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K-RADICAL CLASSES OF ABELIAN LINEARLY ORDERED GROUPS

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The notions of a radical class and a semisimple class of linearly ordered groups were introduced and studied by C. G. Chehata and R. Wiegandt [1]. Further results in that field were obtained in [6], [7] and [10].

Radical classes and semisimple classes of abelian linearly ordered groups were investigated in the papers [9], [11], [12] and [5]. Let $\mathscr{G}_a$ be the class of all abelian linearly ordered groups.

Let X be a radical class of abelian linearly ordered groups. X is said to be a K-radical class if it can be defined by means of the properties of the lattices of convex subgroups of linearly ordered groups belonging to X.

An analogous notion of K-radical class of lattice ordered groups was studied by P. Conrad [2]. Cf. also [8] and [3].

Let $\mathscr{R}_K$ be the collection of all K-radical classes of abelian linearly ordered groups. The collection $\mathscr{R}_K$ is partially ordered by inclusion.

It will be shown that $\mathscr{R}_K$ is a complete lattice. For $\emptyset \neq X \subseteq \mathscr{G}_a$ let $T_K(X)$ be the least element of $\mathscr{R}_K$ containing X as a subclass. A constructive description of $T_K(X)$ will be presented. It will be proved that the relation

$$T_K(X) = \text{Ext Lat Hom } X$$

is valid. (For denotations, cf. Section 1 below.) Let $\mathscr{R}_a$ be the lattice of all radical classes of abelian linearly ordered groups (cf. [9]). It will be proved that $\mathscr{R}_K$ is a closed sublattice of $\mathscr{R}_a$.

1. Preliminaries

The class of all abelian linearly ordered groups will be denoted by $\mathscr{G}_a$. For $G \in \mathscr{G}_a$ let $c(G)$ be the system of all convex l-subgroups of G. The system $c(G)$ is partially ordered by inclusion. Then, in fact, $c(G)$ is a linearly ordered set. Moreover, $c(G)$ is a complete lattice. From this it follows that the lattice operations in $c(G)$ coincide with the corresponding set-theoretical operations i.e., for $\{G_i\}_{i\in I} \subseteq c(G)$ we have

$$\bigwedge_{i\in I} G_i = \bigcap_{i\in I} G_i, \qquad \bigvee_{i\in I} G_i = \bigcup_{i\in I} G_i.$$

We recall some notions concerning radical classes of abelian linearly ordered groups (cf. also [9]).

By considering a subclass X of $\mathscr{G}_a$ we always suppose that X is closed with respect to isomorphisms and that $\{0\} \in X$.

A subclass X of $\mathscr{G}_a$ is said to have the transfinite extension property if, whenever $G \in \mathscr{G}_a$ and

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \ (\alpha < \delta)$$

is an ascending chain of convex subgroups of G such that

$$G_\beta / \bigcup_{\gamma < \beta} G_\gamma \in X \text{ for each } \beta < \delta,$$

then $\bigcup_{\alpha < \delta} G_\alpha$ *belongs to* X. *We express this fact by saying that* X *is closed with respect to transfinite extensions.*

Under the above denotations, the linearly ordered group $\bigcup_{\alpha < \delta} G_\alpha$ *is said to be a transfinite extension of linearly ordered groups* G'_α $(\alpha < \delta)$, *where* G'_α *is isomorphic to* $G_\alpha / \bigcup_{\gamma < \alpha} G_\gamma$ *for each* $\alpha < \delta$

A class $X \subseteq \mathscr{G}_a$ *is said to be a radical class if it is closed with respect to homomorphisms and with respect to transfinite extensions.*

1.1. Definition. *Let X be a radical class of abelian linearly ordered groups. X is said to be a K-radical class, if it satisfies the following condition: whenever $G \in X$ and $G_1 \in G_a$ such that $c(G)$ is isomorphic to $c(G_1)$, then $G_1 \in X$.*

For each $X \in \mathscr{R}_a$ let $\varphi(X) = T_K(X)$. It will be shown that (i) φ is a complete homomorphism with respect to the operation $\bigvee$, and (ii) φ fails to be a homomorphism with respect to the operation $\bigwedge$. The lattice $\mathscr{R}_K$ has no atoms and no antiatoms. It will be proved that if X is hereditary, then the class $T_K(X)$ is hereditary as well.

Let $\mathscr{R}_K$ be the collection of all K-radical classes; the collection $\mathscr{R}_K$ is considered as being partially ordered by inclusion.

If $\{X_j\}_{j\in J}$ is a subcollection of $\mathscr{R}_K$, then we obviously have

$$\bigcap_{j\in J} X_j \in \mathscr{R}_K.$$

Let 0^- be the class of all one-element groups bvelonging to $\mathscr{G}_a$. Then $0^- \in \mathscr{R}_K$; also, 0^- is the least element of $\mathscr{R}_K$ and $\mathscr{G}_a$ is the greatest element of $\mathscr{R}_K$. Thus we obtain:

1.2. Theorem. *$\mathscr{R}_K$ is a complete lattice. If $\{X_j\}_{j\in J}$ is a nonempty subcollection of $\mathscr{R}_K$, then*

$$\bigwedge_{j \in J} X_j = \bigcap_{j \in J} X_j .$$

For $X \subseteq \mathscr{G}_a$ let Ext X be the class of all $G \in \mathscr{G}_a$ which can be expressed as transfinite extensions of linearly ordered groups belonging to X; next let Hom X be the class of all homomorphic images of elements of X.

Let I be a linearly ordered set and for each $i \in I$ let G_i be a linearly ordered group. Let H be the set of all functions $f\colon I \to \bigcup_{i \in I} G_i$ such that (i) $f(i) \in G_i$ for each $i \in I$, and (ii) the set $\{i \in I : f(i) \neq 0\}$ is either empty or dually well-ordered. The operation $+$ in H is defined coordinate-wise and for $f_1, f_2 \in H$ with $f_1 \neq f_2$ we put $f_1 < f_2$ if there exists $i \in I$ such that $f_1(i) < f_2(i)$ and $f_1(j) = f_2(j)$ for each $j \in I$ with $j > i$. Then H is said to be a lexicographic product of linearly ordered groups G_i ($i \in I$) and we write $H = \Gamma_{i \in I} G_i$. If $I = \{1, 2, \ldots, n\}$, then we write also $H = G_1 \circ G_2 \circ \ldots \circ G_n$.

Let us mention an example of a K-radical class which is distinct from 0^- and from $\mathscr{G}_a$.

1.3. Example. Let X be the class of all linearly ordered groups such that $c(G)$ is a well-ordered set. We shall verify that $X \in \mathscr{R}_K$ and that $0^- \neq X \neq \mathscr{G}_a$.

Let Z be the additive group of all integers with the natural linear order. Next let N and Q be the set of all non-negative integers or the set of all rational numbers, respectively (with the natural linear order). For each $i \in N$ and each $j \in Q$ let $H_i = H_j = Z$. Put

$$G_1 = \Gamma_{i \in N} H_i, \qquad G_2 = \Gamma_{j \in Q} H_j .$$

Then $c(G_1)$ is a well-ordered set, while $c(G_2)$ fails to be well-ordered. Since $G_1 \neq \{0\}$, we have $0^- \neq X \neq \mathscr{G}_a$.

If $G \in X$ and $H \in \mathrm{Hom}\, X$, then $c(H)$ is well-ordered (because $c(H)$ is isomorphic to a dual ideal of the linearly ordered set $c(G)$). Hence $H \in X$. Moreover, if $G' \in \mathrm{Ext}\, X$, then $c(G')$ is well-ordered. Therefore X is a radical class of abelian linearly ordered groups. According to the definition of X we infer that X is a K-radical class.

Let I be a linearly ordered set. We denote by $d(I)$ the system of all subsets $I_1 \subseteq I$ which satisfy the following condition: if $i_1 \in I_1$, $i \in I$ and $i < i_1$, then $i \in I_1$. (Hence $d(I)$ is the system of all subsets $I_1 \subseteq I$ such that either $I_1 = \emptyset$ or I_1 is an ideal of I.) The set $d(I)$ is partially ordered by inclusion.

1.4. Lemma. *Let I be a linearly ordered set. For each $i \in I$ let G_i be a nonzero archimedean linearly ordered group. Put $G = \Gamma_{i \in I} G_i$. Then the linearly ordered set $c(G)$ is isomorphic to $d(I)$.*

Proof. The assertion of the lemma is a consequence of the fact that an archimedean linearly ordered group has no nontrivial convex subgroup.

If P is a partially ordered set, x and y are elements of P with $x < y$ and if $[x, y]$ is a prime interval, then we write $x \prec y$.

Let J be a linearly ordered set. For each $j \in J$ let I_j be a linearly ordered set with card $I_j \geqq 2$. Assume that the following condition is satisfied:

(α) *Let $j(1)$ and $j(2)$ be distinct elements of J and let $x \in I_{j(1)}$, $y \in I_{j(2)}$. Then $x = y$ if and only if we have either*

$$j(1) \prec j(2),\ x = \max I_{j(1)} \text{ and } y = \min I_{j(2)},$$

or

$$j(2) \prec j(1),\ y = \max I_{j(2)} \text{ and } x = \min I_{j(1)}.$$

Denote $I = \bigcup_{j \in J} I_j$. For $u, v \in I$ $u \neq v$ we put $u < v$ if either

(i) there is $j \in J$ such that both u and v belong to I_j and $u < v$ in I_j,

or

(ii) there are $j(1)$ and $j(2)$ in J such that $j(1) < j(2)$, $u \in I_{j(1)}$ and $v \in I_{j(2)}$.

Then I is a linearly ordered set under $\leqq$, which will be denoted by

$$\Sigma^0_{j \in J} I_j; \tag{1}$$

I will be said to be the reduced lexicographic sum of linearly ordered sets $I_j (j \in J)$.

From each system of mutually disjoint linearly ordered set $I_j (j \in J')$, where J' is linearly ordered and card $I_j' \geqq 2$ for each $j \in J'$ we can construct a reduced lexicographic sum if some elements of the set $\bigcup_{j \in J} I_j'$ are identified according to the condition (α). The linearly ordered set constructed in this way will be also denoted as in (1).

From the definition of reduced lexicographic sum we immediately obtain the following generalization of Lemma 1.4:

1.5. Lemma. *Let J be a linearly ordered set. For each $j \in J$ let G_j be a linearly ordered group, $G_j \neq \{0\}$. Let $G = \Gamma_{j \in J} G_j$. Then the linearly ordered set $c(G)$ is isomorphic to*

$$d(\Sigma^0_{j \in J} c(G_j)).$$

Next, from the definition of transfinite extension of abelian linearly ordered groups (cf., e.g., [9]) we infer:

1.6. Lemma. *Let $G \in \mathcal{G}_a$. Assume that G is a transfinite extension of linearly ordered groups H_α ($\alpha < \beta$), $H_\alpha \neq \{0\}$. Then the linearly ordered set $c(G)$ is isomorphic to*

$$d(\Sigma^0_{\alpha < \beta} c(H_\alpha)).$$

We conclude this section by an example of a radical class of abelian linearly ordered groups which fails to be a K-radical class.

1.7. Example. Let Z be as above. Let us now denote by Q the additive group of all rational numbers with the natural linear order. According to Propos. 2.2 in [9], $X = \text{Ext Hom}\,\{Z\}$ is a radical class of linearly ordered groups and clearly Q does not belong to X. We have $c(Z) \simeq c(Q)$. Thus X fails to be a K-radical class.

2. The operation $\bigvee$ in the lattice $\mathscr{R}_K$

If P_1 and P_2 are isomorphic partially ordered sets, then we write $P_1 \simeq P_2$. For $X \subseteq \mathscr{G}_a$ we denote by Lat X the class of all abelian linearly ordered groups G such that there exists $G_1 \in X$ with $c(G) \simeq c(G_1)$.

The following lemma is obvious.

2.1. Lemma. *Let $G \in \mathscr{G}_a$, $H \in c(G)$. Then the linearly ordered set $c(G/H)$ is isomorphic to the interval $[H, G]$ of $c(G)$.*

2.2. Corollary. *Let $G \in \mathscr{G}_a$. If G' is a homomorphic image of G, then $c(G')$ is isomorphic to a dual ideal of $c(G)$ having a least element. Conversely, let I be a dual ideal of $c(G)$ having a least element. Then there exists a homomorphic image G' of G such that $I \simeq c(G')$.*

From 2.2 we obtain:

2.3. Lemma. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$,* Hom $X = X$. *Then we have* Hom Lat $X =$ $=$ Lat X.

2.4. Corollary. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Then* Hom Lat Hom $X =$ Lat Hom X.

2.5. Lemma. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Then* Lat Ext Lat $X =$ Ext Lat X.

Proof. Let $G \in$ Lat Ext Lat X. Hence there exists $H \in$ Ext Lat X such that $c(G) \simeq c(H)$. Let φ be an isomorphism of $c(G)$ onto $c(H)$. There exist convex subgroups $H_\alpha (\alpha < \beta)$ of H such that

$\{0\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_\alpha \subseteq \ldots \subseteq H (\alpha < \beta)$, $\bigcup_{\alpha < \beta} H_\alpha = H$ and for each $\alpha < \beta$, $H_\alpha / \bigcup_{\gamma < \alpha} H_\gamma$ either is a zero group or belongs to Lat X. Thus we have

$$\{0\} = \varphi^{-1}(H_0) \subseteq \varphi^{-1}(H_1) \subseteq \ldots \subseteq \varphi^{-1}(H_\alpha) \subseteq \ldots \subseteq G, \bigcup_{\alpha < \beta} \varphi^{-1}(H_\alpha) = G.$$

In view of 2.2 we have $c(\varphi^{-1}(H_\alpha)/\bigcup_{\gamma < \alpha} \varphi^{-1}(H_\gamma)) \simeq c(H_\alpha / \bigcup_{\gamma < \alpha} H_\gamma)$. Hence $G \in$ Ext Lat X and therefore Lat Ext Lat $X \subseteq$ Ext Lat X. Clearly Ext Lat $X \subseteq$ - Lat Ext Lat X.

2.6. Lemma. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Then* Hom Ext Hom $X =$ Ext Hom X.

Proof. This follows from Proposition 2.2 in [9].

2.7. Lemma. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Put* $Y =$ Ext Lat Hom X. *Then we have*

$$\text{Ext } Y = Y, \text{ Lat } Y = Y \textit{ and } \text{Hom } Y = Y.$$

Proof. The first assertion is obvious, because Ext Ext $Z = Z$ for each nonempty subclass Z of $\mathscr{G}_a$. The second assertion follows from 2.5. Finally, in view of 2.4 and 2.6 we have

Hom Y = Hom (Ext Lat Hom X) = Hom Ext (Lat Hom X) =

$=$ Hom Ext (Hom Lat Hom X) $=$ Hom Ext Hom (Lat Hom X) $=$
$=$ Ext Hom (Lat Hom X) $=$ Ext (Hom Lat Hom X) $=$ Ext Lat Hom $X = Y$.

For $\emptyset \neq X \subseteq \mathcal{G}_a$ we denote by $T_K(X)$ the K-radical class generated by X (i.e., $T_K(X)$ is the intersection of all K-radical classes Z with $X \subseteq Z$).

From 2.7 we obtain:

2.8. Theorem. *Let* $\emptyset \neq X \subseteq \mathcal{G}_a$. *Then* $T_K(X) =$ Ext Lat Hom X.

2.9. Theorem. *Let* X_i $(i \in I)$ *be K-radical classes. Then in the lattice* $\mathcal{R}_K$ *the relation*

$$\bigvee_{i \in I} X_i = \operatorname{Ext}\left(\bigcup_{i \in I} X_i\right)$$

is valid.

Proof. We have obviously

$$\bigvee_{i \in I} X_i = T_K\left(\bigcup_{i \in I} X_i\right).$$

Because X_i are K-radical classes, we obtain $X_i =$ Hom X_i and Lat $X_i = X_i$ for each $i \in I$. Hence in view of 2.8,

$T_K\left(\bigcup_{i \in I} X_i\right) = \operatorname{Ext}\left(\bigcup_{i \in I} X_i\right)$.

The above theorem gives a constructive description of the operation $\bigvee$ in the lattice $\mathcal{R}_K$.

3. Another characterization of K-radical classes

We have defined a K-radical class as a radical class of abelian linearly ordered groups fulfilling a particular condition. In this section it will be shown that a K-radical class can be characterized directly, without using the notion of a radical class of an abelian linearly ordered group.

Next we shall prove that the lattice $\mathcal{R}_K$ is a closed sublattice of the lattice of all radical classes of abelian linearly ordered groups.

Let $\emptyset \neq X \subseteq \mathcal{G}_a$. We denote by $l(X)$ the class of all chains $\mathcal{L}$ having the property that there exists $G \in X$ with $c(G) \simeq L$.

3.1. Lemma. *Let X be a K-radical class. Then the following conditions are fulfilled*:

(i) *If J is a well-ordered set and if for each* $j \in J$ *there is given a linearly ordered set* L_j *with* card $L_j \geqq 2$ belonging *to* $l(X)$, *then* $d(\Sigma^0_{j \in J} L_j)$ also belongs to $l(X)$.

(ii) *If* $L \in l(X)$ *and if* L_1 *is a principal dual ideal of* L, *then* $L_1 \in l(X)$.

(iii) *if* $L \in l(X)$, $G \in \mathcal{G}_a$, $C(G) \simeq L$, *then* $G \in X$.

Proof. The assertion (i) follows from the fact that X is closed with respect to transfinite extensions, and from 1.6. Next, because X is closed with respect to homomorphisms, from 2.2 we infer that (ii) is valid. The validity of (iii) is obvious.

3.2. Lemma. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Assume that the conditions* (i), (ii) *and* (iii) *from* 3.1 *are satisfied. Then X is a K-radical class.*

Proof. From (i) and from 1.6 it follows that X is closed with respect to transfinite extensions. According to (ii) and 2.2, X is closed with respect to homomorphisms. Thus X is a radical class of abelian linearly ordered groups. Next, from (iii) we infer that X is a K-radical class.

3.3. Theorem. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Then X is a K-radical class if and only if the conditions* (i), (ii) *and* (iii) *are valid.*

Also, from 1.5, 3.1 and 3.2 we infer:

3.4. Theorem. *Let $\neq \emptyset$ $\mathscr{L}$ be a class of chains. Assume that $\mathscr{L}$ is closed with respect to isomorphisms. Then the following conditions are equivalent*:

(a) *There exists a K-radical class X such that $l(X) = \mathscr{L}$.*

(b) *$\mathscr{L}$ fulfils the conditions* (i), (ii) *and* (iii) *from* 3.1 (with *$l(X)$ replaced by $\mathscr{L}$*).

Let $\mathscr{R}_a$ be the collection of all radical classes of abelian linearly ordered groups; $\mathscr{R}_a$ is partially ordered by inclusion. Then $\mathscr{R}_a$ is a complete lattice; moreover, we have (cf. [9])

3.5. Proposition. *Let $A_i \in \mathscr{R}_a$ $(i \in I)$. Then in the lattice $\mathscr{R}_a$ the relations*

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i, \qquad \bigvee_{i \in I} A_i = \operatorname{Ext}\left(\bigcup_{i \in I} A_i\right)$$

are fulfilled.

The above proposition and Theorem 2.9 yield:

3.6. Theorem. *$\mathscr{R}_K$ is a closed sublattice of the lattice $\mathscr{R}_a$.*

Let $\emptyset \neq X \subseteq \mathscr{G}_a$. The intersection of all elements $Y \in \mathscr{R}_a$ with $X \subseteq Y$ will be denoted by $T_a(X)$ (the radical class of abelian linearly ordered groups generated by X). We have $T_a(X) = \operatorname{Ext} \operatorname{Hom} X$ (cf. [9]).

Let us consider the mapping $\varphi(X) = T_K(X)$, where X runs over the lattice $\mathscr{R}_a$. We can ask whether φ is a $\wedge$-homomorphism or a $\vee$-homomorphism.

3.7. Example. There are X_1 and X_2 in $\mathscr{R}_a$ such that $\varphi(X_1 \wedge X_2) \neq \varphi(X_1) \wedge \varphi(X_2)$.

Let $X_1 = T_a(Z)$ and $X_2 = T_a(Q)$. Then $X_1 \wedge X_2 = X_1 \cap X_2 = 0^-$, hence $\varphi(X_1 \wedge X_2) = 0^-$. On the other hand, $c(Z) \simeq c(Q)$ and thus $\varphi(X_1) = \varphi(X_2)$. Therefore $\varphi(X_1) \wedge \varphi(X_2) \neq 0^-$.

3.8. Theorem. *Let X_i $(i \in I \neq \emptyset)$ be elements of $\mathscr{R}_a$. Then we have*

$$\varphi\left(\bigvee_{i \in I} X_i\right) = \bigvee_{i \in I} \varphi(X_i).$$

Proof. The assertion easily follows from the fact that for $X \subseteq \mathscr{G}_a$ we have $\operatorname{Lat} \operatorname{Ext} X = \operatorname{Ext} \operatorname{Lat} X$, whence $T_K(X) = \operatorname{Lat} \operatorname{Ext} \operatorname{Hom} X$.

By summarizing, we obtain:

3.9. Corollary. *The mapping φ is a complete homomorphism with respect to the operation $\vee$, but it fails to he a homomorphism with respect to the operation $\wedge$.*

Let $\{0\} \neq G \in \mathscr{G}_a$. Let us recall the notion of the skeleton of G (cf.

Fuchs [4]). Let us denote by $c_p(G)$ the system of all principal convex subgroups of G. Let Π be a set indexing the system $c_p(G)$ and inversely ordered. For $\pi \in \Pi$ we denote by C_π the corresponding element of $c_p(G)$. There exists a uniquely determined element D_π of $c(G)$ such that $D_\pi \prec C_\pi$ is valid in $c(G)$. Put $B_\pi = C_\pi / D_\pi$. Then B_π is a real group. The skeleton of G is defined to be the system $[\Pi, B_\pi(\pi \in \Pi)]$. Put $\Pi = \Pi(G)$.

The following consideration shows that by defining the notion of a K-radical class we can apply the linearly ordered set $\Pi(G)$ instead of $c(G)$.

It is easy to verify that a nonzero element H of $c(G)$ belongs to $c_p(G)$ if and only if there exists $H_1 \in c(G)$ with $H_1 \prec H$. Hence for any $G_1, G_2 \in \mathscr{G}_a$ we have

$$c(G_1) \simeq c(G_2) \Rightarrow c_p(G_1) \simeq c_p(G_2).$$

Next, $c(G)$ is isomorphic to the system of all ideals of the linearly ordered set $c_p(G)$ (this system of ideals is linearly ordered by inclusion). In fact, let $H \in c(G)$; we denote by $\varphi(H)$ the system of all principal convex subgroups H_i of G with $H_i \subseteq H$. Then φ is an isomorphism of $c(G)$ onto the system of all ideals of $c_p(G)$. Thus

$$c_p(G_1) \simeq c_p(G_2) \Rightarrow c(G_1) \simeq c(G_2).$$

Therefore $c(G_1)$ is isomorphic to $c(G_2)$ if and only if $\Pi(G_1)$ is isomorphic to $\Pi(G_2)$.

Darnel [3] applied the notion of the skeleton by constructing examples of radical classes (these examples fail to be, in general, K-radical classes).

4. Nonexistence of atoms and antiatoms in $\mathscr{R}_K$

Let α be an infinite cardinal. We denote by $\omega(\alpha)$ the first ordinal with cardinality α. Let $G \in \mathscr{G}_a$, $G \neq \{0\}$. We put

$$G(\alpha) = \Gamma_{i \in I} G_i$$

where $I = \omega(\alpha)$ and $G_i = G$ for each $i \in I$.

Each dual ideal of I is isomorphic to I. Thus we obtain

4.1. Lemma. *Let* $\{0\} \neq H \in \operatorname{Lat\,Hom} Y$ *where* $Y = \{G(\alpha)\}$. *Then* card $H \geqq \alpha$.

In view of 2.8 we infer:

4.2. Corollary. *Let* $\{0\} \neq H \in T_K(Y)$ *where* $Y = \{G(\alpha)\}$. *Then* card $H \geqq \alpha$.

4.3. Lemma. *Let* $\{0\} \neq G \in \mathscr{G}_a$. *Let* α *be a cardinal with* $\alpha >$ card G. *Then* $0^- < T_K(G(\alpha)) < T_K(G)$.

Proof. Since $G(\alpha) \in T_K(G(\alpha))$, we have $0^- < T_K(G(\alpha))$. According to the definition of $G(\alpha)$, the relation $G(\alpha) \in \operatorname{Ext}\{G\}$ is valid. Hence $G(\alpha) \in T_K(G)$

(cf. 2.8) and thus $T_K(G(\alpha)) \leqq T_K(G)$. In view of 4.2, G does not belong to $T_K(G(\alpha))$. Therefore $T_K(G(\alpha)) < T_K(G)$.

4.4. Theorem. *The lattice $\mathscr{R}_K$ has no atom.*

Proof. Let $X \in \mathscr{R}_K$, $X \neq 0^-$. There is $\{0\} \neq G \in X$. Hence $T_K(G) \leqq X$. Now from 4.3 we obtain that X cannot be an atom in $\mathscr{R}_K$.

4.5. Lemma. *Let $\{0\} \neq G \in \mathscr{G}_a$. Let α and β be cardinals,* card $G \leqq \alpha < \beta$. *Then $T_K(G(B)) < T_K(G(\alpha))$.*

Proof. $G(\beta)$ can be represented as a lexicographic product of factors isomorphic to $G(\alpha)$. Hence $G(\beta) \in T_K(G(\alpha))$ and thus $T_K(G(\beta)) < T_K(G(\alpha))$. Now from 4.2 we obtain $G(\alpha) \notin T_K(G(\beta))$. Therefore $T_K(G(\beta)) < T_K((G(\alpha))$.

Let C be a subcollection of $\mathscr{R}_K$. If there exists an injective mapping of the class of all cardinals into C, then C will be said to be a proper collection.

Theorem 4.4 can be strengthened as follows:

4.6. Theorem. *Let $0^- \neq X \in \mathscr{R}_K$. There exists $C \subset [0^-, X]$ such that*

(i) *C is a chain,*

(ii) *C is a proper collection.*

Proof. There exists $G \in X$ with $G \neq \{0\}$. In view of 4.5 it suffices to consider the collection of all K-radical classes $T_K(G(\alpha))$, where α runs over the class of all cardinals larger than card G.

Let $\{0\} \neq G \in \mathscr{G}_a$. Let α be an infinite cardinal and let $\omega(\alpha)$ be as above. Let I be a linearly ordered set dually isomorphic to $\omega(\alpha)$, and for each $i \in I$ let G_i be a linearly ordered group isomorphic to G. We denote $G'(\alpha) = \Gamma_{i \in I} G_i$.

From 2.8 we obtain:

4.7. Lemma. *Let $\{0\} \neq G \in \mathscr{G}_a$ and let α be a cardinal, $\alpha >$* card G. *Then $G'(\alpha)$ does not belong to $T_K(G)$.*

Since $G \in \operatorname{Hom} G'(\alpha)$, we have $G \in T_K(G'(\alpha))$ and thus in view of 4.7 we obtain

4.8. Lemma. *Let G and α be as in* 4.7. *Then $T_K(G) < T_K(G'(\alpha))$.*

A K-radical class of the form $T_K(G)$ is said to be a principal element of $\mathscr{R}_K$. From 4.8 we get

4.9. Corollary. *No principal element of $\mathscr{R}_K$ equals $\mathscr{G}_a$ or is a dual atom of the lattice $\mathscr{R}_K$.*

The second assertion of 4.9 will be strengthened below (cf. Theorem 4.14).

4.10. Proposition. *Let $G \in \mathscr{G}_a$ and $X \in \mathscr{R}_K$. Let $\{H_i\}_{i \in I}$ be the system of all convex subgroups of G which belong to X. Then $\bigcup_{i \in I} H_i$ also belongs to X.*

The proof is analogous to that of [1], Proposition 3; it will be omitted.

Under the denotations as in 4.10 we put $\bigcup_{i \in I} H_i = X[G]$; this linearly ordered group will be called the radical of G with respect to X. We have obviously:

4.11. Lemma. *Let $G \in \mathscr{G}_a$ and $X \in \mathscr{R}_K$. Put $H = G/X[G]$. Then $X[H] = \{0\}$.*

4.12. Lemma. *Let $\{0\} \neq G \in \mathscr{G}_a$, $\{0\} \neq H \in T_K(G)$. Then there exists $H_1 \in c(H)$ such that $H_1 \neq \{0\}$ and* card $c(H_1) \leqq$ card G.

Proof. This is a consequence of 2.8.

In view of the construction of $G'(\alpha)$ we have:

4.13. Lemma. *Let $\{0\} \neq G \in \mathscr{G}_a$ and let α be an infinite cardinal. Let $\{0\} \neq H \in c(G'(\alpha))$. Then card $c(H) \geqq \alpha$.*

4.14. Theorem. *The lattice $\mathscr{R}_K$ has no dual atom.*

Proof. By way of contradiction, assume that X is a dual atom of $\mathscr{R}_K$. Hence there exists $G_1 \in \mathscr{G}_a$ such that G_1 does not belong to X. Thus $X[G_1] \neq G_1$. Put $G = G_1/X[G_1]$. Then $G \neq \{0\}$. In view of 4.11 we have

$$X[G] = \{0\}. \tag{1}$$

Therefore G does not belong to X and hence $T_K(G) \nleq X$. Thus we obtain

$$X \vee T_K(G) = \mathscr{G}_a. \tag{2}$$

Let α be a cardinal, $\alpha > \operatorname{card} G$. In view of (2) and 2.9 we have

$$G'(\alpha) \in X \vee T_K(G) = \operatorname{Ext}(X \cup T_K(G)). \tag{3}$$

From (3) it follows that there exists a nonzero convex subgroup H of $G'(\alpha)$ such that either $H \in X$ or $H \in T_K(G)$. But in view of 4.12 and 4.13, the relation $H \in T_K(G)$ cannot hold. Hence we have $H \in X$.

From the construction of $G'(\alpha)$ we infer that H can be expressed as a lexicographic product $H = H_1 \circ H_2$ such that the following conditions are valid:

(i) H_1 is isomorphic to $G'(\alpha)$;

(ii) H_2 is isomorphic to a convex subgroup of G.

Since $H \in X$ and because H_2 is a homomorphic image of H we get $H_2 \in X$, thus $X[H_2] = H_2$. On the other hand, from (1) and from (ii) we infer that $X[H_2] = \{0\}$, whence $H_2 = \{0\}$. Thus (i) yields that H is isomorphic to $G'(\alpha)$, hence $G'(\alpha) \in X$. Because $G \in \operatorname{Hom} G'(\alpha)$ we obtain $G \in X$, which is a contradiction.

5. Hereditary classes

Let $\emptyset \neq X \subseteq \mathscr{G}_a$. The class X is said to be hereditary if it is closed with respect to isomorphisms and if it fulfils the following condition: whenever $G \in X$ and $H \in c(G)$, then $H \in X$.

In 5.1—5.3 below we suppose that X is a nonempty subclass of $\mathscr{G}_a$.

5.1. Lemma. *Let X be hereditary. Then* Lat X *is hereditary as well.*

Proof. Let $G \in \operatorname{Lat} X$ and $H \in c(G)$. There exists $G_1 \in X$ such that $c(G_1) \simeq c(G)$. Let φ be an isomorphism of $c(G_1)$ onto $c(G)$. Put $H_1 = \varphi^{-1}(H)$. Since X is hereditary, we have $H_1 \in X$. Clearly $c(H_1) \simeq c(H)$ and hence $H \in \operatorname{Lat} X$.

5.2. Lemma. *Let X be hereditary. Then* Hom X *is hereditary.*

Proof. Let $G \in \operatorname{Hom} X$ and $H \in c(G)$. There exists $G_1 \in X$ and $G_2 \in c(G_1)$

such that $G \simeq G_1/G_2$. Let φ be an isomorphism of G onto G_1/G_2. We denote by G_3 the set of all $g_3 \in G_1$ such that $|g_3| \leqq |g_1|$ for some $g_1 \in G_1$ having the property that there exists $h \in H$ with $g_1 \in \varphi(h)$. Then G_3 is a convex subgroup of G_1, $G_2 \subseteq G_3$ and $G_3/G_2 \simeq H$. Now because X is hereditary we infer that G_3 belongs to X and hence $G_3/G_2 \in \operatorname{Hom} X$. Therefore $H \in \operatorname{Hom} X$.

5.3. Lemma. *Let X be hereditary. Then* $\operatorname{Ext} X$ *is hereditary.*

Proof. Let $G \in \operatorname{Ext} X$ and let $H \in c(G)$, $H \neq G$. There exists a chain of convex subgroups of G of the form

$$\{0\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \subseteq G(\alpha < \beta),$$

$$\bigcup_{\alpha < \beta} G_\alpha = G$$

such that for each $\alpha < \beta$ we have $G_\alpha / \bigcup_{\gamma < \alpha} G_\gamma \in X$. Thus

$$H = H \cap G = \bigcup_{\alpha < \beta} (H \cap G_\alpha).$$

There exists the first $\alpha_1 < \beta$ with $H \subset G_{\alpha_1}$. Hence $H \cap G_\alpha = G_\alpha$ for $\alpha < \alpha_1$. The linearly ordered group H is a transfinite extension of linearly ordered groups $G_\alpha / \bigcup_{\gamma < \alpha} G_\gamma (\alpha < \alpha_1)$ and of $H / \bigcup_{\gamma < \alpha_1} G_{\alpha_1}$. We have $H / \bigcup_{\gamma < \alpha_1} G_\gamma \in c(G_{\alpha_1} / \bigcup_{\gamma < \alpha_1} G_\gamma) \in X$. Since X is hereditary, we infer that $H \in \operatorname{Ext} X$.

Remark. Lemma 5.3 can be obtained as a consequence of Theorem 2.1, [10]. The proof of this theorem is more involved than the proof of 5.3, since the commutativity of linearly ordered groups under consideration is not assumed.

5.4. Theorem. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Assume that X is hereditary. Then the class $T_K(X)$ is hereditary as well.*

Proof. This follows from 5.1, 5.2, 5.3 and 2.8.

Also, from 5.1, 5.2 and [9], Propos. 2.2 we obtain (cf. also [5], Corollary 2.2)

5.5. Corollary. *Let $\emptyset \neq X \subseteq \mathscr{G}_a$. Let X be hereditary. Then $T_a(X)$ is hereditary.*

REFERENCES

[1] CHEHATA, C. G., WIEGANDT, R.: Radical theory for fully ordered groups. Mathematica — Rév. d'Anal. Numér. Théor. Approx. 20(43), 1979, 143—157.

[2] CONRAD, P.: K-radical classes of lattice ordered groups. Conf. Algebra Carbondale, 1980, Lect. Notes in Mathematics 848, Springer Verlag 1981, 186—207.

[3] DARNEL, M.: Closure operators on radicals of lattice ordered groups. Czech. Math. J. 37, 1987, 51—64.

[4] FUKS, L.: Častično uporjadočennye algebraičeskie sistemy. Moskva, 1972.

[5] GARDNER, B. J.: Some aspects of radical theory for fully ordered abelian groups. The University of Tasmania, Technical Report No 203, 1985.

[6] JAKUBÍK, J.: On the lattice of radical classes of linearly ordered groups. Studia scient. mathem. Hungar. 19, 1981, 76—86.

[7] JAKUBÍK, J.: On the lattice of semisimple classes of linearly ordered groups. Čas. pěst. matem. 107, 1982, 183—190.

[8] JAKUBÍK, J.: On K-radical classes of lattice ordered groups. Czechoslov. Math. J. 33, 1983, 149—163.
[9] JAKUBÍK, J.: On radical classes of abelian lattice ordered groups. Math. Slovaca 35, 1985, 141—154.
[10] JAKUBÍKOVÁ, M.: Hereditary radical classes of linearly ordered groups. Čas. pěst. matem. 107, 1982, 199—207.
[11] PRINGEROVÁ, G.: Covering condition in the lattice of radical classes of linearly ordered groups. Math. Slovaca 33, 1983, 363—369.
[12] PRINGEROVÁ, G.: On semisimple classes of abelian linearly ordered groups. Čas. pěst. matem. 108, 1983, 40—52.

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K-РАДИКАЛЬНЫЕ КЛАССЫ АБЕЛЕВЫХ ЛИНЕЙНО УПОРЯДОЧЕННЫХ ГРУПП

Ján Jakubík

Резюме

Для линейно упорядоченной группы G обозначим через $c\ (G)$ линейно упорядоченное множество всех выпуклых подгрупп. G. Радикальный класс X (всмысле Чехаты-Вигандта) называетса K-радикальным, если для каждой абелевой линейно упорядоченной группы G_1 и каждого $G_2 \in X$ из $c(G_1) \simeq c(G_2)$ вытекает $G_1 \in X$. В статье исследуется решетка всех K-радикальных классов.