcal{GL}_n$.

The set $\mathcal{E}$ is a *cone* if it is nonempty and $\mathbb{F}\mathcal{E} := \{\lambda\mathbf{A} : \ \lambda \in \mathbb{F}, \ \mathbf{A} \in \mathcal{E}\} \subseteq \mathcal{E}$.

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Let $\mathbb{F}[\mathcal{M}_n]$ be the coordinate ring of the space $\mathcal{M}_n$, i.e. the polynomial ring $\mathbb{F}[T_{11}, \ldots, T_{nn}]$ in n^2 variables which are the entries of the "generic matrix" $\mathsf{T} = [T_{kl}]_{k,l=1,\ldots,n}$. For a positive integer $j \leq n$ we define $\mathrm{s}_n^j \in \mathbb{F}[\mathcal{M}_n]$ to be the sum of all principal minors of size j of the matrix T. Notice that $\mathrm{s}_n^1(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$, $\mathrm{s}_n^n(\mathbf{A}) = \det(\mathbf{A})$, and $T^n + \sum_{j=1}^{n}(-1)^j \mathrm{s}_n^j(\mathbf{A})T^{n-j} \in \mathbb{F}[T]$ is the characteristic polynomial of a matrix $\mathbf{A} \in \mathcal{M}_n$.

We consider an arbitrary finite dimensional vector space X over $\mathbb{F}$ and all their subsets as topological spaces equipped with the Zariski topology induced by any linear isomorphism $X \to \mathbb{F}^d$, where $d = \dim X$. This is the only topology we deal with in the text.

Let X and Z be finite dimensional vector spaces over $\mathbb{F}$, let $d = \dim X$, $w = \dim Z$, and let $f\colon X \to \mathbb{F}^d$, $g\colon Z \to \mathbb{F}^w$ be linear isomorphisms. A map $\Phi\colon X \supseteq E \to Z$, where E is a closed set, is *regular* if there is $\widetilde{\Phi}\colon X \to Z$ such that $\Phi = \widetilde{\Phi}|_E$ and $g \circ \widetilde{\Phi} \circ f^{-1}\colon \mathbb{F}^d \to \mathbb{F}^w$ is a polynomial map.

Let E be an irreducible closed subset of X. A regular map $\Phi\colon E \to Z$ is *dominant* if the range $\Phi(E)$ is dense in Z. Notice that the map Φ is dominant if and only if the rank of the differential $\mathrm{d}_x\Phi \in \mathcal{M}_{w\times v}$, where $v = \dim E$, is equal to w for a smooth point $x \in E$.

We refer to [5], [7], [8] for all information needed about Algebraic Geometry, to [6], [7] for Algebra, and to [3] for Matrix Theory.

From now on n stands for an integer not smaller than 2.

In the present note we deal with the *characteristic map* $\chi_{\mathbf{A}}$ associated with a matrix $\mathbf{A} \in \mathcal{M}_n$, i.e. $\chi_{\mathbf{A}}\colon \mathcal{M}_n \to \mathbb{F}^n$ is defined by $\chi_{\mathbf{A}}(\mathbf{B}) = \big(\mathrm{s}_n^j(\mathbf{A}+\mathbf{B})\big)_{j=1}^n$. We write χ instead of $\chi_{\mathbf{0}}$.

In paper [4] it is shown that, under natural assumptions on a fixed linear subspace $\mathcal{L} \subseteq \mathcal{M}_n$, the set of all matrices $\mathbf{A} \in \mathcal{M}_n$ such that the restriction $\chi_{\mathbf{A}}|_{\mathcal{L}}\colon \mathcal{L} \to \mathbb{F}^n$ is a dominant map contains a nonempty open subset of $\mathcal{M}_n$. The authors work over the field of complex numbers. Our purpose is to provide basic geometrical characterizations of the "singular set" $\{\mathbf{A} \in \mathcal{M}_n : \chi_{\mathbf{A}}|_{\mathcal{L}}$ is not dominant$\}$. We also aim at completely describing that singular set in certain simple cases.

The present note is a self-contained continuation of the work on the characteristic map and on the Helton-Rosenthal-Wang theorem originated in [9].

Notice that our definition of the characteristic map differs a bit (in the signs of coordinates) from that given in [4].

1. Basic properties

From now on, throughout the text, $\mathcal{L}$ stands for a linear subspace of $\mathcal{M}_n$. The subject of the note are the sets $\mathcal{S}(\mathcal{L})$ and $\mathcal{R}(\mathcal{L})$ defined as follows:

$$\mathcal{S}(\mathcal{L}) = \{\mathbf{A} \in \mathcal{M}_n : \ \chi_{\mathbf{A}}(\mathcal{L}) \text{ is not dense in } \mathbb{F}^n\}$$

and $\mathcal{R}(\mathcal{L}) = \mathcal{M}_n \setminus \mathcal{S}(\mathcal{L})$. The set $\mathcal{S}(\mathcal{L})$ is referred to as the *singular set (for the restriction of the characteristic map to* $\mathcal{L}$). It is easy to see that $\mathcal{S}(\mathcal{M}_n) = \mathcal{S}(\mathcal{T}_n) = \emptyset$. (In a more detailed way, $\chi_{\mathbf{A}}(\mathcal{M}_n) = \chi_{\mathbf{A}}(\mathcal{T}_n) = \mathbb{F}^n$ for all $\mathbf{A} \in \mathcal{M}_n$.)

Let us remark that

$$\mathcal{R}(\mathcal{L}) \neq \emptyset \implies (\dim \mathcal{L} \geq n \ \& \ \mathcal{L} \not\subseteq \mathfrak{sl}_n).$$

The main result of [4] may be rephrased now as follows.

THEOREM 1.1 (HELTON-ROSENTHAL-WANG). *Assume that* $\mathbb{F}$ *coincides with the field of complex numbers. Then* $\mathcal{R}(\mathcal{L})$ *contains a nonempty open subset of* $\mathcal{M}_n$ *provided* $\dim \mathcal{L} \geq n$ *and* $\mathcal{L} \not\subseteq \mathfrak{sl}_n$.

In the present approach to the sets $\mathcal{S}(\mathcal{L})$ and $\mathcal{R}(\mathcal{L})$ we will not make use of the above theorem. We begin with a topological property that is not surprising but seems to be worth giving an explicit proof.

PROPOSITION 1.2. *The set* $\mathcal{R}(\mathcal{L})$ *is open in* $\mathcal{M}_n$.

Proof. If $\mathcal{L} = \{\mathbf{0}\}$, then $\mathcal{R}(\mathcal{L}) = \emptyset$. Assume that $\mathcal{L} \neq \{\mathbf{0}\}$. Put $m = \dim \mathcal{L}$, fix a basis in $\mathcal{L}$, and consider the regular map $\Phi\colon \mathcal{M}_n \times \mathcal{L} \to \mathcal{M}_{n\times m}$ defined by $\Phi(\mathbf{A}, \mathbf{B}) = \mathrm{d}_{\mathbf{B}}(\chi_{\mathbf{A}}|_{\mathcal{L}})$. (We take into consideration the canonical basis in $\mathbb{F}^n$.) Define $\mathcal{H} = \{\mathbf{C} \in \mathcal{M}_{n\times m} : \ \mathrm{rank}(\mathbf{C}) = n\}$. Then $\mathcal{H}$ is an open subset of $\mathcal{M}_{n\times m}$. By the definition of a dominant map via the differential, $\mathcal{R}(\mathcal{L}) = \Pi(\Phi^{-1}(\mathcal{H}))$, where $\Pi\colon \mathcal{M}_n \times \mathcal{L} \to \mathcal{M}_n$ is the natural projection. Since Π is an open map (with respect to the Zariski topology in $\mathcal{M}_n \times \mathcal{L} = \mathbb{F}^{n^2+m}$), the openess of $\mathcal{R}(\mathcal{L})$ follows. □

Let us note three useful technical properties of the singular set $\mathcal{S}(\mathcal{L})$.

PROPOSITION 1.3.

(i) *If* $\mathcal{K} \subseteq \mathcal{L}$ *is a linear subspace, then* $\mathcal{S}(\mathcal{K}) \supseteq \mathcal{S}(\mathcal{L})$.

(ii) *If* $\mathbf{U} \in \mathcal{GL}_n$, *then* $\mathcal{S}(\mathbf{U}^{-1}\mathcal{L}\mathbf{U}) = \mathbf{U}^{-1}\mathcal{S}(\mathcal{L})\mathbf{U}$.

(iii) *If* $\mathcal{S}(\mathcal{L}) \neq \emptyset$, *then* $\mathcal{S}(\mathcal{L})$ *is a closed cone.*

Proof. Property (i) is obvious. Property (ii) follows from the equality $\chi_{\mathbf{A}}(\mathcal{L}) = \chi(\mathbf{U}^{-1}(\mathbf{A} + \mathcal{L})\mathbf{U})$ that holds true for all $\mathbf{A} \in \mathcal{M}_n$. In order to prove (iii), pick $\lambda \in \mathbb{F}^*$ and $\mathbf{A} \in \mathcal{M}_n$ and observe that $\chi(\lambda\mathbf{A} + \mathcal{L}) = \chi(\lambda(\mathbf{A} + \mathcal{L})) =$

$\Psi\big(\chi(\mathbf{A}+\mathcal{L})\big)$, where $\Psi\colon \mathbb{F}^n \to \mathbb{F}^n$ is defined by $\Psi\big((x_j)_{j=1}^n\big) = (\lambda^j x_j)_{j=1}^n$. (Recall that s_n^j is a homogeneous polynomial of degree j.) Since Ψ is a linear automorphism, $\chi(\lambda\mathbf{A}+\mathcal{L})$ is dense in $\mathbb{F}^n$ if and only if so is $\chi(\mathbf{A}+\mathcal{L})$. Therefore, $\mathcal{S}(\mathcal{L}) \supseteq \mathbb{F}^*\mathcal{S}(\mathcal{L})$. Since $\mathcal{S}(\mathcal{L})$ is a closed set (cf. Proposition 1.2), the latter inclusion implies that $\mathcal{S}(\mathcal{L})$ is a cone provided it is nonempty. The proof is complete. $\square$

LEMMA 1.4. *For an arbitrary set* $\mathcal{V} \subseteq \mathcal{M}_n$, *the image* $\chi(\mathcal{V})$ *is dense in* $\mathbb{F}^n$ *if and only if so is* $\chi(\mathbf{I}+\mathcal{V})$.

P r o o f. Observe that

$$\begin{aligned}\lambda^n + \sum_{j=1}^{n}(-1)^j\,\mathrm{s}_n^j(\mathbf{I}+\mathbf{A})\lambda^{n-j} &= \det\big(\lambda\mathbf{I} - (\mathbf{I}+\mathbf{A})\big)\\ &= \det\big((\lambda-1)\mathbf{I} - \mathbf{A}\big)\\ &= (\lambda-1)^n + \sum_{j=1}^{n}(-1)^j\,\mathrm{s}_n^j(\mathbf{A})(\lambda-1)^{n-j}\end{aligned}$$

for all $\lambda \in \mathbb{F}$ and for all $\mathbf{A} \in \mathcal{M}_n$. Reducing the polynomial on the right hand side to the form $\lambda^n + \sum_{j=1}^{n} c_j\lambda^{n-j}$ one can see that there is a triangular affine automorphism $\Xi\colon \mathbb{F}^n \to \mathbb{F}^n$ such that $\big(\mathrm{s}_n^j(\mathbf{I}+\mathbf{A})\big)_{j=1}^n = \Xi\big((\mathrm{s}_n^j(\mathbf{A}))_{j=1}^n\big)$ for all $\mathbf{A} \in \mathcal{M}_n$. The assertion follows. $\square$

Let us turn to the main result of the section. For a closed set $\mathcal{E} \subseteq \mathcal{M}_n$ we define $\widehat{\mathcal{E}} \subseteq \mathcal{M}_n$ to be the Zariski closure of the union $\bigcup_{\mathbf{U}\in\mathcal{GL}_n} \mathbf{U}^{-1}\mathcal{E}\mathbf{U}$. Notice that $\widehat{\mathcal{E}}$ is irreducible whenever so is $\mathcal{E}$.

THEOREM 1.5. *The following conditions are equivalent:*

(1) $\mathcal{S}(\mathcal{L}) = \emptyset$,
(2) $\mathbf{0} \in \mathcal{R}(\mathcal{L})$,
(3) $\mathbf{I} \in \mathcal{R}(\mathcal{L})$,
(4) $\widehat{\mathcal{L}} = \mathcal{M}_n$.

P r o o f. Implication (1) $\Longrightarrow$ (2) is obvious. Implication (2) $\Longrightarrow$ (1) is a consequence of Proposition 1.3(iii). Equivalence (2) $\Longleftrightarrow$ (3) follows from Lemma 1.4. Assume that condition (4) is satisfied. Then, by the continuity and the $\mathcal{GL}_n$-invariancy of the map χ, we have

$$\mathbb{F}^n = \chi(\widehat{\mathcal{L}}) \subseteq \overline{\chi\Big(\bigcup_{\mathbf{U}\in\mathcal{GL}_n} \mathbf{U}^{-1}\mathcal{L}\mathbf{U}\Big)} = \overline{\chi(\mathcal{L})},$$

where the bars denote Zariski closures. Condition (2) follows. Assume that (2) is satisfied. Then the inverse image $\chi^{-1}\big(\chi(\mathcal{L})\big)$ contains a nonempty open subset of $\mathcal{M}_n$. Since the set $\mathcal{V} := \{\mathbf{A} \in \mathcal{M}_n :$ all eigenvalues of $\mathbf{A}$ are of multiplicity $1\}$ is open, the intersection $\chi^{-1}(\chi(\mathcal{L})) \cap \mathcal{V}$ is dense in $\mathcal{M}_n$. Pick an $\mathbf{A} \in \chi^{-1}(\chi(\mathcal{L})) \cap \mathcal{V}$. Then there is a $\mathbf{B} \in \mathcal{L}$ such that $\chi(\mathbf{A}) = \chi(\mathbf{B})$. Therefore, the eigenvalues of $\mathbf{B}$ coincide with those of $\mathbf{A}$. It turns out that $\mathbf{A} = \mathbf{U}^{-1}\mathbf{B}\mathbf{U}$ for a $\mathbf{U} \in \mathcal{GL}_n$. Hence, $\chi^{-1}(\chi(\mathcal{L})) \cap \mathcal{V} \subseteq \widehat{\mathcal{L}}$. Condition (4) follows. The proof is complete. □

The above theorem extends, in some sense, the main results of [1], [2].

Corollary 1.6. *If $\mathcal{S}(\mathcal{L}) \neq \emptyset$, then $\mathcal{S}(\mathcal{L}) \supseteq \mathbb{F}\mathbf{I} + \mathcal{L}$.*

Proof. Observe that $\chi(\lambda\mathbf{I} + \mathbf{A} + \mathcal{L}) = \chi(\lambda\mathbf{I} + \mathcal{L})$ for all $\lambda \in \mathbb{F}$ and for all $\mathbf{A} \in \mathcal{L}$. Furthermore, by Theorem 1.5 and Proposition 1.3 (iii), the assumption $\mathcal{S}(\mathcal{L}) \neq \emptyset$ yields $\mathbb{F}\mathbf{I} \subset \mathcal{S}(\mathcal{L})$. The claim follows. □

Corollary 1.7. *If $\mathcal{L}$ is a hyperplane in $\mathcal{M}_n$ and $\mathcal{L} \neq \mathfrak{sl}_n$, then $\mathcal{S}(\mathcal{L}) = \emptyset$.*

Proof. Notice that $\mathfrak{sl}_n$ is the only $\mathcal{GL}_n$-invariant linear hyperplane in $\mathcal{M}_n$ and that $\widehat{\mathcal{L}}$ is a $\mathcal{GL}_n$-invariant irreducible closed subset of $\mathcal{M}_n$ containing $\mathcal{L}$. Thus, the above assumptions on $\mathcal{L}$ imply $\mathcal{L} \neq \widehat{\mathcal{L}}$, which yields $\dim \widehat{\mathcal{L}} > \dim \mathcal{L}$. Therefore, $\widehat{\mathcal{L}} = \mathcal{M}_n$. Using Theorem 1.5 completes the proof. □

Let $\mathcal{D}_n \subset \mathcal{M}_n$ be the set of all diagonal matrices. Notice that by Theorem 1.5 and Proposition 1.3 (i), if $\mathcal{L} \supseteq \mathcal{D}_n$, then $\mathcal{S}(\mathcal{L}) = \emptyset$. This is a counterpart of the main result of [2].

It seems to be of some interest to provide a characterization of the singular sets $\mathcal{S}(\mathcal{L})$ for the *triangularizable* subspaces $\mathcal{L}$.

Proposition 1.8. *If $\mathcal{L}$ is such that $\mathbf{U}^{-1}\mathcal{L}\mathbf{U} \subseteq \mathcal{T}_n$ for a $\mathbf{U} \in \mathcal{GL}_n$ and $\mathcal{S}(\mathcal{L}) \neq \emptyset$, then $\mathcal{S}(\mathcal{L}) \supseteq \mathbf{U}\mathcal{T}_n\mathbf{U}^{-1}$.*

Proof. Put $\mathcal{K} = \mathbf{U}^{-1}\mathcal{L}\mathbf{U}$. In virtue of Proposition 1.3 (ii), it is enough to show that $\mathcal{T}_n \subseteq \mathcal{S}(\mathcal{K})$. Consider the map $\Theta \colon \mathcal{M}_n \to \mathcal{D}_n$ defined by $\Theta(\mathbf{A}) = \mathrm{Diag}(\mathbf{A})$. By the assumptions, we have $\mathcal{K} \subseteq \mathcal{T}_n$ and $\mathcal{S}(\mathcal{K}) = \mathbf{U}^{-1}\mathcal{S}(\mathcal{L})\mathbf{U} \neq \emptyset$. Consequently, the image $\chi(\mathcal{K}) = \chi(\Theta(\mathcal{K}))$ is not dense in $\mathbb{F}^n$ (cf. Theorem 1.5). This yields $\dim \Theta(\mathcal{K}) < n$. Pick an $\mathbf{A} \in \mathcal{T}_n$. The image $\chi(\mathbf{A}+\mathcal{K}) = \chi(\mathbf{A}+\Theta(\mathcal{K}))$ is not dense in $\mathbb{F}^n$ because $\mathbf{A} + \Theta(\mathcal{K}) \subset \mathcal{M}_n$ is an affine subspace of dimension smaller than n. Hence, $\mathcal{T}_n \subseteq \mathcal{S}(\mathcal{K})$. The proof is complete. □

One may ask finally the question about the surjectivity of restrictions $\chi_{\mathbf{A}}|_{\mathcal{L}} : \mathcal{L} \to \mathbb{F}^n$. Consider the following example.

EXAMPLE 1.9. Let $\mathcal{L} \subset \mathcal{M}_2$ be defined by

$$\mathcal{L} = \left\{ \begin{bmatrix} \alpha & \alpha \\ \beta & \alpha \end{bmatrix} : \ \alpha, \beta \in \mathbb{F} \right\} .$$

It is not difficult to verify that $\mathcal{S}(\mathcal{L}) = \emptyset$. Pick an $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_2$. Then $\chi_{\mathbf{A}}(\mathcal{L})$ does not contain the product $\{a + d - 2b\} \times (\mathbb{F} \setminus \{(a-b)(d-b)\})$.

Thus even if $\mathcal{R}(\mathcal{L}) = \mathcal{M}_n$, the set $\{\mathbf{A} \in \mathcal{M}_n : \ \chi_{\mathbf{A}}(\mathcal{L}) = \mathbb{F}^n\}$ may be empty.

2. Low dimensional case

We begin with a counterpart of Helton-Rosenthal-Wang's Theorem 1.1 for the size $n = 2$. Notice that only two-dimensional subspaces of $\mathcal{M}_2$ can be of interest.

THEOREM 2.1. *Let $\mathcal{L}$ be a linear subspace of $\mathcal{M}_2$ such that* $\dim \mathcal{L} = 2$ *and* $\mathcal{L} \not\subset \mathfrak{sl}_2$. *Then the singular set $\mathcal{S}(\mathcal{L})$ is a linear hyperplane in $\mathcal{M}_2$ provided it is nonempty.*

Proof. Pick matrices $\mathbf{B} = [b_{kl}], \mathbf{C} = [c_{kl}] \in \mathcal{M}_2 \setminus \mathfrak{sl}_2$ which form a basis of the subspace $\mathcal{L}$. Put $\tau_1 = \operatorname{tr}(\mathbf{B})$, $\tau_2 = \operatorname{tr}(\mathbf{C})$, $\delta_1 = \det(\mathbf{B})$, and $\delta_2 = \det(\mathbf{C})$. A matrix $\mathbf{A} = [a_{kl}] \in \mathcal{M}_2$ is an element of $\mathcal{S}(\mathcal{L})$ if and only if the regular map $\Psi_{\mathbf{A}} \colon \mathbb{F}^2 \to \mathbb{F}^2$ defined by $\Psi_{\mathbf{A}}(\lambda, \mu) = \chi_{\mathbf{A}}(\lambda \mathbf{B} + \mu \mathbf{C})$ is not dominant. Observe that this is the case if and only if the polynomial $\psi_{\mathbf{A}} \colon \mathbb{F}^2 \to \mathbb{F}$ defined by $\psi_{\mathbf{A}}(\lambda, \mu) = \det\big(\mathrm{d}_{(\lambda,\mu)} \Psi_{\mathbf{A}}\big)$ is identically equal to zero. A direct calculation reveals that

$$\begin{aligned} \psi_{\mathbf{A}}(\lambda, \mu) = {} & (\tau_1 \omega - 2\tau_2 \delta_1)\lambda + (2\tau_1 \delta_2 - \tau_2 \omega)\mu + (\tau_1 c_{22} - \tau_2 b_{22}) a_{11} \\ & - (\tau_1 c_{21} - \tau_2 b_{21}) a_{12} - (\tau_1 c_{12} - \tau_2 b_{12}) a_{21} + (\tau_1 c_{11} - \tau_2 b_{11}) a_{22} , \end{aligned}$$

where $\omega = b_{11}c_{22} + b_{22}c_{11} - b_{12}c_{21} - b_{21}c_{12}$. The assumption $\mathcal{S}(\mathcal{L}) \neq \emptyset$ yields $\tau_1 \omega - 2\tau_2 \delta_1 = 0 = 2\tau_1 \delta_2 - \tau_2 \omega$. Consequently,

$$\begin{aligned} \mathcal{S}(\mathcal{L}) = \big\{ \mathbf{A} = [a_{kl}] \in \mathcal{M}_2 : {} & (\tau_1 c_{22} - \tau_2 b_{22}) a_{11} - (\tau_1 c_{21} - \tau_2 b_{21}) a_{12} \\ & - (\tau_1 c_{12} - \tau_2 b_{12}) a_{21} + (\tau_1 c_{11} - \tau_2 b_{11}) a_{22} = 0 \big\} . \end{aligned}$$

Since $\mathbf{B}$ and $\mathbf{C}$ are linearly independent and $\tau_1 \neq 0 \neq \tau_2$, at least one of the coefficients of the linear form which describes the set $\mathcal{S}(\mathcal{L})$ is nonzero. The proof is complete. □

The following corollary is an immediate consequence of the above theorem and Proposition 1.8.

COROLLARY 2.2. *If $\mathcal{L} \subset \mathcal{M}_2$, $\dim \mathcal{L} = 2$, $\mathcal{L} \not\subset \mathfrak{sl}_2$, and $\mathbf{U}^{-1}\mathcal{L}\mathbf{U} \subset \mathcal{T}_2$ for a $\mathbf{U} \in \mathcal{GL}_2$, then either $\mathcal{S}(\mathcal{L}) = \emptyset$ or $\mathcal{S}(\mathcal{L}) = \mathbf{U}\mathcal{T}_2\mathbf{U}^{-1}$.*

Revising the proof of Theorem 2.1 leads to stating effective formulae.

PROPOSITION 2.3. *Let $\mathbf{B} = [b_{kl}]$, $\mathbf{C} = [c_{kl}] \in \mathcal{M}_2$ be two linearly independent matrices such that $\tau_1 := \operatorname{tr}(\mathbf{B}) \neq 0 \neq \tau_2 := \operatorname{tr}(\mathbf{C})$. Denote by $\mathcal{K}$ the linear subspace of $\mathcal{M}_2$ generated by $\mathbf{B}$ and $\mathbf{C}$. Define $\omega = b_{11}c_{22} + b_{22}c_{11} - b_{12}c_{21} - b_{21}c_{12}$. Then the following conditions are equivalent:*

(1) $\mathcal{S}(\mathcal{K}) \neq \emptyset$,

(2) $\tau_1\omega - 2\tau_2 \det(\mathbf{B}) = 0 = 2\tau_1 \det(\mathbf{C}) - \tau_2\omega$.

Moreover, if $\mathcal{S}(\mathcal{K}) \neq \emptyset$, then

$$\begin{aligned}\mathcal{S}(\mathcal{K}) = \big\{\mathbf{A} = [a_{kl}] \in \mathcal{M}_2 : (\tau_1 c_{22} - \tau_2 b_{22})a_{11} - (\tau_1 c_{21} - \tau_2 b_{21})a_{12} \\ - (\tau_1 c_{12} - \tau_2 b_{12})a_{21} + (\tau_1 c_{11} - \tau_2 b_{11})a_{22} = 0\big\}.\end{aligned}$$

In the case of $n = 3$ the situation is much more complicated. To see that, consider an example of a linear subspace $\mathcal{L} \subset \mathcal{M}_3$ whose singular set $\mathcal{S}(\mathcal{L})$ is reducible and of pure codimension two.

EXAMPLE 2.4. Define

$$\mathcal{L} = \left\{ \begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{F} \right\}.$$

Let $\mathbf{A} = [a_{kl}] \in \mathcal{M}_3$. A straightforward calculation (cf. the proof of Theorem 2.1) shows that $\mathbf{A} \in \mathcal{S}(\mathcal{L})$ if and only if either $a_{31} = a_{21} = 0$ or $a_{31} = a_{32} = 0$. Hence $\mathcal{S}(\mathcal{L}) = \mathcal{K}_1 \cup \mathcal{K}_2$, where $\mathcal{K}_1 = \{\mathbf{A} = [a_{kl}] \in \mathcal{M}_3 : a_{31} = a_{21} = 0\}$ and $\mathcal{K}_2 = \{\mathbf{A} = [a_{kl}] \in \mathcal{M}_3 : a_{31} = a_{32} = 0\}$.

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