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Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

REMARKS ON REPRESENTATION OF FUZZY QUANTUM POSETS

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(*Communicated by Sylvia Pulmannová*)

ABSTRACT. We present a new look at representation of fuzzy quantum posets via a family of q-σ-algebras (σ-algebras) of crisp subsets of an original universe. In particular, we give a representation of observables and states of fuzzy quantum spaces in these q-σ-algebras via pointwisely defined functions and special types of probability measures.

1. Introduction

The Kolmogorovian probability model [13] completely describes situation which appears in measurement of quantities of different kinds, which are important in physics, biology, economy, measurement theory, etc. However, there are situations, for example in quantum mechanics, psychology of human brain, computer science or sociometry (for details see [11]), where Kolmogorov's model is not adequate. Therefore, many efforts have been done to describe the probability world of quantum mechanics, where, in particular, quantum logics [25], [24] play an important role.

For our aims there is a very important model introduced by P. Suppes [23], called quantum probability space. The latter means a couple $(\Omega, \mathcal{Q})$, where $\mathcal{Q}$ is a collection of subsets (= quantum mechanical events) of the crisp set Ω, called a *q-σ-algebra*, closed with respect to countable disjoint unions and with

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respect to the set-theoretical complementation, and it was introduced in order to describe the position and the momentum of a quantum mechanical particle.

Using Zadeh's fuzzy ideas [26] and a one-to-one correspondence between subsets and their characteristic functions, the quantum space of Suppes may be uniquely represented by a system of its characteristic functions defined on a non-empty set Ω with values in the closed interval $[0,1]$.

The fruitful ideas of K. Piasecki's fuzzy set model [17]–[19], called a *fuzzy soft σ-algebra*, have found a reflection for quantum mechanical models by B. Riečan [21], J. Pykacz [20], and by the author [2] showing the similar structures with quantum logics. These models are known as fuzzy measurable spaces (F-quantum spaces) and fuzzy quantum posets, respectively.

In the present remarks, we shall study the fuzzy quantum posets which are analogues of Suppes' quantum spaces. For these models we show that they can be represented by a family of q-σ-algebras of the original universe. Moreover, we present that all observables of fuzzy quantum posets are in a "one-to-one" correspondence with pointwisely defined mappings on the universe, as well as, we show that states of fuzzy quantum posets are represented by probability measures of special types on these q-σ-algebras.

We note that the presented methods give us the opportunity to describe fuzzy situation by means of the classical Kolmogorov and Suppes models, which on the other hand, restricts the usage of fuzzy set ideas for quantum mechanics to the classical ones in our situations.

2. Fuzzy quantum posets

Suppose that Ω is a non-empty set called the *universum*. Using the language of fuzzy set theory, we shall say that

$$\bigcap_i f_i := \inf_i f_i \,, \tag{1}$$

$$\bigcup_i f_i := \sup_i f_i \,, \tag{2}$$

$$f^{\perp} := 1 - f \tag{3}$$

are called the *fuzzy intersection* and the *fuzzy union* of the fuzzy sets f_i's, and the *fuzzy complement* of the fuzzy set f, respectively.

According to [1], we introduce the *intuitionistic complement* $f^{\sim}$ of the fuzzy set f via

$$f^{\sim}(\omega) = \begin{cases} 1 & \text{if } f(\omega) = 0\,, \\ 0 & \text{if } f(\omega) \neq 0\,. \end{cases}$$

Two fuzzy sets f and g are called *orthogonal* or *fuzzy orthogonal*, and we write $f \perp g$ or $f \perp_F g$, respectively, if $f+g \le 1$ or $f \cap g \le 1/2$, correspondingly.

We say that for two fuzzy sets f and g we have $f \le g$ if $f(\omega) \le g(\omega)$ for any $\omega \in \Omega$.

For any fuzzy set $a \in [0,1]^\Omega$ we define by [15] $H(a) = a^{-1}\big((1/2,1]\big)$ (*high values*), $L(a) = a^{-1}\big([0,1/2)\big)$ (*low values*), and $M(a) = a^{-1}\big(\{1/2\}\big)$ (*middle values*).

K. Piasecki in [17]–[19] studied a *fuzzy soft σ-algebra* as a non-empty family M of fuzzy subsets of a crisp set Ω for which we have:

(i) if $1(\omega) = 1$ for any $\omega \in \Omega$, then $1 \in M$;

(ii) if $f \in M$, then $f^\perp \in M$;

(iii) if $f_i \in M$, $i \ge 1$, then $\bigcup\limits_{i=1}^{\infty} f_i \in M$;

(iv) if $1/2(\omega) = 1/2$ for any $\omega \in \Omega$, then $1/2 \notin M$.

It is clear that fuzzy soft σ-algebra is a distributive, de Morgan, σ-complete lattice with the minimal and maximal elements 0 and 1, respectively, and with the unary operation $\perp\colon M \to M$. We recall that $a \cup a^\perp$ is not necessary 1.

If we change (iii) to the requirement

(iii)* $\bigcup\limits_i f_i \in M$ whenever $f_i \perp_F f_j$ for $i \ne j$,

then the couple (Ω, M) is said to be a *fuzzy quantum poset* ([2]).

In particular, if $(\Omega, \mathcal{Q})$ is a quantum probability space of Suppes, then the couple (Ω, M), where M consists of all characteristic functions of all subsets from $\mathcal{Q}$, is a fuzzy quantum poset. The overlook of this theory is given in [8]. We recall that according to [1], fuzzy quantum posets can be understood also as a particular case of Brower-Zadeh posets (see also [10]) if we take into account the intuitionistic complement.

Let (Ω, M) be a fuzzy quantum poset. In M, there are two special families $W_0(M)$ and $W_1(M)$ consisting of all fuzzy sets $a \in M$ such that $a \le 1/2$ and $a \ge 1/2$, respectively. For any $a \in M$ we have $a \cap a^\perp \in W_0(M)$ and $a \cup a^\perp \in W_1(M)$, and $W_0(M)$ and $W_1(M)$ consist only of elements of those forms. Moreover, $\bigcap\limits_i c_i \in W_1(M)$ whenever $c_i \in W_1(M)$, $i \ge 1$.

A non-void subset $I \subset M$ is said to be an F-*σ-ideal* of (Ω, M) if

(i) $a \cap a^\perp \in I$ for any $a \in M$;

(ii) if $a \le b$, $a \in M$, $b \in I$, then $a \in I$;

(iii) if $a_i \perp_F a_j$ for $i \ne j$, $\{a_i\} \subseteq I$, then $\bigcup\limits_i a_i \in I$;

(iii)* $\bigcup\limits_i a_i \in I$ whenever $\{a_i\} \subseteq I$, for the case of a fuzzy measurable space;

(iv) if $a \cap c \in I$ for some $c \in W_1(M)$, then $a \in I$.

Put

$$I_0 = \{a \in M : \exists c \in W_1(M),\ a \cap c \leq 1/2\} \tag{4}$$

Then I_0 is a proper F-σ-ideal of (Ω, M) containing $W_0(M)$. Moreover, if I is any F-σ-ideal of (Ω, M), then $I_0 \subseteq I$, ([5], [7]).

Define a relation $\sim \subseteq M \times M$ via

$$a \sim b \iff a \perp_F b^\perp,\ a^\perp \perp_F b. \tag{5}$$

Then for $\sim$ we have the properties ([5], [7]):

(i) $a \sim a$ for any $a \in M$;

(ii) if $a \sim b$, then $a^\perp \sim b^\perp$;

(iii) if $a_i \perp_F a_j$, $b_i \perp_F b_j$ for $i \neq j$, $a_i \sim b_i$, $i \geq 1$, then $\bigcup_i a_i \sim \bigcup_i b_i$;

(iii)* if $a_i \sim b_i$, then $\bigcup_i a_i \sim \bigcup_i b_i$, for a fuzzy measurable space.

We note that $\sim$ cannot be transitive, and for its transitive closure $\approx$ we have $a \approx b \iff \exists c \in W_1(M)$ such that $a \cap b^\perp \cap c$, $a^\perp \cap b \cap c \leq 1/2 \iff \exists c_1, \dots, c_n \in W_1(M)$ such that $H(a \cap b^\perp) \cup H(a^\perp \cap b) \subseteq \bigcup_{i=1}^{n} M(c_i) \iff \exists \{c_n\}_{n=1}^{\infty} \in W_1(M)^\omega$ such that $H(a \cap b^\perp) \cup H(a^\perp \cap b) \subseteq \bigcup_{n=1}^{\infty} M(c_n)$. It is simple to verify that $I_0 = \{a \in M : a \approx 0\}$.

3. Representation of fuzzy quantum posets

In the present section, we give characterizations of fuzzy quantum posets via a family of q-σ-algebras (σ-algebras) of crisp subsets of the universe Ω.

According to [5] and [7], we define a quotient $\mathcal{M} := M/I_0$ as the set of all $\bar{a} = \{b \in M : b \approx a\}$, $a \in M$. Then $\mathcal{M}$ is a quantum logic ([5]), see below for definition, (Boolean σ-algebra, [7]) with respect to the partial ordering $\leq_M$ defined via $\bar{a} \leq_M \bar{b}$ if and only if there is a $c \in W_1(M)$ with $a \cap b^\perp \cap c \leq 1/2$, and a unary operation $\perp_M : \mathcal{M} \to \mathcal{M}$ via $\bar{a}^{\perp_M} := \overline{a^\perp}$, and with the minimal and maximal elements $\bar{0}$ and $\bar{1}$, respectively. Denote by ϕ the canonical embedding from M onto $\mathcal{M}$ defined via $\phi(a) = \bar{a}$, $a \in M$.

We recall that a *quantum logic* ([24]) is a poset L with the least and greatest elements 0 and 1, respectively, with a unary operation $\perp : L \to L$ such that:

(i) $a^{\perp\perp} = a$, $a \in L$;

(ii) if $a \leq b$, then $b^\perp \leq a^\perp$;

(iii) $a \vee a^\perp = 1$, $a \in L$;

(iv) if $a_i \le a_j^{\perp}$, $i \ne j$, then $\bigvee_{i=1}^{\infty} a_i \in L$;

(v) if $a \le b$, then $b = a \vee (b \wedge a^{\perp})$ (orthomodularity law).

We shall say that a q-σ-algebra (a σ-algebra) $\mathcal{Q}$ of crisp subset of the universe Ω is a *crisp representation* of a fuzzy poset (a fuzzy measurable space) (Ω, M) if and only if there is a σ-homomorphism h from $\mathcal{Q}$ onto M/I_0 such that:

(i) $h(\emptyset) = \bar{0}$;

(ii) $h(\Omega \setminus A) = h(A)^{\perp_M}$, $A \in \mathcal{Q}$;

(iii) $h\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \bigvee_{i=1}^{\infty} h(A_i)$ whenever $A_i \cap A_j = \emptyset$ for $i \ne j$, $A_i \in \mathcal{Q}$, $i \ge 1$.

Following to [5] and [15] (for the case of fuzzy measurable space), we introduce the next two families of subsets of Ω:

$$\mathcal{K}(M) = \big\{A \subseteq \Omega :\ \exists a \in M \text{ such that } \{a > 1/2\} \subseteq A \subseteq \{a \ge 1/2\}\big\}, \tag{6}$$

and let $\mathcal{A}(M)$ be the minimal q-σ-algebra generated by $\{H(a) :\ a \in M\}$. As it has been proved in [5], $\mathcal{K}(M)$ is a q-σ-algebra (σ-algebra). It is evident that $\mathcal{A}(M) \subseteq \mathcal{K}(M)$, and $\mathcal{A}(M)$ can be a proper subset of $\mathcal{K}(M)$, moreover, in view of $N(a \cap b) = N(a) \cap N(b)$, $\mathcal{A}(M)$ is a σ-algebra whenever M is a fuzzy soft σ-algebra.

THEOREM 3.1. *Let (Ω, M) be a fuzzy quantum poset (a fuzzy measurable space). Then $\mathcal{K}(M)$ and $\mathcal{A}(M)$ are crisp representations of (Ω, M).*

P r o o f. The statement on $\mathcal{K}(M)$ has been proved in [4] and [5], and to be self-contained, we recall that a σ-homomorphism $h_{\mathcal{K}}$ from $\mathcal{K}(M)$ onto $\mathcal{M}$ is defined as follows: $h_{\mathcal{K}}(A) = \bar{a}$ whenever

$$\{a > 1/2\} \subseteq A \subseteq \ \{a \ge 1/2\}. \tag{7}$$

The σ-homomorphism $h_{\mathcal{A}}$ from $\mathcal{A}(M)$ onto $\mathcal{M}$ is defined as the restriction of $h_{\mathcal{K}}$ onto $\mathcal{A}(M)$, i.e., $h_{\mathcal{A}} = h_{\mathcal{K}}\big|_{\mathcal{A}(M)}$. The surjectivity of $h_{\mathcal{A}}$ follows from the following. Let a be any element of M, then $\{a > 1/2\} \subseteq H(a) \subseteq \{a \ge 1/2\}$, so that $h_{\mathcal{K}}\big(H(a)\big) = \bar{a}$, which gives $h_{\mathcal{A}}\big(H(a)\big) = \bar{a}$. □

Suppose that L is any Boolean σ-algebra. Due to L o o m i s - S i k o r s k i's theorem ([22]), there is a non-void set Ω with a σ-algebra $\mathcal{Q}$ and a σ-homomorphism h from $\mathcal{Q}$ onto L. Using the result of N a v a r a and P t á k [15], we can find a fuzzy soft σ-algebra M such that M/I_0 corresponds to L, and $\mathcal{Q}$ is a crisp representation of (Ω, M). Indeed, define M as the set of all functions from Ω into the set $\{0, 1/2, 1\}$ such that $h\big(M(a)\big) = \bar{0}$ for any $a \in M$ and a is measurable with respect to $\mathcal{Q}$. An easy verification gives us that $\mathcal{A}(M) = \mathcal{Q}$.

4. Representation of observables

A fuzzy analogue of a random variable is an *observable*, i.e., a mapping x from the Borel σ-algebra $B(\mathbb{R})$ of the real line $\mathbb{R}$ into M such that:

(i) $x(\mathbb{R} \setminus E) = x(E)^{\perp}$, $E \in B(\mathbb{R})$;

(ii) $x\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \bigcup_{i=1}^{\infty} x(E_i)$ if $E_i \cap E_j = \emptyset$ for $i \neq j$.

For example, if a is a fuzzy set from M, then a *question observable*, x_a, is a mapping $x_a\colon B(\mathbb{R}) \to M$ such that

$$x_a(E) = \begin{cases} a \cap a^{\perp} & \text{if } 0,1 \notin E, \\ a^{\perp} & \text{if } 0 \in E,\ 1 \notin E, \\ a & \text{if } 0 \notin E,\ 1 \in E, \\ a \cup a^{\perp} & \text{if } 0,1 \in E \end{cases}$$

for any $E \in B(\mathbb{R})$. It is evident that x_a is an observable and it plays the role of the indicator of the fuzzy set $a \in M$.

In the present section, we show that for any observable x of a fuzzy quantum poset (Ω, M) and any crisp representation $\mathcal{Q}$ of (Ω, M) we can find in some sense unique $\mathcal{Q}$-measurable mapping $f\colon \Omega \to \mathbb{R}$ which represents x, and vice versa.

The following result has been proved in [4] and [6].

THEOREM 4.1. *Let x be an observable of a fuzzy quantum poset (Ω, M), and let S be any countable, dense subset of $\mathbb{R}$ (or $S = \mathbb{R}$). Denote, for any $r \in S$, $B_x(r) = x\big((-\infty, r)\big)$. The system $\{B_x(r) : r \in S\}$ fulfills the following conditions:*

1. $B_x(s) \leq B_x(t)$ *if* $s < t$ *for* $s,t \in S$,
2. $\bigcup_{r\in S} B_x(r) = a$, $\bigcap_{r \in S} B_x(r) = a^{\perp}$,
3. $\bigcup_{s<r} B_x(s) = B_x(r)$ *for any* $r \in S$,
4. $B_x(r) \cup B_x(r)^{\perp} = a$, $r \in S$,

where $a = x(\mathbb{R})$.

Conversely, if a is a system $\{B(r) : r \in S\}$ of fuzzy sets from M fulfilling 1.–4. for some $a \in M$, then there is a unique observable x of (Ω, M) such that $B_x(r) = B(r)$ for any $r \in S$ and $x(\mathbb{R}) = a$.

THEOREM 4.2. *Let $\mathcal{Q}$ be any crisp representation of a fuzzy measurable space (Ω, M), and let h be a σ-homomorphism from $\mathcal{Q}$ onto M/I_0. If x is an observable of (Ω, M), then there is a $\mathcal{Q}$-measurable, real-valued function $f\colon \Omega \to \mathbb{R}$*

such that

$$h(f^{-1}(E)) = \phi(x(E)) \tag{8}$$

for any $E \in B(\mathbb{R})$. *If* g *is any* $\mathcal{Q}$*-measurable, real-valued mapping on* Ω *with* (8), *then* $\{\omega \in \Omega : f(\omega) \neq g(\omega)\} \in \mathcal{Q}$ *and* $h(\{f \neq g\}) = \bar{0}$.

Conversely, let $f: \Omega \to \mathbb{R}$ *be any* $\mathcal{Q}$*-measurable function. Then there is an observable* x *of* (Ω, M) *with* (8). *If* y *is any observable of* (Ω, M) *with* (8), *then*

$$x(E) \perp_F y(E)^{\perp} \tag{9}$$

for any $E \in B(\mathbb{R})$.

P r o o f. Suppose that x is an observable of (Ω, M). The set $\{\phi(x(E)) : E \in B(\mathbb{R})\}$ is a Boolean sub-σ-algebra of M/I_0 with a countable generator. Due to [25; Theorem 6.9], there is a mapping $f: \Omega \to \mathbb{R}$, $\mathcal{Q}$-measurable such that (8) holds.

Let $\mathbb{Q}$ be the set of all rationals. If g is any $\mathcal{Q}$-measurable, real-valued function on Ω, then $\{f < g\} = \bigcup_{r\in\mathbb{Q}} \{\omega : f(\omega) < r < g(\omega)\} = \bigcup_{r\in\mathbb{Q}} \{\omega : f(\omega) < r\} \cap \{\omega : g(\omega) > r\}$. Hence, $h(\{f < g\}) = \bigvee_{r\in\mathbb{Q}} h(f^{-1}((-\infty, r))) \cap h(g^{-1}([r, \infty))) = \bigvee_{r\in\mathbb{Q}} \phi(x(\emptyset)) = \bar{0}$. Similarly, we have $h(\{g < f\}) = \bar{0}$.

Suppose now f is any $\mathcal{Q}$-measurable, real-valued function defined on Ω. The set $\mathcal{R} := \{h \circ f^{-1}(E) : E \in B(\mathbb{R})\}$ is a Boolean sub-σ-algebra of $\mathcal{M}$ with a countable generator, and $h \circ f^{-1}$ is a σ-homomorphism from $B(\mathbb{R})$ onto $\mathcal{R}$. Hence, as in the first part of the present proof, there is a $\mathcal{K}(M)$-measurable, real-valued function $f_{\mathcal{K}}$ on Ω such that $h(f^{-1}(E)) = h_{\mathcal{K}}(f_{\mathcal{K}}^{-1}(E))$ for any $E \in B(\mathbb{R})$.

Assume that $\mathbb{Q} = \{r_1, r_2, \dots\}$. For any integer $n \geq 1$ we find a fuzzy set $a_n \in M$ such that $h(f^{-1}((-\infty, r_n))) = \bar{a}_n$ and put $a = \bigcap_n (a_n \cup a_n^{\perp})$. For the system $\{b_n : n \geq 1\}$ defined via $b_n = (a_n \cap a) \cup a^{\perp}$, $n \geq 1$, we have $b_n \cup b_n^{\perp} = a$ for any $n \geq 1$.

We claim to construct a system $\{B(r) : r \in \mathbb{Q}\}$ fulfilling 1.–4. of Theorem 4.1. We put $B(r_1) = b_1$. Suppose that $B(r_1), \dots, B(r_n)$ have been constructed. A fuzzy set $B(r_{n+1})$ is constructed as follows: Let $(i_1, \dots, i_n)$ be a permutation of $(1, \dots, n)$ such that $r_{i_1} < \dots < r_{i_n}$. Then there are only three possibilities:

(a) $r_{n+1} < r_{i_1}$,
(b) $r_{n+1} > r_{i_n}$,
(c) $\exists k = 1, \dots, n-1$, such that $r_{i_k} < r_{n+1} < r_{i_{k+1}}$.

We define

$$B(r_{n+1}) = \begin{cases} b_{n+1} \cap B(r_{i_1}) & \text{if (a) holds,} \\ b_{n+1} \cup B(r_{i_n}) & \text{if (b) holds,} \\ B(r_{i_k}) \cup B(r_{i_{k+1}}) \cap b_{n+1} & \text{if (c) holds.} \end{cases}$$

By the induction, the system $\{B(r) : r \in \mathbb{Q}\}$ fulfills 1. and 4. of Theorem 4.1. Moreover, for any $r \in \mathbb{Q}$ we have $\phi(B(r)) = h(f^{-1}((-\infty, r))) = h_{\mathcal{K}}(f_{\mathcal{K}}^{-1}((-\infty, r)))$, which gives

$$\Omega = \bigcup_n f_{\mathcal{K}}^{-1}((-\infty, r_n)) \subseteq \bigcup_n \{B(r_n) \geq 1/2\} \subseteq \Big\{ \bigcup_n B(r_n) \geq 1/2 \Big\},$$

therefore $\bigcup_n B(r_n) = a$. Similarly,

$$H\Big(\bigcap_n B(r_n) \Big) \subseteq \bigcap_n H(B(r_n)) \subseteq \bigcap_n f_{\mathcal{K}}^{-1}((-\infty, r_n)) = \emptyset,$$

so that $\bigcap_n B(r_n) = a^{\perp}$.

In the same way, we have

$$\bigcup_{s<r} H(B(s)) \subseteq \bigcup_{s<r} f_{\mathcal{K}}^{-1}((-\infty, s)) \subseteq \bigcup_{s<r} \{B(s) \geq 1/2\}.$$

Since for any $r \in \mathbb{Q}$ and any $\omega \in \Omega$ we have $B(r)(\omega) \in \{a(\omega), a^{\perp}(\omega)\}$, we conclude $\bigcup_{s<r} B(s) = B(r)$.

In other words, we have proved that $\{B(r) : r \in \mathbb{Q}\}$ satisfies the conditions 1.–4. of Theorem 4.1. Therefore, there is a unique observable x of (Ω, M) such that $x((-\infty, r)) = B(r)$ for any $r \in \mathbb{Q}$. Hence, for any $r \in \mathbb{Q}$ we have $h(f^{-1}((-\infty, r))) = \phi(B(r)) = \phi(x((-\infty, r)))$. In a standard way we have that (8) holds for any Borel set $E \in B(\mathbb{R})$.

The property (9) follows from the following. Let y fulfill (8). Then we have $h(f^{-1}(E)) = \phi(y(E)) = \phi(x(E)) = h_{\mathcal{K}}(f_{\mathcal{K}}^{-1}(E))$ for any $E \in B(\mathbb{R})$ which gives $\{y(E)^{\perp} > 1/2\} \subseteq \Omega \setminus f_{\mathcal{K}}^{-1}(E) \subseteq \{y(E)^{\perp} \geq 1/2\}$. So that $\{x(E) \cap y(E^c) > 1/2\} \subseteq H(x(E)) \cap H(y(E^c)) \subseteq f_{\mathcal{K}}^{-1}(E) \cap f_{\mathcal{K}}^{-1}(E^c) = \emptyset$, which gives $x(E) \cap y(E^c) \leq 1/2$ for any $E \in B(\mathbb{R})$, and Theorem is completely proved. □

Remark 4.1. Using the same method as that in the proof of Theorem 4.2, we can prove Theorem 4.2 also for any crisp representation of a fuzzy quantum poset without the "uniqueness property" in the first direction. For $\mathcal{Q} = \mathcal{K}(M)$ the uniqueness holds in both directions.

For another look at the representation of observables for fuzzy measurable spaces see e.g. [12].

In [9], there has been introduced the notion of the sum for fuzzy quantum measurable spaces: we say that an observable z is a *sum* of observables x and y if, for any $t \in \mathbb{R}$, we have

$$B_z(t) = \bigcup_{r\in\mathbb{Q}} B_x(r) \cap B_y(t-r)\,. \tag{10}$$

It has been shown that the sum always exists. The representation theorem for observables (Theorem 4.2) allows us to build up the calculus of observables in a more straight way.

Let $\mathcal{Q}$ be a crisp representation of a fuzzy measurable space (Ω, M), and let f be a $\mathcal{Q}$-measurable, real-valued function such that (8) holds, then we shall write $x \sim_{\mathcal{Q}} f$.

For example, if a is a fuzzy observable of M, and $A \in \mathcal{Q}$ such that $h(A) = \bar{a}$, then $x_a \sim_{\mathcal{Q}} I_A$, where I_A is the indicator of the set A. Moreover, if ψ is a Borel function, then by $\psi(x)$ we mean an observable $x \circ \psi^{-1}$, and we have if $x \sim_{\mathcal{Q}} f$, then $\psi(x) \sim_{\mathcal{Q}} \psi(f)$.

The calculus for observables $x_1, \dots, x_N$ is introduced as follows. Let $x_i \sim_{\mathcal{Q}} f_i$ for $i = 1, \dots, N$, where N can be either an integer or $+\infty$, and let $\rho\colon \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable function. We define $\rho(x_1, \dots, x_N)$ as any observable x of (Ω, M) such that:

(i) $\rho(f_1, \dots, f_N)$ is $\mathcal{Q}$-measurable;

(ii) $x \sim_{\mathcal{Q}} \rho(f_1, \dots, f_N)$;

(iii) $x(\mathbb{R}) = \bigcap_{i=1}^{N} x_i(\mathbb{R})$.

It is simple to verify that x satisfying (i) – (iii) is unique, and if $x_i \sim_{\mathcal{Q}} g_i$ for $i = 1, \dots, N$, and $y \sim_{\mathcal{Q}} \rho(g_1, \dots, g_N)$, then $\phi\big(y(E)\big) = h\big(\rho(g_1, \dots, g_N)^{-1}(E)\big) = h\Big(\rho(g_1, \dots, g_N)^{-1}(E) \cap \bigcup_{i=1}^{N} \{f_i \neq g_i\}\Big) \vee h\Big(\rho(g_1, \dots, g_N)^{-1}(E) \cap \bigcap_{i=1}^{N} \{f_i = g_i\}\Big) = \bar{0} \vee h\big(\rho(f_1, \dots, f_N)^{-1}(E)\big) = \phi\big(x(E)\big)$.

We recall that if $N = \infty$, then by $\rho(x_1, \dots, x_N)$ we mean some limit expression, or convergence, respectively. Moreover, if $\mathcal{Q}$ and $\mathcal{G}$ are two crisp representations, and if $x_i \sim_{\mathcal{Q}} f_i$, $x_i \sim_{\mathcal{G}} g_i$, $i = 1, \dots, N$, and $x \sim_{\mathcal{Q}} \rho(f_1, \dots f_N)$, $y \sim_{\mathcal{G}} \rho(g_1, \dots, g_N)$, then $\overline{x(E)} = \overline{y(E)}$ for any $E \in B(\mathbb{R})$.

For example, let $\rho(u,v) = u+v$, $u,v \in \mathbb{R}$, then $x+y \sim \rho(f,g)$ if $x \sim f$ and $y \sim g$ and $\rho(x,y) = x+y$.

5. States on fuzzy quantum posets

An analogue of probability measure for fuzzy quantum poset (Ω, M) is a *state*, i.e., any mapping $m\colon M \to [0,1]$ such that

$$m\left(\bigcup_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} m(f_i), \qquad f_i \perp_F f_j, \ (f_i \perp f_j) \ i \neq j. \tag{11}$$

$$m(f \cup f^{\perp}) = 1, \qquad f \in M. \tag{12}$$

Now we present two representation theorems for states. For a fuzzy measurable space the first one has been proved in [3], see also [15].

THEOREM 5.1. *For any state m on M, the mapping $\bar{m}\colon \mathcal{M} \to [0,1]$ defined via $\bar{m}(\bar{a}) = m(a)$, $a \in M$, is a state on a quantum logic $\mathcal{M}$. Conversely, for any state s on the quantum logic $\mathcal{M}$, there is a unique state m on M such that $\bar{m} = s$.*

Proof. It is clear that $\bar{m}$ is a well-defined mapping. Suppose that $\bar{a}_i \leq_M \bar{a}_j^{\perp_M}$ for $i \neq j$. We can find a sequence $\{\hat{a}_i\}$ of mutually orthogonal elements from M such that $\overline{\hat{a}_i} = \bar{a}$ for any i. For this, it suffices to find a sequence $\{c_{ij}\} \in W_1(M)^{\aleph_0}$ such that $a_i \cap a_j \cap c_{ij} \leq 1/2$. Putting $c = \bigcap_{i,j} c_{ij} \in W_1(M)$ and $\hat{a}_i = a_i \cap c$, $i \geq 1$, we obtain the elements in question. Moreover, $\bigvee_i \bar{a}_i = \bigvee_i \overline{\hat{a}_i} = \bar{a}$, where $a = \bigcup_i \hat{a}_i$ so that $\bar{m}\left(\bigvee_i \bar{a}_i\right) = \bar{m}\left(\bigvee_i \overline{\hat{a}_i}\right) = \bar{m}(\bar{a}) = m(a) = \sum_i m(\hat{a}_i) = \sum_i m(a_i) = \sum_i \bar{m}(\bar{a}_i)$.

The second part of the assertion is evident. □

The following theorem extends representation for $\mathcal{Q} = \mathcal{K}(M)$ by Piasecki [17] and Le Ba Long [14]:

THEOREM 5.2. *Let $\mathcal{Q}$ be a crisp representation of a fuzzy quantum space (Ω, M). For any state m on M, a mapping P_m on $\mathcal{Q}$ defined via*

$$P_m(A) = m(a), \qquad A \in \mathcal{Q}, \tag{13}$$

where $\bar{a} = h(A)$ is a probability measure on $\mathcal{Q}$ such that

$$P_m(A) = 0, \qquad A \in h^{-1}(\bar{0}). \tag{14}$$

Conversely, for any probability measure P on $\mathcal{Q}$ with (14), there is a unique state m_P on M such that $P_{m_P} = P$.

P r o o f. If $\bar{a} = h(A) = \bar{b}$, then $m(a) = m(b)$, so that P_m is well defined. Moreover, as in the proof of Theorem 5.1, we can prove that P_m is a probability measure on $\mathcal{Q}$. The proof of (14) is evident.

For the converse, we put $m_P(a) = P(A)$ whenever $h(A) = \bar{a}$. In the standard way, we can show that m_P is a state on M. □

R e m a r k 5.1. N a v a r a and P t á k [15] using the representation as in Theorem 5.1 for a fuzzy measurable space show that it is possible to find (Ω, M) such that M has no states. Now we present a new example of this statement. Let $\Omega = \mathbb{R}$, $\mathcal{Q} = B(\mathbb{R})$, and let Δ be the σ-ideal of $B(\mathbb{R})$ consisting of all subsets of the first category. Let h be the canonical σ-homomorphism from $\mathcal{Q}$ onto $\mathcal{Q}/\Delta$. If we define a fuzzy soft σ-algebra M via a manner described in [15] (see also the end of the third section of the present paper), then M has no state since $M/I_0 = \mathcal{Q}/\Delta$ has no states (see [22; § 21, Example E]).

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