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Dedicated to Professor Tibor Katriňák

PRIME AND MAXIMAL IDEALS OF PARTIALLY ORDERED SETS

MARCEL ERNÉ

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. We exhibit a broad spectrum of classes of ordered sets with the property that in **ZF** set theory, the Ultrafilter Principle (**UP**) is equivalent to the validity of a Prime Ideal Theorem (**PIT**) or of a Maximal Ideal Theorem (**MIT**), respectively, in the specified class. Weak forms of (semi-)distributivity together with **UP** yield the desired Prime Ideal Theorems, while weak forms of complementation are responsible for the corresponding Maximal Ideal Theorems. We also study stronger versions, like the extension or separation by prime and maximal ideals, or meet representations by such ideals. Moreover, we investigate slight variations in the definition of prime ideals, which coincide in the case of lattices, but lead to quite different results in the case of posets. Also, rather small changes of the class of posets under consideration may turn a **PIT** or **MIT** that was equivalent to **UP** in one class into a statement equivalent to the full Axiom of Choice (**AC**) in another class. For example, in the class of arbitrary lower pseudocomplemented posets, **PIT** is false, while **MIT** is equivalent to **UP**, and **MIT** for upper pseudocomplemented posets is equivalent to **AC**. Our results extend many known algebraic, lattice-theoretical or topological facts concerning prime and maximal ideals to the setting of partially ordered sets.

0. Introduction

In the literature, we find dozens, if not hundreds of results saying that one or another form of a prime or maximal ideal theorem holds in certain structures, or, if set-theoretical principles are respected carefully, that the statement in question is equivalent to or at least a consequence of some choice principle not derivable

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in **ZF** set theory. It is well known that certain distributive laws are the crucial ingredient in proving the existence of "enough" prime ideals in diverse algebraic structures, especially in lattices and semilattices. As a general phenomenon, (semi-)distributivity is often decisive for the conclusion that maximal ideals are prime, while certain forms of complementation ensure that, conversely, prime ideals are maximal. To Tibor Katriňák we owe a great part of the modern research on distributive laws and pseudocomplementation in semilattices and lattices; may it suffice here to mention only three of his early papers in that area, quite different in content and size: [23], [24], [25].

The present note is devoted to a systematic investigation and comparison of various prime and maximal ideal theorems in *partially ordered sets* (*posets*). Whereas the case of *lattices* has been settled since quite a long time (see e.g. [29], [30], [31]), some questions remained open for the case of partially ordered sets or semilattices. Ordered sets that fail to be (semi-)lattices receive more and more attention in modern branches of pure and applied mathematics, for example in Formal Concept Analysis ([15]), in Domain Theory ([16]) and in diverse parts of Theoretical Computer Sciences. Therefore, it is certainly a worthwhile undertaking to discover when and to what extent classical prime or maximal ideal theorems may be transferred to the setting of rather general classes of ordered sets, and to discern which choice principles are needed in each case. Of course, optimal results are obtained if

- the Axiom of Choice (**AC**), the Ultrafilter Principle (**UP**) or a similar principle *implies* the ideal theorems for a *large* class of structures,
- **AC** or some weaker choice principle *follows* from the ideal theorems in a *small* class.

Let us mention two typical examples: the class of 1*-semidistributive posets* (see Section 1) is the largest one in which every proper ideal may be extended to a prime ideal — provided the Ultrafilter Principle holds, which is equivalent to the corresponding prime ideal theorem for the much smaller class of power set lattices. Or, the Axiom of Choice implies the existence of maximal ideals in all posets having a top element, while already the existence of maximal ideals in spatial frames (i.e. lattices of open sets of topological spaces) returns the full force of **AC**, as was discovered recently by Herrlich [20]; in contrast to that fact, the existence of maximal ideals in spatial coframes (i.e. lattices of closed sets of topological spaces) is equivalent to **UP** (see [28]).

The notions of prime and maximal ideals or filters, respectively, are well known for lattices but perhaps not for the more general setting of posets, so that it appears opportune to recall here the main definitions.

An *ideal* (*in the sense of Frink* [13]) of a poset S is a subset I such that for any finite subset F of I, the *cut closure*

$$\Delta F = \bigcap \{{\downarrow} b : \ F \subseteq {\downarrow} b\}$$

is still contained in I, where

$${\downarrow} b = \{a \in S : \ a \leq b\}$$

is the *principal ideal* generated by b. For an arbitrary subset B of S, the ideal generated by B is the union of all ΔF where F runs through the finite subsets of B. The ideals form the least algebraic closure system containing all principal ideals, hence an algebraic complete lattice, the (*Frink*) *ideal completion* $\mathcal{I}S$; and the ideals disjoint from a fixed subset build an *algebraic* $\cap$*-system* (closed under unions of directed subsystems and intersections of nonempty subsystems; cf. [5]). But, unlike the ideals in [16], Frink ideals need not be directed! A *filter* of S is an ideal of the dual poset. An ideal or filter is *proper* if it is not the entire poset; of course, *"maximal ideal"* means *"maximal among all proper ideals"*, and analogously for filters.

An element p of a $\wedge$-semilattice is $\wedge$*-prime* if it is not the greatest element and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. A quick computation shows that the *prime ideals*, i.e. the $\wedge$-prime members of the ideal lattice $\mathcal{I}S$, are precisely those ideals which have a down-directed (in particular nonempty) complement; similarly, the $\wedge$-prime members of the *Alexandroff completion* ([1]) or *down-set frame*

$$\mathcal{A}S = \{{\downarrow} B : \ B \subseteq S\},$$

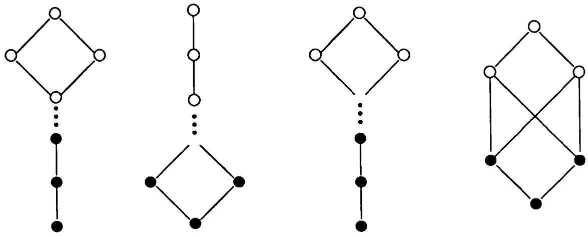
where

$${\downarrow} B = \bigcup \{{\downarrow} b : \ b \in B\}$$

is the *down-set* or *lower set* generated by B, are precisely those down-sets whose complement is down-directed (cf. [8]). *Prime filters* are defined dually. While in bounded lattices the prime filters are just the complements of prime ideals, that up-down duality fails in bounded semilattices!

In the last two sections of this paper, we shall regard a weaker notion of prime ideals that is "invariant under dualization" (see also [8] and [27]):
a *weak prime ideal* is an ideal whose complement is a filter. In the subsequent four diagrams, the bold dots indicate

(1) a non-principal directed prime ideal,
(2) a non-directed prime ideal,
(3) a directed weak prime ideal that is not prime,
(4) a non-directed weak prime ideal that is not prime.



(1) (2) (3) (4)

The central theme of the present note are the following four types of prime ideal theorems for a class **A** of partially ordered sets:

PITA *Prime Ideal Theorem*:
Each non-empty and non-singleton member of **A** has a prime ideal.

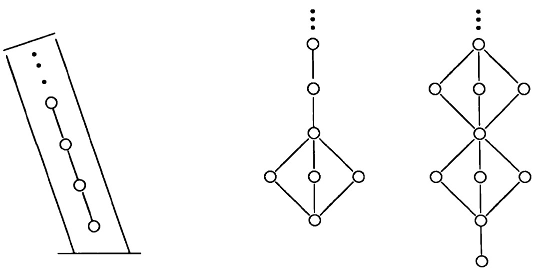
PICA *Prime Ideal Containment Theorem*:
Each proper ideal in any member of **A** is contained in a prime ideal.

PIRA *Prime Ideal Representation Theorem*:
Each ideal in any member of **A** is an intersection of prime ideals.

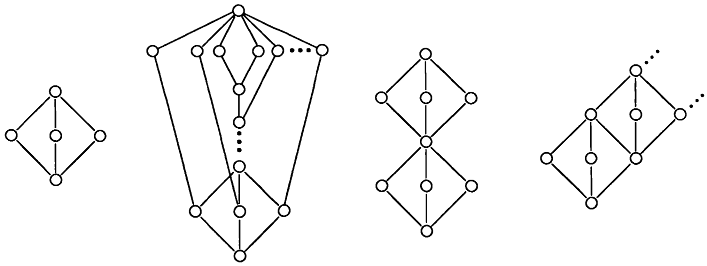
PISA *Prime Ideal Separation Theorem*:
Each ideal disjoint from a down-directed set D in any member of **A** is contained in a prime ideal still disjoint from D.

The corresponding Maximal Ideal Theorems **MITA**, **MICA**, **MIRA** and **MISA** are obtained by replacing "prime" with "maximal". The following implications are obvious:

$$\textbf{PISA} \Longrightarrow \textbf{PIRA} \Longrightarrow \textbf{PICA} \Longrightarrow \textbf{PITA}$$



$$\mathbf{MISA} \Longrightarrow \mathbf{MIRA} \Longrightarrow \mathbf{MICA} \Longrightarrow \mathbf{MITA}$$



The sketched examples show that no other implications hold (not even for lattices), except **PIRA** $\Longrightarrow$ **PISA**, which we shall derive from the Ultrafilter Principle (**UP**), assuring that every (proper) set-theoretical filter is contained in an ultrafilter (see Theorem 4). Moreover, the previous examples demonstrate that no implication generally holds between one of the maximal ideal theorems and one of the prime ideal theorems, nor conversely.

It is well known since half a century (see Rubin, Scott and Tarski [29], [30], [31]) that for the class **B** of all Boolean algebras, each of the eight statements **PITB**, **MITB** etc. is equivalent to **UP**. Since in Boolean algebras the maximal ideals coincide with the prime ones, one generally speaks of "the" (Boolean) Prime Ideal Theorem when one of these equivalent statements is meant. From the ingenious work of Halpern and Lévy [19] we know that in **ZF** and similar set theories, **UP** is effectively weaker than the Axiom of Choice (**AC**). Meanwhile, there exists an extensive literature on topological, algebraic, logical and combinatorial equivalents of the Boolean Prime Ideal Theorem (see for example [10], [11] and [21] for recent developments and more references).

Note that if a class **A** of posets contains all *Boolean frames*, that is, all complete Boolean lattices, then **PITA** implies **PITB** and so **UP**. Indeed, the Dedekind-MacNeille completion (by cuts) of any Boolean lattice is again Boolean (see, e.g., [3]), and the embedding in the completion preserves joins and meets; thus, the preimage of any prime ideal in the completion is a prime ideal in the original Boolean lattice. We shall establish the following results about such a class **A**:

- if **A** consists of 1-semidistributive posets and includes all distributive lattices, then
  UP $\Longleftrightarrow$ **PITA** $\Longleftrightarrow$ **PICA**, but **AC** $\Longleftrightarrow$ **MITA** $\Longleftrightarrow$ **MICA**;
- if **A** consists of lower pseudocomplemented posets and contains all Alexandroff topologies, then **UP** $\Longleftrightarrow$ **MITA** $\Longleftrightarrow$ **MICA**;
- if **A** consists of ideal distributive posets, then
  UP $\Longleftrightarrow$ **PITA** $\Longleftrightarrow$ **PICA** $\Longleftrightarrow$ **PIRA** $\Longleftrightarrow$ **PISA**;

- if **A** consists of weakly distributive posets, then
  UP $\Longleftrightarrow$ **PIRA** $\Longleftrightarrow$ **PISA**
  for *weak* prime ideals (whose complements are filters);
- if **A** consists of 0-semidistributive $\wedge$-semilattices, then **UP** $\Longleftrightarrow$ **PITA** for *directed* prime ideals.

These results cover most of the prime or maximal ideal theorems known for ordered structures. With the help of the Separation Lemma for Quantales, which is equivalent to **UP** (see Banaschewski and Erné [2]), it is possible to include also many algebraic prime ideal theorems (see [11]).

1. Semidistributivity and pseudocomplements

Crucial for the investigation of prime elements and prime ideals is a weak form of distributivity, the so-called *semidistributivity* (see e.g. [17]). In [8], we have extended that notion from lattices to posets. An element s of a *lattice* L is $\vee$-*semidistributive* if and only if for all $a, b, c \in L$, the equation $s = a \vee b = a \vee c$ implies $s = a \vee (b \wedge c)$; in $\vee$-*semilattices*, the latter equation has to be replaced with $s = a \vee d$ for some $d \leq b, c$; and in an arbitrary *poset* S, an element s is called $\vee$-*semidistributive* if and only if it has the equivalent properties stated below.

Lemma 1. *For any element s of a poset S, the following are equivalent:*

(a) *Given a finite $A \subseteq S$ and $b, c \in S$ with $s = \bigvee(A \cup {\downarrow}b) = \bigvee(A \cup {\downarrow}c)$, there is a finite $F \subseteq {\downarrow}b \cap {\downarrow}c$ with $s = \bigvee(A \cup F)$.*

(b) *For all finite $B, C \subseteq S$ with $s = \bigvee B = \bigvee C$, there is a finite $F \subseteq {\downarrow}B \cap {\downarrow}C$ with $s = \bigvee F$.*

(c) *For all finite $A, B \subseteq S$ with $s = \bigvee\big(A \cup {\downarrow}b\big)$ for all $b \in B$, there is a finite $F \subseteq B_{\downarrow} = \bigcap\{{\downarrow}b : \ b \in B\}$ with $s = \bigvee(A \cup F)$.*

(d) *The principal ideal ${\downarrow}s$ is $\vee$-semidistributive in the ideal lattice $\mathcal{I}S$.*

We call a poset S 1-*semidistributive* if it has a $\vee$-semidistributive greatest element 1. In such a poset, every maximal ideal must be prime. From [8; Proposition 4] (applied to $L = \mathcal{I}S$) one immediately deduces:

Theorem 1. *For any class* **A** *of* 1*-semidistributive posets that includes all bounded distributive lattices,*

UP $\Longleftrightarrow$ **PITA** $\Longleftrightarrow$ **PICA**, **AC** $\Longleftrightarrow$ **MITA** $\Longleftrightarrow$ **MICA**.

Note that the 1-semidistributive posets constitute the *largest* class of posets S with 1 for which the Prime Ideal Containment Theorem holds. Indeed, if each proper ideal of S is contained in a prime ideal, then S must be 1-semidistributive (see [8] again). But, of course, there are large classes **A** of (complete) lattices satisfying **PITA** but not **PICA**, for example, the class of all non-distributive (complete) lattices having a $\wedge$-prime least element.

Similar phenomena as in Theorem 1 have been observed in various other structures: while the respective Maximal Ideal Theorem is equivalent to the full **AC**, the corresponding Prime Ideal Theorem is equivalent to **UP**. However, there are also certain structures in which the Maximal Ideal Theorem is still equivalent to **UP** (hence *not* to **AC**). Frequently, the reason for that coincidence is a certain kind of (pseudo-)complementation. Let us analyze briefly the order-theoretical background for such situations.

A poset is *lower pseudocomplemented* if and only if it has a greatest element 1 and for each element a, there is a least element a_* (called the *lower pseudocomplement* of a) having only one upper bound in common with a (that is, $a \vee a_* = 1$). Observe that any lower pseudocomplemented poset has not only a greatest element 1, but also a least element $0 = 1_*$. The proof of the following remark is straightforward:

Lower pseudocomplemented $\vee$-semilattices are 1-semidistributive.

Unfortunately, this implication cannot be extended to arbitrary posets.

Example 1. Pick top, bottom, all atoms and all coatoms from an infinite power set lattice $\mathcal{P}X$. This gives a lower pseudocomplemented poset

$$S = \{X, \emptyset\} \cup \big\{\{x\} : \ x \in X\big\} \cup \big\{X \setminus \{x\} : x \in X\big\}.$$

For distinct elements $y, z \in X$ and $A = \big\{\{y\}, \{z\}\big\}$, $b = X \setminus \{y\}$, $c = X \setminus \{z\}$, we have $X = \bigvee(A \cup {\downarrow} b) = \bigvee(A \cup {\downarrow} c)$, but there is no finite $F \subseteq {\downarrow} b \cap {\downarrow} c$ with $X = \bigvee(A \cup F)$, because F would have to consist of finite sets, and consequently $\bigcup(A \cup F)$ would be a finite set, hence contained in a coatom.

In this example, the atoms together with the least element $\emptyset$ form a maximal but not prime ideal, although the completion by cuts is the whole power set, hence a complete atomic Boolean algebra (cf. [6]). The prime ideals of S are precisely the principal maximal ideals. Thus, S has many prime ideals, and every proper ideal is contained in a maximal one, but the Prime Ideal Containment Theorem fails for the class $\{S\}$.

Under certain completeness assumptions, the above connection between semidistributivity and pseudocomplementation may be strengthened. A $\vee$-semilattice is called *down-complete* if every down-directed subset D has a meet, and $\vee$-*continuous* ([16]) if $a \vee \bigwedge D = \bigwedge\{a \vee d : \ d \in D\}$ for any such D.

PROPOSITION 1. *A down-complete $\vee$-semilattice S is lower pseudocomplemented if and only if it is 1-semidistributive and satisfies $a \vee \bigwedge D = 1$ for each down-directed subset D with $a \vee d = 1$ for all $d \in D$ ($\vee$-continuity at 1). Thus, for $\vee$-continuous and, in particular, for finite $\vee$-semilattices, lower pseudocomplementation and 1-semidistributivity are equivalent properties.*

Proof. If S is down-complete and lower pseudocomplemented, then $a \vee d = 1$ for all $d \in D$ implies $a_* \leq \bigwedge D$ and so $a \vee \bigwedge D = 1$. Conversely, if S is down-complete, 1-semidistributive and $\vee$-continuous at 1, then for each $a \in S$, the set $D = \{b \in S : a \vee b = 1\}$ is down-directed, and its meet is the lower pseudocomplement of a. □

Whereas (lower) pseudocomplementation is transferred from a poset to the Dedekind-MacNeille completion by cuts (see e.g. [12]), this is not the case with Frink's ideal completion: in the ideal completion of an infinite power set, the ideal of all finite subsets has no lower pseudocomplement. Of course, the Alexandroff completion of a poset is *always* lower pseudocomplemented: the lower pseudocomplement of a down-set is just the down-set generated by the set-theoretical complement.

2. Maximal ideals in pseudocomplemented posets

For a convenient description of maximal ideals in lower pseudocomplemented posets, it appears helpful to provide a more general framework. Recall that a system $\mathcal{X}$ of sets is said to be *of finite character* if it contains a set precisely when all finite subsets of that set belong to $\mathcal{X}$. A *cutset* for $\mathcal{X}$ is a set C such that each member of $\mathcal{X}$ is contained in some member of $\mathcal{X}$ that meets C. The maximal members of $\mathcal{X}$ are exactly those which intersect every cutset of $\mathcal{X}$. Therefore, we call a member of $\mathcal{X}$ *almost maximal* if it meets every *finite* cutset of $\mathcal{X}$ (in [22], *"almost maximal"* has a related but different meaning). It was shown in [10] that **UP** is equivalent to the

> *Finite Cutset Lemma: Every member of a system of finite character is contained in an almost maximal one,*

whereas **AC** is known to be equivalent to

> *Tukey's Lemma: Every member of a system of finite character is contained in a maximal one.*

A comparison of these statements shows that the Ultrafilter Principle may be regarded as a finitary version of Tukey's Lemma, or equivalently, of **AC**. For various applications of the Finite Cutset Lemma, see [10]. Note that for any poset S with top element 1, the collection $\mathcal{X}S$ of those subsets which generate a proper ideal is of finite character.

It turns out that many existence theorems for maximal ideals rely, explicitly or implicitly, on the next three general order-theoretical lemmas.

Lemma 2. *Either a subset $X \cup \{a\}$ of a lower pseudocomplemented poset S generates a proper ideal, or the lower pseudocomplement a_* belongs to the ideal generated by X. Hence, each of the doubletons $\{a, a_*\}$ is a cutset for $\mathcal{X}S$.*

Proof. If S is the ideal generated by $X \cup \{a\}$, we find a finite $F \subseteq X$ such that $1 \in \Delta(F \cup \{a\})$. In particular, each upper bound b of F must satisfy $a \vee b = 1$, i.e. $a_* \le b$, and it follows that a_* belongs to ΔF, hence to the ideal generated by X. □

Lemma 3. *For an ideal I of a lower pseudocomplemented poset S, the following three conditions are equivalent:*

(a) *I is maximal.*
(b) *I is almost maximal (in $\mathcal{X}S$).*
(c) *For each $a \in S$, either a or a_* belongs to I, but not both.*

If, moreover, S is complemented (i.e. $a \wedge a_* = 0$), then every prime ideal is maximal.

Proof. That (a) implies (b) is clear by definition, and (b) implies (c) by Lemma 2. For (c) $\Longrightarrow$ (a), note that if I is not maximal, say $I \subset J \subset S$ for another ideal J, then for $a \in J \setminus I$, the element a_* cannot belong to I either (otherwise $1 = a \vee a_* \in J$). □

A map $\varphi\colon S \to S'$ between posets is *ideal continuous* if preimages of ideals are ideals, or equivalently, if $\varphi[\Delta F] \subseteq \Delta\varphi[F]$ for all finite $F \subseteq S$ ([4]).

Lemma 4. *Suppose $\varphi\colon S \to S'$ is a map between lower pseudocomplemented posets such that*

(1) *φ is ideal continuous,*
(2) *φ preserves pseudocomplements, i.e. $\varphi(a_*) = \varphi(a)_*$ for all $a \in S$.*

Then the preimages of maximal ideals under φ are again maximal ideals.

Proof. Let M' be a maximal ideal of S'. Then $M = \varphi^{-1}[M']$ is an ideal of S. As the top element $1'$ of S' is not an element of M', the top element 1 of S cannot belong to M, since $\varphi(1) = \varphi(0_*) = \varphi(0)_* = 0'_* = 1'$. For $a \in S$, we have $\varphi(a) \in M'$ or $\varphi(a_*) = \varphi(a)_* \in M'$, hence $a \in M$ or $a_* \in M$, proving maximality of M (see Lemma 3). □

A map between $\vee$-semilattices is ideal continuous if and only if it preserves finite joins. Thus, Lemma 4 ensures that if a map between lower pseudocomplemented $\vee$-semilattices preserves finite joins and pseudocomplements, then preimages of maximal ideals are maximal ideals. This includes the known fact that preimages of ultrafilters under homomorphisms between Boolean algebras are ultrafilters. Now we are ready for the main result of this section.

THEOREM 2. *Let* **A** *be any class of lower pseudocomplemented posets.*

(1) *If* **A** *contains at least all power set lattices, then*
UP $\iff$ **MICA**.

(2) *If* **A** *contains all down-set frames of algebraic* $\cap$*-systems, then*
UP $\iff$ **MITA** $\iff$ **MICA**.

(3) *If* **A** *consists of* $\vee$*-semilattices and contains all Boolean frames, then*
UP $\iff$ **MITA** $\iff$ **MICA** $\iff$ **PITA** $\iff$ **PICA**.

Proof. It is easy to see that **UP** or, more directly, **MICB** implies **MICA** for the class **A** of all lower pseudocomplemented $\vee$-semilattices S, by applying Lemma 4 to the Booleanization map $\psi\colon S \to S_* = \{a_* : a \in S\}$, $a \mapsto a_{**}$ (see [3], [14], [18], [25] for the dual situation). A short and elegant proof of the fact that the skeleton S_* of a pseudocomplemented semilattice is a Boolean algebra was given by Katriňák [25]. The case of a lower pseudocomplemented *poset* S, however, is more subtle: the skeleton S_* is a Boolean poset (see Niederle [27]), but not always a Boolean lattice. Note first that the arguments at the end of Section 1 together with Lemma 4, applied to the down-set operator $\downarrow\colon \mathcal{P}S \to \mathcal{A}S$, ensure that

> **UP** *implies the Maximal Ideal Containment Theorem for all down-set frames.*

Given a proper ideal I of S, consider the algebraic $\cap$-system S_I of all proper ideals of S containing I, and the down-set frame $L = \mathcal{A}S_I$. Define a map

$$\varphi\colon S \to L, \quad a \mapsto \{J \in S_I : a \notin J\}.$$

In order to prove ideal continuity of φ, choose a finite $F \subseteq S$ and an upper bound $\mathcal{U}$ of $\varphi[F]$ in L. We have to verify that $b \in \Delta F$ implies $\varphi(b) \subseteq \mathcal{U}$. By definition, $J \in \varphi(b)$ means $b \notin J$, which implies $a \notin J$ for some $a \in F$ (otherwise $b \in \Delta F \subseteq J$). It follows that $J \in \varphi(a) \subseteq \mathcal{U}$.

Next, we show that φ preserves pseudocomplements. For $a \in S$ and $\mathcal{B} \in L$, we obtain the equivalences

$$\begin{aligned} \varphi(a) \cup \mathcal{B} = S_I &\iff (\forall J \in S_I)(a \in J \implies J \in \mathcal{B}) \\ &\overset{*}{\iff} (\forall J \in S_I)(a_* \notin J \implies J \in \mathcal{B}) \iff \varphi(a_*) \subseteq \mathcal{B}. \end{aligned}$$

(for the equivalence $\overset{*}{\Longleftrightarrow}$, use Lemma 2). Thus, $\varphi(a_*)$ is actually the lower pseudocomplement of $\varphi(a)$ in L. Now, if M is a maximal ideal in L, then, by Lemma 4, its preimage under φ is a maximal ideal in S containing I (because for $a \in I$, the image $\varphi(a)$ is empty, hence a member of M).

(1) By the preceding clues, **UP** implies **MICA**, and the Maximal Ideal Containment Theorem for power sets gives back the Ultrafilter Principle.

An entirely different deduction of **MICA** uses the equivalence of **UP** to the Finite Cutset Lemma, which allows to extend every proper ideal to an almost maximal one; by Lemma 3, this is already a maximal ideal.

(2) As we saw before, the Maximal Ideal Theorem for all down-set frames of algebraic $\cap$-systems implies the Maximal Ideal Containment Theorem for all lower pseudocomplemented posets, which in turn entails **UP**.

(3) By (1), **UP** implies **MICA**, which implies **PICA** by the remark that maximal ideals in 1-semidistributive (*a fortiori* in lower pseudocomplemented) $\vee$-semilattices are prime. By the same reason, **MITA** entails **PITA**. □

Since the cotopologies (i.e. the lattices of closed sets of topological spaces) form a class of lower pseudocomplemented lattices containing all down-set frames (Alexandroff topologies), it follows from the dual of Theorem 2 that **UP** is equivalent to the existence of maximal *filters* in all nontrivial topologies (cf. [28]). In contrast to that fact, the existence of maximal *ideals* in nontrivial topologies already implies **AC**, and it suffices to consider products of sobrified cofinite topologies ([20]).

3. Some transfer principles

For any prime ideal in a bounded lattice, the trace on any $0-1$-sublattice is a prime ideal of that sublattice; the same is true with "maximal" instead of "prime" at least for pseudocomplementation-closed $0-1$-sublattices of pseudocomplemented lattices. More generally, prime and maximal ideals are transferred by suitable morphisms. By an *ideal adjoint*, we mean an ideal continuous map φ having a lower adjoint $\psi\colon S' \to S$ (i.e. $\psi(a') \le a \iff a' \le \varphi(a)$; see [16]) such that $\downarrow\psi[S'] = S$. The latter certainly holds if ψ is surjective or, equivalently, if φ is injective. An injective map between complete lattices is an ideal adjoint if and only if it preserves finite joins and arbitrary meets. In particular, for any continuous surjection f between topological spaces, the preimage map f^{-1} between the closed set lattices is an ideal adjoint.

THEOREM 3. *Let $\varphi\colon S \to S'$ be an ideal adjoint.*

(1) *If I is a (proper) ideal in S, then $\downarrow\varphi[I]$ is a (proper) ideal in S'.*
(2) *If I' is a (proper) ideal in S', then $\varphi^{-1}[I']$ is a (proper) ideal in S.*
(3) *If I' is a prime ideal in S', then $\varphi^{-1}[I']$ is a prime ideal in S.*
(4) *If S' is 1-semidistributive, then so is S.*
(5) *If S' is lower pseudocomplemented, then so is S.*
(6) *If the Prime Ideal (Containment) Theorem holds for S', then it also holds for S.*
(7) *If S' is lower pseudocomplemented, φ preserves pseudocomplements and S' enjoys the Maximal Ideal (Containment) Theorem, then the same holds for S.*

Proof.

(1) For finite $F' \subseteq \downarrow\varphi[I]$, the image $\psi[F']$ is contained in the ideal I. It follows that $\Delta\psi[F'] \subseteq I$ and $\Delta F' \subseteq \downarrow\varphi[\psi[\Delta F']] \subseteq \downarrow\varphi[\Delta\psi[F']] \subseteq \downarrow\varphi[I]$, because lower adjoints are ideal continuous. Thus, $\downarrow\varphi[I]$ is an ideal, and if I is proper, then so is $\downarrow\varphi[I]$; indeed, $S' = \downarrow\varphi[I] \iff \downarrow\psi[S'] \subseteq I \iff S = I$.

(2) For $a' \in S' \setminus I'$, the element $a = \psi(a') \in S$ cannot be in $\varphi^{-1}[I']$, because $\varphi(a) \in I'$ together with $a' \leq \varphi(a)$ would entail $a' \in I'$.

(3) Let D' be the down-directed complement of I' in S' and put $D = \varphi^{-1}[D'] = S \setminus \varphi^{-1}[I']$. Given a finite subset F of D, we find a lower bound b' of $\varphi[F]$ in D', and then $b = \psi(b')$ is a lower bound of F in D (note that $\varphi(b) \geq b' \in D'$ implies $\varphi(b) \in D'$, i.e. $b \in D$). Thus, D is down-directed, too.

(4) The image $\psi(1')$ of the top element of S' is the top element of S, by the equation $S = \downarrow\psi[S']$. Furthermore, φ maps the element $1 = \psi(1')$ onto $1'$, since $\psi(a') \leq 1$ is equivalent to $a' \leq \varphi(1)$. Now, 1-semidistributivity is transferred from S' to S, because the map $\overline{\varphi}\colon \mathcal{I}S \to \mathcal{I}S'$, $I \mapsto \downarrow\varphi[I]$ is a complete lattice homomorphism (preserving meets by the adjointness between ψ and φ, and joins by the adjointness between $\overline{\varphi}$ and $\varphi^{-1}\colon \mathcal{I}S' \to \mathcal{I}S$); moreover, $\overline{\varphi}(I) = S'$ is equivalent to $I = S$, as we saw in (1).

(5) For $a \in S$, put $a_* := \psi\big(\varphi(a)_*\big)$, where $\varphi(a)_*$ denotes the lower pseudocomplement of $\varphi(a)$ in S'. For $b \in S$, $a \vee b = 1$ implies $\varphi(a) \vee \varphi(b) = 1'$ (by ideal continuity of φ), hence $\varphi(a)_* \leq \varphi(b)$ and $a_* = \psi\big(\varphi(a)_*\big) \leq b$. On the other hand, if $a_* = \psi\big(\varphi(a)_*\big) \leq c$ in S, then $\varphi(a)_* \leq \varphi(c)$ and so $\varphi(a) \vee \varphi(c) = 1'$. If also $a \leq c$, then $\varphi(a) \leq \varphi(c) = 1'$, hence $1 = \psi(1') \leq c$. Thus, a_* is in fact the lower pseudocomplement of a in S.

Finally, (6) follows from (1), (2), (3), while (7) follows from (1), (2) (5) and Lemma 4. $\square$

The above claim about meet-closed sublattices S of a lower pseudocomplemented complete lattice is now an immediate consequence, by taking for φ the inclusion map and for ψ the corresponding closure operator.

A similar application is obtained for meet-dense inclusions, since

every meet-dense embedding $\varphi\colon S \to S'$ *is ideal continuous.*

Indeed, if $\varphi(a) \notin \Delta\varphi[F]$ for some $a \in S$, then there is a $b' \in S'$ with $\varphi[F] \subseteq {\downarrow}b'$, but $\varphi(a) \not\leq b'$, and by meet-density of $\varphi[S]$ in S', we may assume $b' = \varphi(b)$ for some $b \in S$. But then $a \not\leq b$ and $F \subseteq {\downarrow}b$ by the embedding property, hence $a \notin \Delta F$. By contraposition, we get the inclusion $\varphi[\Delta F] \subseteq \Delta\varphi[F]$.

Corollary 1. *Let P' be a prime ideal and S any subset of a poset S'. Then the trace $P = S \cap P'$ is a prime ideal of S, provided one of the following hypotheses is fulfilled:*

(1) *The inclusion map of S in S' is an ideal adjoint,*
(2) *S has a top element and is a meet-dense $\wedge$-subsemilattice of S',*
(3) *S is join- and meet-dense in S', and P' is a principal ideal ${\downarrow}p$.*

In the third case, the complement of $P = S \cap {\downarrow}p$ in S is down-directed, since each finite $E \subseteq S \setminus P$ has a lower bound s in $S' \setminus P'$, whence $s \not\leq p$, and by join-density of S in S', we may assume $s \in S \setminus P' = S \setminus P$.

Recall that any poset is both join- and meet-dense in its completion by cuts, and that a complete lattice L is *spatial* if and only if for $a \not\leq b$ in L, there is a $\wedge$-prime $p \in L$ with $a \not\leq p$, but $b \leq p$ (i.e. the set of $\wedge$-primes is meet-dense).

Corollary 2. *If the completion by cuts of a poset S is spatial, then for $a \not\leq b$ in S there is a prime ideal of S containing b but not a.*

Posets with this and stronger separation properties will be investigated more thoroughly in the next section.

4. Representation and separation by prime ideals

Sometimes, the mere existence of prime ideals is much weaker than the separation by prime ideals in the sense of **PISA**, which in turn leads to important subdirect representations by irreducible objects. One would expect that "distributive" posets should have such separation properties in **ZF** with **UP**. However, the absence of joins and meets causes some serious difficulties. Various kinds of distributive laws for posets (and even more general structures) have been introduced, analyzed and compared with each other in [6], [9] and [12]. Of particular interest in the present context is the so-called *ideal distributivity*. It demands that for each finite subset F and each $a \in \Delta F$, there is a *finite* subset E of ${\downarrow}F$ whose join is a. The chosen terminology is justified by the fact that a poset S is ideal distributive if and only if its ideal lattice $\mathcal{I}S$ is distributive, hence an algebraic frame (see [6]). From this equivalence and Lemma 1, we see that each element of an ideal distributive poset is $\vee$-semidistributive. For $\vee$-semilattices, ideal distributivity coincides with the notion of distributivity considered by Katriňák [23], which requires that for a, b, c with $a \leq b \vee c$, but neither $a \leq b$ nor $a \leq c$, there exist $b_0 \leq b$ and $c_0 \leq c$ with $a = b_0 \vee c_0$. Distributive $\wedge$-semilattices are defined dually. It was shown in [6] that

for $\wedge$-semilattices, distributivity implies ideal distributivity, but not vice versa.

Concerning more details on ideal continuous maps and ideal distributive posets, we refer to [4]. Here, we focus on separation properties:

THEOREM 4. *Consider the following conditions on a poset S:*

(a) *For each ideal I and each down-directed subset D of S disjoint from I there is a prime ideal containing I and disjoint from D.*

(b) *Each ideal of S is an intersection of prime ideals.*

(c) *For $a, b \in S$ with $a \not\leq b$, there is a prime ideal I with $a \notin I$ but $b \in I$.*

(d) *S is ideal distributive.*

While the implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) *and* (b) $\Longrightarrow$ (d) *always hold,* (c) $\Longrightarrow$ (d) *fails. In the converse direction, each of the three implications* (d) $\Longrightarrow$ (a), (d) $\Longrightarrow$ (b) *and* (d) $\Longrightarrow$ (c) *is equivalent to* **UP**. *Hence,* **UP** *entails the equivalence of* (a), (b) *and* (d), *and for any class* $\mathbf{A}$ *of ideal distributive posets containing all complete Boolean lattices, e.g. for distributive $\vee$- or $\wedge$-semilattices,*

$$\mathbf{UP} \iff \mathbf{PITA} \iff \mathbf{PICA} \iff \mathbf{PIRA} \iff \mathbf{PISA}.$$

Proof. It is clear that (a) implies (b), which in turn implies (c) and (d). In fact, (b) says that the ideal lattice $\mathcal{I}S$ is *spatial*, hence distributive.

Now, we deduce (d) $\Longrightarrow$ (a), *a fortiori* (d) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (c), from **UP**. The ideals intersecting D form a Scott-open filter $\mathcal{U}$ in the frame $\mathcal{I}S$ (see [2] or [11], where one may find a general discussion of separation properties; "Scott-open" means that the complement is closed under directed joins). Hence, by the *Separation Lemma for Quantales*, which is equivalent to **UP** (see [2]), I is contained in a prime ideal outside $\mathcal{U}$, i.e. disjoint from D.

The implication (d) $\Longrightarrow$ (c), applied to Boolean frames, in turn entails **UP**. □

That (c) does not imply (d) is witnessed by a modification of Example 1.

EXAMPLE 2. Let X be an infinite set, and consider the system S of all subsets that are either finite or have a complement with at most one element. Obviously, this $\vee$-semilattice S has property (c) (with principal prime ideals generated by coatoms), but it even fails to be 1-semidistributive. This is checked in the same way as in Example 1. The finite subsets form a maximal but not prime ideal. As observed in [6], the dual of S is an ideal distributive but not distributive $\wedge$-semilattice, whereas S itself cannot be ideal distributive, because it is not even a $\vee$-semidistributive ideal. This example also shows that **PICA** fails for the class of dually ideal distributive posets.

5. Weak prime ideals and weak distributivity

In this section, we have a look at weaker (but self-dual) notions of distributivity and primeness in posets. *Weakly distributive posets* may be defined by dropping the finiteness restriction on E (but not on F!) in the definition of ideal distributivity. Several alternative descriptions of weak distributivity for posets, like the identity $\downarrow a \cap \Delta\{b,c\} = \Delta(\downarrow a \cap \downarrow\{b,c\})$, were given in [6]; some of them have been rediscovered and discussed by Larmerová and Rachůnek [26], Niederle [27] and others (see also [9]). Particularly convenient is the following relation to the *lattice extension* or *characteristic lattice*, the sublattice generated by a poset in its completion by cuts:

A poset is weakly distributive iff its lattice extension is distributive.

Another useful characterization, established in [4] and [9], is the following *"anti-blocking property"*, which makes it evident that (in contrast to ideal distributivity) weak distributivity is a self-dual property:

A poset is weakly distributive iff $\downarrow a \cap \downarrow c \subseteq \downarrow b$ and $\uparrow b \cap \uparrow c \subseteq \uparrow a$ imply $a \leq b$.

A complemented weakly distributive poset is called *Boolean*. Like the completion by cuts, the lattice extension of a Boolean poset is Boolean (cf. [12], [27]); but the Boolean poset in Example 1 is not even 1-semidistributive!

By a *weak prime ideal* of a poset S we mean an ideal I that is complementary to a filter (dual ideal). In Example 2, the maximal ideal of all finite subsets is a weak prime ideal, though not prime. The same example shows that a weakly distributive $\vee$-semilattice need not be distributive in Katriňák's sense. On the other hand, it was shown in [6] that for $\wedge$-semilattices, weak distributivity and ideal distributivity are equivalent properties. Niederle [27] derived the following separation lemma from **AC**:

For any two elements a, b in a weakly distributive poset with $a \not\leq b$, there is a weak prime ideal containing b but not a.

At the end of his paper, he mentioned that Paseka deduced the same conclusion from **UP**. But even more is true (compare this with Theorem 4):

THEOREM 5. *Consider the following conditions on a poset S:*

(a) *For each ideal I and each down-directed subset D of S disjoint from I, there is a weak prime ideal containing I and disjoint from D.*

(b) *Each ideal of S is an intersection of weak prime ideals.*

(c) *For $a, b \in S$ with $a \not\leq b$, there is a weak prime ideal I with $a \notin I$, $b \in I$.*

(d) *S is weakly distributive.*

The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) *are valid in* **ZF**.

Each of the implications (d) $\Longrightarrow$ (a), (d) $\Longrightarrow$ (b), (d) $\Longrightarrow$ (c) *is equivalent to* **UP**. *Hence, under the assumption of* **UP**, *all four statements are equivalent.*

Proof. The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are obvious. For (c) $\Longrightarrow$ (d), we show that (c) entails the "anti-blocking property" (for an alternate proof, see [27]). If $a \not\leq b$, choose a weak prime ideal I with $a \notin I$ but $b \in I$. If $c \in I$, then $\Delta\{b, c\} \subseteq I$, hence $a \notin \Delta\{b, c\}$, i.e. $\uparrow b \cap \uparrow c \not\subseteq \uparrow a$. On the other hand, if c belongs to the dual ideal $S \setminus I$, a dual argument yields $\downarrow a \cap \downarrow c \not\subseteq \downarrow b$.

Next, let us derive the implication (d) $\Longrightarrow$ (a), *a fortiori* (d) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (c), from **UP**. Let I be an ideal and D a down-directed subset of a weakly distributive poset S with $I \cap D = \emptyset$. Denote by S' the lattice extension of S and by I' the ideal generated by I in S'. If there would exist an element $d \in I' \cap D$, we could find a finite subset E of I such that $d \leq \bigvee E$ in S'. But then d would be a member of $\Delta E \subseteq I$, which is impossible. Thus, I' and D are still disjoint, and since S' is a distributive lattice, **UP** yields a prime ideal P' of S' containing I' and disjoint from D' (that **UP** implies the Prime Ideal Separation Theorem for distributive lattices is well-known). Now, we use the fact that the canonical embedding of S in S' is ideal continuous and dually ideal continuous. Hence, the inverse image P of P' under the embedding is an ideal, and its complement is a dual ideal. Thus, we have found a weak prime ideal containing I and disjoint from D (cf. Corollaries 1 and 2).

Conversely, it is clear that each of the implications (d) $\Longrightarrow$ (a), (d) $\Longrightarrow$ (b) or (d) $\Longrightarrow$ (c), applied to Boolean lattices, returns some versions of the Boolean Prime Ideal Theorem, hence **UP**. □

6. Prime ideals in 0-semidistributive posets

Example 2 demonstrates that in Theorem 5, "weak prime" cannot be substituted by "prime" without affecting the implications (d) $\Longrightarrow$ (a) and (d) $\Longrightarrow$ (b). But what about (d) $\Longrightarrow$ (c)? Here Example 2 does not serve as a counterexample, and the following questions remain open:

If a nontrivial poset is

weakly distributive,

filter distributive (dual to an ideal distributive poset),

– *0-semidistributive (dual to a 1-semidistributive poset),*

upper pseudocomplemented (dual to a lower pseudocomplemented poset),

does it always have a prime ideal (under the assumption of **AC** *or* **UP***)?*

For some specific classes of posets, we have a positive answer. Notice that the result (4) below does not follow directly via dualization from the Prime Ideal Theorem for 1-semidistributive posets!

THEOREM 6. *Let S be a nontrivial 0-semidistributive poset.*

(1) *The complement of any maximal filter is a weak prime ideal.*
(2) *The complement of a down-directed maximal filter is a prime ideal.*
(3) *If S is a $\wedge$-semilattice, then* **UP** *guarantees a directed prime ideal in S.*
(4) *If the intersection of any two principal ideals in S is a finitely generated down-set, then* **AC** *ensures that S has a prime ideal.*

Proof.

(1) Let D be a maximal (proper) filter in S. If F is a finite subset of $S \setminus D$, then for each $b \in F$, the filter generated by $D \cup \{b\}$ coincides with S. Put $F^{\uparrow} = \bigcup\{\uparrow b : b \in K\}$. By $\vee$-semidistributivity of $S = \uparrow 0$ in the filter lattice, one obtains $D \vee F^{\uparrow} = S$, so there cannot exist any $d \in D$ with $F^{\uparrow} \subseteq \uparrow d$. Thus $\Delta F \subseteq S \setminus D$.

(2) is an immediate consequence of (1).

(3) Dualize Theorem 1 (nonempty filters of S are down-directed).

(4) Using **AC**, let D be a maximal down-directed subset of $S \setminus \{0\}$ (hence a proper filter). Assume $a \in S \setminus D$, but $F_d = \downarrow a \cap \downarrow d \setminus \{0\} \neq \emptyset$ for all $d \in D$. Since each F_d is a finitely generated down-set in $S \setminus \{0\}$, Rudin's Lemma (which is a consequence of **AC**, see [16]) yields a down-directed subset D' of the union $\bigcup\{F_d : d \in D\}$ with $D' \cap F_d \neq \emptyset$ for all $d \in D$. But then $\uparrow D'$ would be a larger down-directed proper filter than D (since $D \cup \{a\} \subseteq \uparrow D'$). By way of contradiction, for each $a \in S \setminus D$ there is a $d \in D$ with $d \wedge a = 0$, and as D is down-directed, we find for any finite $F \subseteq S \setminus D$ some $d \in D$ with $d \wedge b = 0$ for all $b \in F$. As in (1), we get $\uparrow d \vee F^{\uparrow} = S$, hence $d \wedge c = 0$ for all $c \in \Delta F$ and so $\Delta F \subseteq S \setminus D$ (since $c \in \Delta F \cap D$ would entail $0 < e \leq c, d$ for some $e \in D$). Hence, $S \setminus D$ is a prime ideal. $\square$

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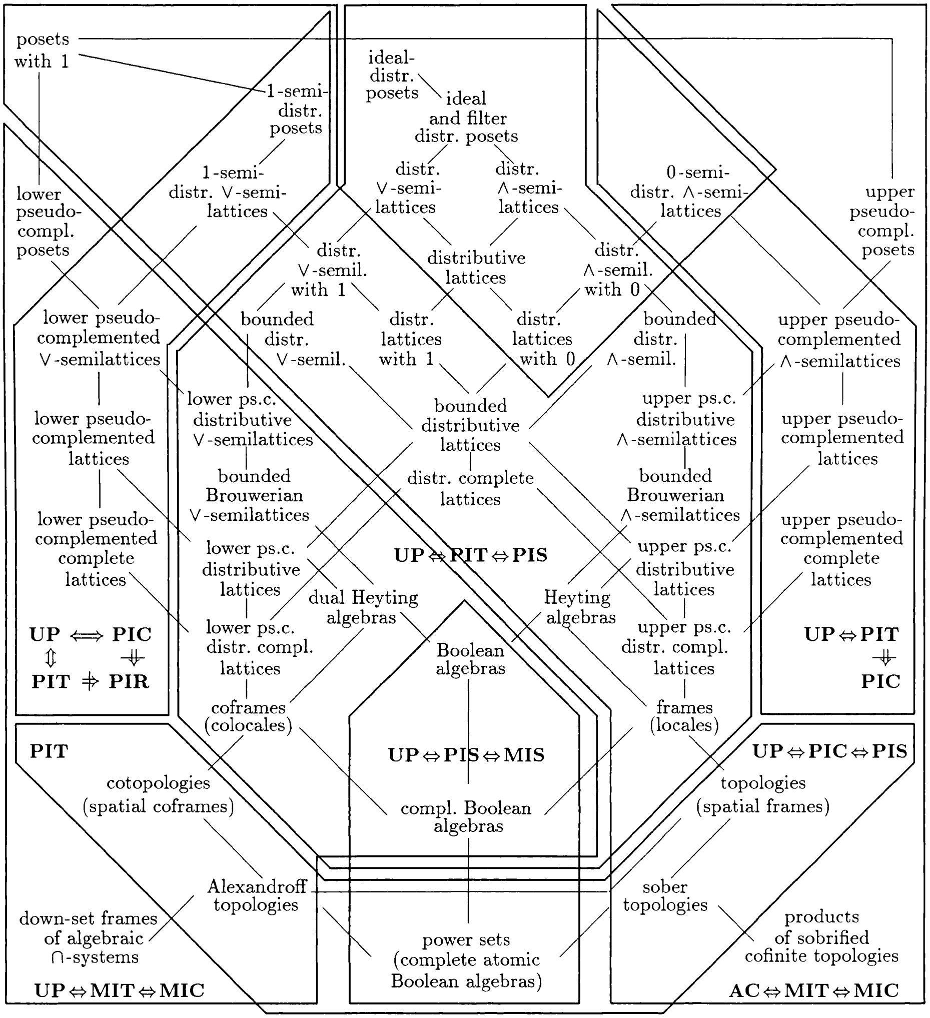
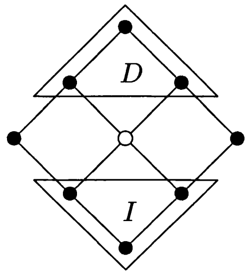


For all classes included in one and the same framed area, the equivalences indicated by bold letters hold (and the crossed implications fail). Note that there are overlapping regions! Nontrivial spatial frames and coframes have prime ideals.

7. A symmetric prime ideal separation theorem

Condition (a) in Theorem 5 is not entirely satisfactory in that the symmetric version would involve a *dual ideal* D, rather than a down-directed set. However, even for finite weakly distributive posets (whose completion by cuts coincides with the ideal completion and is a distributive lattice), statement (a) becomes false if down-directed sets are replaced with dual ideals (filters).



EXAMPLE 3. The previous diagram represents an eight-element poset and its completion (by cuts or ideals) obtained by inserting the "middle" point; this completion is a product of two three-element chains, hence (completely) distributive. The indicated ideal I and the disjoint filter D cannot be separated by any weak prime ideal. The reason for that failure is that in the completion, the ideal generated by I meets the filter generated by D. However, it is clear that any two points may be separated by (principal) prime ideals, both in the poset and in its completion.

More generally, it was shown in [7] that a poset is *principally separated*, i.e. any two elements may be separated by a principal (prime) ideal and a complementary principal filter, if and only if its completion by cuts is a superalgebraic lattice (in which every element is a join of $\bigvee$-prime elements and a meet of $\bigwedge$-prime elements). Every principally separated poset is weakly distributive, but as Example 2 shows, it need not be ideal distributive. Hence, even in a principally separated $\vee$-semilattice, an up-directed ideal need not be an intersection of prime ideals and cannot always be separated from a down-directed filter by a prime ideal. The mentioned difficulties vanish in distributive semilattices (see Theorem 4).

Using the cut closure $\Delta F = \bigcap\{\downarrow b : \ F \subseteq \downarrow b\}$ and the dual cut closure $\nabla F = \bigcap\{\uparrow b : \ F \subseteq \uparrow b\}$, the anti-blocking property may be written as follows:

$$\big(c \in \nabla\{a,b\} \ \& \ a \in \Delta\{b,c\}\big) \implies \nabla\{a\} \cap \Delta\{c\} \neq \emptyset .$$

Replacing singletons with finite sets A, C, we obtain the *strong anti-blocking property*

$$\nabla(A \cup \{b\}) \cap C \neq \emptyset \neq A \cap \Delta(\{b\} \cup C) \implies \nabla A \cap \Delta C \neq \emptyset .$$

And now we are in a position to prove a symmetric version of the Prime Ideal Separation Theorem for posets:

THEOREM 7. *Consider the following statements on a poset S:*

(a) *For each ideal I and each dual ideal D disjoint from I, there is a weak prime ideal P containing I and disjoint from D.*
(b) *S has the strong anti-blocking property.*
(c) *S has the anti-blocking property, i.e. S is weakly distributive.*

The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) *hold without any further hypotheses.*
For semilattices, (b) *and* (c) *are equivalent.*
Under assumption of **AC**, (a) *and* (b) *are equivalent for arbitrary posets.*

Proof.

(a) $\Longrightarrow$ (b): If the dual ideal ∇A does not intersect the ideal ΔC, then we may choose a weak prime ideal P with $\Delta C \subseteq P$ and $P \cap \nabla A = \emptyset$. For $b \in P$ we get $\Delta(\{b\} \cup C) \subseteq P$, hence $A \cap \Delta(\{b\} \cup C) = \emptyset$, while if b belongs to the dual ideal $S \setminus P$, it follows that $\nabla(A \cup \{b\}) \subseteq S \setminus P$ and consequently $\nabla(A \cup \{b\}) \cap C = \emptyset$.

The implication (b) $\Longrightarrow$ (c) is clear.

For (c) $\Longrightarrow$ (b), suppose that S is a $\vee$-semilattice and put $c = \bigvee C$. Then $A \cap \Delta(\{b\} \cup C) \neq \emptyset$ means $a \leq b \vee c$, i.e. $a \in \Delta\{b, c\}$ for some $a \in A$, and $\nabla(A \cup \{b\}) \cap C \neq \emptyset$ implies $c \in \nabla(A \cup \{b\})$. Now, if $A \subseteq \uparrow d$, then $c \in \nabla\{d, b\}$, and this together with $d \leq a \in \Delta\{b, c\}$ forces $d \leq c$. Thus, we obtain $c \in \nabla A \cap \Delta C \neq \emptyset$. For $\wedge$-semilattices, a dual argument works.

Finally, assume **AC** and (b) hold. Then, for an ideal I and a dual ideal D with $I \cap D = \emptyset$, Zorn's Lemma provides a maximal pair (P, Q) such that P is an ideal containing I and Q is a dual ideal containing D, disjoint from P. If Q was not the complement of P, we could choose an element b in $S \setminus (P \cup Q)$. By the maximality assumption, the ideal generated by $P \cup \{b\}$ meets Q, so there is a finite $F \subseteq P$ and an element $q \in Q$ with $q \in \Delta(\{b\} \cup F)$. Similarly, the dual ideal generated by $Q \cup \{b\}$ meets P, and we find a finite set $E \subseteq Q$ and a $p \in P$ with $p \in \nabla(E \cup \{b\})$. Now, $A = E \cup \{q\}$ and $C = F \cup \{p\}$ are finite sets such that $q \in A \cap \Delta(\{b\} \cup C) \neq \emptyset$ and $p \in \nabla(A \cup \{b\}) \cap C \neq \emptyset$. The strong anti-blocking property entails $\nabla A \cap \Delta C \neq \emptyset$, contradicting the disjointness of P and Q, because ΔC was contained in P and ∇A in Q. □

The following concluding question is now obvious:

Can **AC** *be substituted by* **UP** *in Theorem 7?*

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