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THE $\bar{\partial}$-NEUMANN OPERATOR ON STRONGLY PSEUDOCONVEX DOMAIN WITH PIECEWISE SMOOTH BOUNDARY

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ABSTRACT. On a bounded strongly pseudoconvex domain D in $\mathbb{C}^n$ with a piecewise smooth boundary, we prove that the $\bar{\partial}$-Neumann operator N can be extended as a bounded operator from Sobolev $\left(-\frac{1}{2}\right)$-spaces to the Sobolev $\left(\frac{1}{2}\right)$-spaces. In particular, N is a compact operator on Sobolev $\left(-\frac{1}{2}\right)$-spaces.

0. Introduction

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with the standard Hermitian metric. Let $\bar{\partial}$ be the maximal extension of the Cauchy-Riemann operator on the space $L^2_{(r,q)}(D)$ of square integrable (r,q)-forms $(0 \le r \le n,\ 0 \le q \le n)$ and $\partial^\star$ its Hilbert space adjoint. The $\bar{\partial}$-Neumann problem consists in proving existence and regularity for the solutions of the equation

$$\Box\varphi = \psi\,, \qquad \Box = \bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}\,.$$

The ∂-Neumann problem has been studied extensively when the domain D has smooth boundary (see [1], [3], [10], [14], [15], [17], and [18]). If D has smooth boundary and has a C^∞-plurisubharmonic defining function on ∂D, Boas and Straube [2] showed that the $\bar{\partial}$-Neumann operator is bounded on Sobolev (s)-spaces with $s \ge 0$. If D is bounded domain with piecewise smooth strongly pseudoconvex boundary, Henkin, Iordan and Kohn [12] and Michel and Shaw [19] showed that the $\bar{\partial}$-Neumann operator is bounded from $L^2_{(r,q)}(D)$ to $H^{\frac{1}{2}}_{(r,q)}(D)$ by two different method. If D is a bounded pseudoconvex Lipschitz domain with plurisubharmonic defining function on ∂D, Michel and Shaw [20] showed that the $\bar{\partial}$-Neumann operator is bounded on Sobolev $\left(\frac{1}{2}\right)$-spaces.

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Straube [23] has extended the subelliptic estimates of N to domains with piecewise smooth boundaries of finite type. Other results in this direction belong to Bonami and Charpentier [4], Engliš [9], and Ehsani [6], [7], and [8]. In fact, the main aim of this work is to establish the following:

THEOREM. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with piecewise smooth boundary. For each $0 \le r \le n$, $1 \le q \le n-1$, the $\bar\partial$-Neumann operator*

$$N\colon L^2_{(r,q)}(D) \to L^2_{(r,q)}(D)$$

satisfies the following estimate: for any $\varphi \in L^2_{(r,q)}(D)$, there exists a constant $C > 0$ such that

$$\|N\varphi\|_{\frac{1}{2}(D)} \le C\|\varphi\|_{-\frac{1}{2}(D)}, \tag{0.1}$$

where $C = C(D)$ is independent of φ; i.e., N can be extended as a bounded operator from $H^{-\frac{1}{2}}_{(r,q)}(D)$ into $H^{\frac{1}{2}}_{(r,q)}(D)$. In particular, N is a compact operator on $L^2_{(r,q)}(D)$ and $H^{-\frac{1}{2}}_{(r,q)}(D)$.

In this paper we shall apply Michel and Shaw technique [19] with suitable modifications required. The plan of this paper is as follows: In Section 1 we first recall the L^2 existence theorem of the $\bar\partial$-Neumann operator on any bounded pseudoconvex domains. In Section 2 we prove a priori estimates on each smooth subdomain. In Section 3 we prove the main theorem.

1. Preliminaries

Let D be a bounded domain of $\mathbb{C}^n$. We express a (r,q)-form φ on D as follows:

$$\varphi = \sum_{A_r, B_q} \varphi_{A_r B_q}\, \mathrm{d}z^{A_r} \wedge \mathrm{d}\bar z^{B_q},$$

where $A_r = (\alpha_1, \dots, \alpha_r)$; $1 \le \alpha_1 < \dots < \alpha_r \le n$, $B_q = (\beta_1, \dots, \beta_q)$; $1 \le \beta_1 < \dots < \beta_q \le n$. We denote by $C^\infty_{(r,q)}(D)$ the space of differential forms of class C^∞ and of type (r,q) on D. Let

$$C^\infty_{(r,q)}(\bar D) = \left\{ \varphi|_{\bar D} \colon\ \varphi \in C^\infty_{(r,q)}(\mathbb{C}^n) \right\}$$

be the subspace of $C^\infty_{(r,q)}(D)$ whose elements can be extended smoothly up to the boundary ∂D of D. For $\varphi, \psi \in C^\infty_{(r,q)}(\bar D)$, we define

$$(\varphi, \psi) = \sum_{A_r, B_q} \varphi_{A_r B_q} \overline{\psi_{A_r B_q}}, \qquad |\varphi|^2 = (\varphi, \varphi),$$

$$\langle \varphi, \psi \rangle = \int_D (\varphi, \psi)\, \mathrm{d}v, \qquad \|\varphi\|^2 = \langle \varphi, \varphi \rangle,$$

where dv is the Lebesgue measure. Let $C^{\infty}_{0,(r,q)}(D)$ be the subspace of $C^{\infty}_{(r,q)}(\bar{D})$ whose elements have compact support in D. The formal adjoint operator ϑ of

$$\bar{\partial}\colon C^{\infty}_{(r,q-1)}(D) \to C^{\infty}_{(r,q)}(D)$$

is defined by :

$$\langle \vartheta\varphi, \psi\rangle = \langle \varphi, \bar{\partial}\psi\rangle$$

for any $\varphi \in C^{\infty}_{(r,q)}(D)$ and $\psi \in C^{\infty}_{0,(r,q-1)}(D)$. It is easily seen that $\bar{\partial}$ is a closed, linear, densely defined operator, and $\bar{\partial}$ forms a complex, i.e., $\bar{\partial}^2 = 0$. We denote by $L^2_{(r,q)}(D)$ the Hilbert space of all (r,q) forms with square integrable coefficients. Let $\bar{\partial}\colon L^2_{(r,q-1)}(D) \to L^2_{(r,q)}(D)$ be the maximal closure of the original $\bar{\partial}$; thus a form $\varphi \in L^2_{(r,q)}(D)$ is in the domain of $\bar{\partial}$ if and only if $\bar{\partial}\varphi$ is defined in the sense of distributions, belongs to $L^2_{(r,q+1)}(D)$. Then $\bar{\partial}$ is a closed, linear, densely defined operator, and forms a complex, i.e., $\bar{\partial}^2 = 0$. We denote the domain and the range of $\bar{\partial}$ in $L^2_{(r,q)}(D)$ by $\mathrm{dom}_{(r,q)}(\bar{\partial})$ and $\mathrm{Rang}_{(r,q)}(\bar{\partial})$ respectively. The adjoint operator

$$\bar{\partial}^{\star}\colon L^2_{(r,q)}(D) \to L^2_{(r,q-1)}(D)$$

of $\bar{\partial}$ also a closed, linear, densely defined operator. Hence, φ is in the domain of $\partial^{\star}$ if there is a $\psi \in L^2_{(r,q-1)}(D)$ such that for any $\chi \in \mathrm{dom}_{(r,q-1)}(\bar{\partial}) \cap L^2_{(r,q-1)}(D)$, we have

$$\langle \varphi, \bar{\partial}\chi\rangle = \langle \psi, \chi\rangle\,.$$

We then define $\bar{\partial}^{\star}\varphi = \psi$. Clearly, $\bar{\partial}^{\star}$ also forms a complex.

Definition 1.1. A domain $D \Subset \mathbb{C}^n$ is said to be *strongly pseudoconvex with C^{∞}-boundary* if there exist an open neighborhood U of ∂D and a C^{∞} function $\lambda\colon U \to \mathbb{R}$ having the following properties:

(i) $D \cap U = \{z \in U : \lambda(z) < 0\}$.

(ii) $\sum\limits_{\alpha,\beta=1}^{n} \frac{\partial^2\lambda(z)}{\partial z^{\alpha}\partial\bar{z}^{\beta}}\zeta^{\alpha}\bar{\zeta}^{\beta} \geq L(z)|\zeta|^2$;
$z \in U$, $\zeta = (\zeta^1,\dots,\zeta^n) \in \mathbb{C}^n$ and $L(z) > 0$.

(iii) The gradient $\nabla\lambda(z) = \left(\frac{\partial\lambda(z)}{\partial x^1}, \frac{\partial\lambda(z)}{\partial y^1},\dots,\frac{\partial\lambda(z)}{\partial x^n}, \frac{\partial\lambda(z)}{\partial y^n}\right) \neq 0$
for $z = (z^1,\dots,z^n) \in U$; $z^{\alpha} = x^{\alpha} + \mathrm{i}y^{\alpha}$.

Definition 1.2. Let D be a bounded domain in $\mathbb{C}^n$. The boundary ∂D of D will be called *piecewise smooth strongly pseudoconvex* if there exists:

(i) A finite open covering $\{V_j\}_{j=1}^{k}$ of an open neighborhood V of ∂D.

(ii) C^2-strongly plurisubharmonic functions $\varrho_j\colon V_j \to \mathbb{R}$, $j = 1,\dots,k$, such that the following conditions hold:

(a) A point $z \in V_1 \cup \dots \cup V_k$ belongs to D if and only if, for every $1 \le j \le k$, $z \notin V_j$ or $\varrho_j(z) < 0$.
(b) For every collection of indices $1 \le j_1 < \dots < j_m \le k$ we have $\mathrm{d}\varrho_{j_1} \wedge \dots \wedge \mathrm{d}\varrho_{j_m} \neq 0$ for all $z \in V_{j_1} \cap \dots \cap V_{j_m}$.

Let $H^s(D)$, $s \ge 0$, be defined as the space of all $u|_D$ such that $u \in H^s(\mathbb{C}^n)$, where $H^s(\mathbb{C}^n) = H^s(\mathbb{R}^{2n})$ is the Sobolev space of $\mathbb{R}^{2n}$. We define the *norm* of $H^s(D)$ by

$$\|u\|_{s(D)} = \inf\left\{\|v\|_{s(\mathbb{C}^n)} : v \in H^s(\mathbb{C}^n),\ v|_D = u\right\}.$$

Let $C_0^\infty(D)$ be the space of C^∞-functions with compact support in D and $H_0^s(D)$ be the completion of $C_0^\infty(D)$ under the $H^s(D)$-norm. When $s = 0$, since $C_0^\infty(D)$ is dense in $L^2(D)$, it follows that $H_0^0(D) = H^0(D) = L^2(D)$. If D is a Lipschitz domain, then $C^\infty(\bar{D})$ are dense in $H^s(D)$ in the $H^s(D)$-norm. If $s \le \frac{1}{2}$, we also have $C_0^\infty(D)$ is dense in $H^s(D)$. Thus

$$H^s(D) = H_0^s(D) \qquad \text{for} \quad s \le \frac{1}{2}. \tag{1.1}$$

We define $H^{-s}(D)$ to be the *dual* of $H_0^s(D)$ when $s > 0$ and the norm of $H^{-s}(D)$ is defined by

$$\|f\|_{-s(D)} = \sup \frac{|\langle f, g\rangle|}{\|g\|_{s(D)}},$$

where the supremum is taken over all functions $g \in C_0^\infty(D)$.

We use $H^s_{(r,q)}(D)$ to denote Hilbert spaces of (r,q)-forms with $H^s(D)$-coefficients and their norms are denoted by $\|\cdot\|_{s(D)}$.

2. A priori estimates

In this section we prove a priori estimates on each smooth subdomain of D. We then prove the estimates on each smooth strongly pseudoconvex domain with good control of the constants in each subdomain. Let $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ be the Laplace-Beltrami operator from $L^2_{(r,q)}(D)$ to $L^2_{(r,q)}(D)$ such that $\mathrm{dom}_{(r,q)}(\square) = \{\varphi \in \mathrm{dom}_{(r,q)}(\bar\partial) \cap \mathrm{dom}_{(r,q)}(\bar\partial^\star) : \bar\partial\varphi \in \mathrm{dom}_{(r,q+1)}(\bar\partial^\star)$ and $\bar\partial^\star\varphi \in \mathrm{dom}_{(r,q-1)}(\bar\partial)\}$. Let $\mathrm{Ker}_{(r,q)}(\square) = \{\varphi \in \mathrm{dom}_{(r,q)}(\bar\partial) \cap \mathrm{dom}_{(r,q)}(\bar\partial^\star) : \bar\partial\varphi = 0$ and $\bar\partial^\star\varphi = 0\}$. Then $\square$ is a linear, closed, densely defined self-adjoint operator from $L^2_{(r,q)}(D)$ to $L^2_{(r,q)}(D)$. Following H ö r m a n d e r L^2-estimates for $\bar\partial$ on any bounded pseudoconvex domains, one can prove that $\square$ has closed range and $\mathrm{Ker}_{(r,q)}(\square) = \{0\}$. The $\bar\partial$-Neumann operator N is the inverse of $\square$. The following L^2-existence of N on D is proved in H ö r m a n d e r [13] and S h a w [21; Proposition 2.3].

PROPOSITION 2.1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq r \leq n$ and $1 \leq q \leq n$, there exists a bounded linear operator*

$$N \colon L^2_{(r,q)}(D) \to L^2_{(r,q)}(D)$$

such that

(i) $\mathrm{Rang}_{(r,q)}(N) \subset \mathrm{dom}_{(r,q)}(\Box)$, $\Box N = N\Box = I$ *on* $\mathrm{dom}_{(r,q)}(\Box)$.

(ii) *For any* $\varphi \in L^2_{(r,q)}(D)$, $\varphi = \partial\bar{\partial}^\star N\varphi + \bar{\partial}^\star\partial N\varphi$.

(iii) *Let δ be the diameter of D. The following estimates hold for any* $\varphi \in L^2_{(r,q)}(D)$:

$$\|N\varphi\| \leq \frac{\mathrm{e}\,\delta^2}{q}\|\varphi\|,$$

$$\|\bar{\partial}N\varphi\| \leq \sqrt{\frac{\mathrm{e}\,\delta^2}{q}}\|\varphi\|,$$

$$\|\bar{\partial}^\star N\varphi\| \leq \sqrt{\frac{\mathrm{e}\,\delta^2}{q}}\|\varphi\|.$$

The following lemma is proved by Michel and Shaw [19]:

LEMMA 2.2. *Let D be a bounded domain in $\mathbb{C}^n$ with a piecewise smooth strongly pseudoconvex boundary. Then, there exists an exhaustion $\{D_\kappa\}_{\kappa=1}^\infty$ of D such that we have the following conditions:*

(i) $\{D_\kappa\}_{\kappa=1}^\infty$ *is an increasing sequence of relatively compact subsets of D and* $\bigcup_{\kappa=1}^\infty D_\kappa = D$.

(ii) *Each $\{D_\kappa\}_{\kappa=1}^\infty$ has a C^∞ plurisubharmonic defining function λ_κ, such that*

$$\sum_{\alpha,\beta=1}^n \frac{\partial^2 \lambda_\kappa}{\partial z^\alpha \partial \bar{z}^\beta}\zeta^\alpha\bar{\zeta}^\beta \geq c_1|\zeta|^2 \qquad \text{for} \quad z \in \partial D_\kappa, \quad \zeta \in \mathbb{C}^n,$$

where $c_1 > 0$ is a constant independent of κ.

(iii) *There exist positive constants c_2, c_3 such that $c_2 \leq |\nabla\lambda_\kappa| \leq c_3$ on ∂D_κ, where c_2, c_3 are independent of κ.*

Lemma 2.2 implies that D can be approximated by a family of strongly pseudoconvex domains with smooth boundaries which are uniformly Lipschitz.

By using the identity of Morrey-Kohn-Hörmander which is proved in Chen and Shaw [5; Proposition 4.3.1], we prove the following lemma:

Lemma 2.3. *Let D and $\{D_\kappa\}_{\kappa=1}^{\infty}$ be the same as in Lemma 2.2. There exists a constant $c_4 > 0$ such that for any $\varphi \in C^\infty_{(r,q)}(\bar D_\kappa) \cap \mathrm{dom}_{(r,q)}(\bar\partial^\star_\kappa)$, $0 \le r \le n$, $1 \le q \le n-1$, we have*

$$\sum_{A_r, B_q}\sum_{k=1}^{n}\left\|\frac{\partial\varphi_{A_r B_q}}{\partial \bar z^k}\right\|^2 + \int_{\partial D_\kappa} |\varphi|^2 \, \mathrm{d}s_\kappa \le c_4\big(\|\bar\partial\varphi\|^2_{D_\kappa} + \|\bar\partial^\star_\kappa\varphi\|^2_{D_\kappa}\big),$$

where $\mathrm{d}s_\kappa$ is the surface element on ∂D_κ and c_4 is independent of κ.

Proof. Since $|\nabla\lambda_\kappa| \neq 0$ on a neighborhood W of ∂D_κ, then the function $\eta_\kappa = \lambda_\kappa/|\nabla\lambda_\kappa|$ is defined on W. We extend η_κ to be negative smoothly inside D_κ. Then η_κ is a defining function in a neighborhood of $\bar D_\kappa$ such that $\eta_\kappa < 0$ on D_κ, $\eta_\kappa = 0$ on ∂D_κ and $|\nabla\eta_\kappa| = 1$ on W. The following identity is proved in Hörmander [13] or in Chen and Shaw [5; Proposition 4.3.1]: for any $\varphi \in C^\infty_{(r,q)}(\bar D_\kappa) \cap \mathrm{dom}_{(r,q)}(\bar\partial^\star_\kappa)$,

$$\begin{aligned}\|\bar\partial\varphi\|^2_{D_\kappa} + \|\bar\partial^\star_\kappa\varphi\|^2_{D_\kappa} &= \sum_{A_r, B_q}\sum_{k=1}^{n}\left\|\frac{\partial\varphi_{A_r B_q}}{\partial \bar z^k}\right\|^2 \\ &\quad + \sum_{A_r B_{q-1}}\sum_{\alpha,\beta=1}^{n}\int_{\partial D_\kappa}\frac{\partial^2\eta_\kappa}{\partial z^\alpha\partial\bar z^\beta}\varphi_{A_r\alpha B_{q-1}}\bar\varphi_{A_r\beta B_{q-1}}\,\mathrm{d}s_\kappa.\end{aligned} \tag{2.1}$$

By simple calculation, for each $z \in \partial D_\kappa$ and $\zeta \in \mathbb{C}^n$, we have

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\eta_\kappa}{\partial z^\alpha\partial\bar z^\beta}\zeta^\alpha\bar\zeta^\beta \\ &\qquad = \frac{1}{|\nabla\lambda_\kappa|}\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\lambda_\kappa}{\partial z^\alpha\partial\bar z^\beta}\zeta^\alpha\bar\zeta^\beta + 2\mathrm{Re}\sum_{\alpha=1}^{n}\left(\frac{\partial\lambda_\kappa}{\partial z^\alpha}\zeta^\alpha\right)\sum_{\beta=1}^{n}\frac{\partial\big(1/|\nabla\lambda_\kappa|\big)}{\partial\bar z^\beta}\bar\zeta^\beta.\end{aligned}$$

Then, if $\sum\limits_{\alpha=1}^{n}\frac{\partial\lambda_\kappa}{\partial z^\alpha}\zeta^\alpha = 0$, it follows from Lemma 2.2(ii) and (iii) that there exists a constant $c_1 > 0$ independent of κ such that on ∂D_κ,

$$\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\lambda_\kappa}{\partial z^\alpha\partial\bar z^\beta}\zeta^\alpha\bar\zeta^\beta \ge c_1|\zeta|^2.$$

Since $\varphi \in C^\infty_{(r,q)}(\bar D_\kappa) \cap \mathrm{dom}_{(r,q)}(\bar\partial^\star_\kappa)$, it follows that φ verifies the Neumann condition

$$\sum_{\beta=1}^{n}\frac{\partial\lambda_\kappa}{\partial z^\beta}\varphi_{A_r\beta B_{q-1}} = 0 \quad \text{on } \partial D_\kappa \qquad \text{for each} \quad A_r, B_{q-1}.$$

Substituting these into (2.1), we have

$$\sum_{A_r,B_q}\sum_{j=1}^{n}\left\|\frac{\partial\varphi_{A_rB_q}}{\partial\bar{z}^j}\right\|^2+c_1\int_{\partial D_\kappa}|\varphi|^2\,\mathrm{d}s_\kappa\le\|\bar{\partial}\varphi\|^2_{D_\kappa}+\|\bar{\partial}^\star_\kappa\varphi\|^2_{D_\kappa}\,.$$

Then, the lemma is proved by taking $c_4=1/\min\{1,c_1\}$. □

PROPOSITION 2.4. *Let D and $\{D_\kappa\}_{\kappa=1}^\infty$ be the same as in Lemma 2.2. There exists a constant $c_5>0$ such that for any $\varphi\in C^\infty_{(r,q)}(\bar{D}_\kappa)\cap\mathrm{dom}_{(r,q)}(\bar{\partial}^\star_\kappa)$, $0\le r\le n$, $1\le q\le n-1$,*

$$\|\varphi\|^2_{\frac{1}{2}(D_\kappa)}\le c_5\big(\|\bar{\partial}\varphi\|^2_{D_\kappa}+\|\bar{\partial}^\star_\kappa\varphi\|^2_{D_\kappa}\big)\,. \tag{2.2}$$

Moreover, if $\varphi\in C^\infty_{(r,q)}(\bar{D}_\kappa)\cap\mathrm{dom}_{(r,q)}(\Box_\kappa)$,

$$\|\varphi\|^2_{\frac{1}{2}(D_\kappa)}\le c_5\|\Box_\kappa\varphi\|^2_{D_\kappa}\,, \tag{2.3}$$

where c_5 is independent of φ and κ.

Proof. Let $z\in\partial D_\kappa$ and u be a special boundary chart containing z. From Kohn [16; Proposition 3.10] and Chen and Shaw [5; Lemma 5.2.2], the tangential Sobolev norm $\sum_{j=1}^{n}|||\,\mathrm{D}^j\varphi\,|||_{\varepsilon-1}$, and the ordinary Sobolev norm $|\,\varphi\|_\varepsilon$ are equivalent for $\varphi\in\mathrm{dom}(\bar{\partial})\cap\mathrm{dom}(\bar{\partial}^\star)$ where the support of φ lies in $u\cap D_\kappa$, $\mathrm{D}^j\varphi=\partial\varphi/\partial x_j$ $(j=1,2,\dots,2n)$, and $\varepsilon>0$. Then, from Folland and Kohn [10; Theorems 2.4.4, 2.4.5], it follows that there exist a neighborhood $w\subset u$ of z and a positive constant c_6 such that

$$\|\varphi\|^2_{\frac{1}{2}(D_\kappa)}\le c_6\bigg(\sum_{A_r,B_q}\sum_{j=1}^{n}\left\|\frac{\partial\varphi_{A_rB_q}}{\partial\bar{z}^j}\right\|^2+\|\varphi\|^2_{D_\kappa}+\int_{\partial D_\kappa}|\varphi|^2\,\mathrm{d}s_\kappa\bigg) \tag{2.4}$$

for $\varphi\in C^\infty_{0,(r,q)}(w\cap\bar{D}_\kappa)\cap\mathrm{dom}_{(r,q)}(\Box_\kappa)$. Since D_κ is a Lipschitz domain, then c_6 depends only on the Lipschitz constant. Also from Lemma 2.2, $\{D_\kappa\}_{\kappa=1}^\infty$ is uniformly Lipschitz, then the constant c_6 can be chosen to depend only on the Lipschitz character of ∂D_κ, which is independent of κ. Now cover ∂D_κ by finite charts $\{w_i\}_{i=1}^m$ such that this conclusion holds on each chart and choose w_0 so that $D_\kappa-\bigcup_{i=1}^{m}w_i\subset w_0\subset\bar{w}_0\subset D_\kappa$. Then, the estimate (2.4) holds for all $\varphi\in C^\infty_{0,(r,q)}(w_0)$. Using a partition of unity subordinate to $\{w_i\}_{i=0}^m$, the estimate (2.4) becomes

$$\|\varphi\|^2_{\frac{1}{2}(D_\kappa)}\le c_6\bigg(\sum_{A_r,B_q}\sum_{j=1}^{n}\left\|\frac{\partial\varphi_{A_rB_q}}{\partial\bar{z}^j}\right\|^2+\|\varphi\|^2_{D_\kappa}+\int_{\partial D_\kappa}|\varphi|^2\,\mathrm{d}s_\kappa\bigg) \tag{2.5}$$

for any $\varphi \in C^\infty_{(r,q)}(\bar{D}_\kappa) \cap \mathrm{dom}_{(r,q)}(\bar{\partial}^\star_\kappa)$, c_6 is independent of κ. It follows from Proposition 2.1 that

$$\|\varphi\|^2_{D_\kappa} \le \frac{\mathrm{e}\,\delta^2}{q}\left(\|\bar{\partial}\varphi\|^2_{D_\kappa} + \|\bar{\partial}^\star\varphi\|^2_{D_\kappa}\right).$$

Then, by using Lemma 2.3 and (2.5) and by taking $c_5 = c_6\left(\frac{\mathrm{e}\,\delta^2}{q} + c_4\right)$, the inequality (2.2) is proved. Also

$$\|\bar{\partial}\varphi\|^2_{D_\kappa} + \|\bar{\partial}^\star_\kappa\varphi\|^2_{D_\kappa} \le \|\Box_\kappa\varphi\|_{D_\kappa}\|\varphi\|_{D_\kappa}$$

whenever $\varphi \in C^\infty_{(r,q)}(\bar{D}_\kappa) \cap \mathrm{dom}_{(r,q)}(\Box_\kappa)$. Then, (2.3) is proved, too. □

Theorem 2.5 (Rellich Theorem). *Let D be a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary. If $s_1 > s_2 \ge 0$, the inclusion $H^{s_1}(D) \hookrightarrow H^{s_2}(D)$ is compact.*

The description, the construction and the properties of the linear extension operator P follows from [22; Chap. VI]. Also it is evident that:

Theorem 2.6. *Let D be a bounded open subset of $\mathbb{C}^n$ with Lipschitz boundary; then for every $s > 0$ there exists a continuous linear extension operator P from $H^s(D)$ into $H^s(\mathbb{C}^n)$ such that $Pg|_D = g$, Pg is C^∞ on $\mathbb{C}^n \setminus \bar{D}$, and*

$$\|Pg\|_{s(\mathbb{C}^n)} \le c\|g\|_{s(D)}$$

for some constant c independent of g.

3. The proof of the main theorem

Let D and $\{D_\kappa\}^\infty_{\kappa=1}$ be the same as in Lemma 2.2 and N_κ denote the $\bar{\partial}$-Neumann operator on $L^2_{(r,q)}(D_\kappa)$. To prove the main theorem, it suffices to prove (0.1) for any $\varphi \in C^\infty_{(r,q)}(\bar{D})$. By using the boundary regularity for N_κ which was established by Kohn [15], we have $N_\kappa\varphi \in C^\infty_{(r,q)}(\bar{D}_\kappa) \cap \mathrm{dom}_{(r,q)}(\Box_\kappa)$. By using (iii) and (ii) in Proposition 2.1, we have

$$\|N_\kappa\varphi\|_{D_\kappa} \le \frac{\mathrm{e}\,\delta^2}{q}\|\varphi\|_{D_\kappa} \le \frac{\mathrm{e}\,\delta^2}{q}\|\varphi\|_D, \tag{3.1}$$

$$\|\bar{\partial}N_\kappa\varphi\|_{D_\kappa} + \|\bar{\partial}^\star_\kappa N_\kappa\varphi\|_{D_\kappa} \le 2\sqrt{\frac{\mathrm{e}\,\delta^2}{q}}\|\varphi\|_{D_\kappa} \le 2\sqrt{\frac{\mathrm{e}\,\delta^2}{q}}\|\varphi\|_D, \tag{3.2}$$

and

$$\|\bar{\partial}\bar{\partial}^\star_\kappa N_\kappa\varphi\|^2_{D_\kappa} + \|\bar{\partial}^\star_\kappa\bar{\partial}N_\kappa\varphi\|^2_{D_\kappa} = \|\varphi\|^2_{D_\kappa} \le \|\varphi\|^2_D. \tag{3.3}$$

Let us extend $N_\kappa\varphi$ to all of D by setting $N_\kappa\varphi = 0$ in $D \setminus D_\kappa$, thus by the Rellich and Sobolev lemmas we can choose a subsequence (still denoted by $N_\kappa\varphi$) converging weakly to some element $\psi \in L^2_{(r,q)}(D)$ and $\bar\partial\psi \in L^2_{(r,q+1)}(D)$. In view of (3.1), (3.2) and (3.3), we can assume that $N_\kappa\varphi$, $\bar\partial N_\kappa\varphi$, $\bar\partial^\star_\kappa N_\kappa\varphi$, $\partial^\star_\kappa\bar\partial N_\kappa\varphi$, and $\partial\partial^\star_\kappa N_\kappa\varphi$ converge weakly to some elements ψ, ψ_1, ψ_2, ψ_3 and ψ_4 of $L^2_{(r,q)}(D)$, respectively (here again extending $\partial N_\kappa\varphi$ etc. by zero on $D\setminus D_\kappa$). We claim that $\psi \in \mathrm{dom}_{(r,q)}(\bar\partial) \cap \mathrm{dom}_{(r,q)}(\bar\partial^\star)$ and $\bar\partial\psi = \psi_1$, $\bar\partial^\star\psi = \psi_2$. Indeed, for any $u \in \mathrm{dom}_{(r,q-1)}(\bar\partial) \cap L^2_{(r,q-1)}(D)$,

$$\begin{aligned} |\langle\psi, \bar\partial u\rangle_D| &= \lim_{\kappa\to\infty} |\langle N_\kappa\varphi, \bar\partial u\rangle_{D_\kappa}| = \lim_{\kappa\to\infty} |\langle\bar\partial^\star_\kappa N_\kappa\varphi, u\rangle_{D_\kappa}| \\ &\le 2\sqrt{\frac{e\,\delta^2}{q}}\|\varphi\|_D\|u\|_D\,. \end{aligned} \tag{3.4}$$

Thus $\psi \in \mathrm{dom}_{(r,q)}(\bar\partial^\star)$. The proof for $\bar\partial$ is the same. Using the same arguments as in (3.4) we obtain $\psi_1 \in \mathrm{dom}(\bar\partial^\star)$, $\psi_2 \in \mathrm{dom}(\bar\partial)$ and $\bar\partial^\star\psi_1 = \psi_3$, $\bar\partial\psi_2 = \psi_4$. Thus $\psi \in \mathrm{dom}(\Box)$ and $\Box\psi$ is the weak limit of $\Box_\kappa N_\kappa\varphi = \varphi$; that is, $\psi = N\varphi$ and $N_\kappa\varphi \to N\varphi$ weakly in L^2. Then, from (1.1), we have

$$H^{\frac12}(D) = H_0^{\frac12}(D)\,.$$

Then it follows from the Generalized Schwartz inequality (see F o l l a n d and K o h n [10; Proposition (A.1.1)] or C h e n and S h a w [5; p. 340]) that

$$|\langle h, f\rangle_{D_\kappa}| \le \|h\|_{\frac12(D_\kappa)}\|f\|_{-\frac12(D_\kappa)}$$

for any $h \in H^{\frac12}_{(r,q)}(D_\kappa)$ and $f \in H^{-\frac12}_{(r,q)}(D_\kappa)$. By using (2.2), there exists a constant $c_5 > 0$ such that for any $\varphi \in C^\infty_{(r,q)}(\bar D_\kappa) \cap \mathrm{dom}_{(r,q)}(\Box_\kappa)$, $0 \le r \le n$ and $1 \le q \le n$,

$$\begin{aligned} \|\varphi\|^2_{\frac12(D_\kappa)} &\le c_5\big(\|\bar\partial\varphi\|^2_{D_\kappa} + \|\partial^\star_\kappa\varphi\|^2_{D_\kappa}\big) = c_5\langle\varphi, \Box_\kappa\varphi\rangle_{D_\kappa} \\ &\le c_5\|\varphi\|_{\frac12(D_\kappa)}\|\Box_\kappa\varphi\|_{-\frac12(D_\kappa)}\,, \end{aligned} \tag{3.5}$$

where c_5 is independent of φ and κ. Substituting $N_\kappa\varphi$ into (3.5), we have

$$\|N_\kappa\varphi\|_{\frac12(D_\kappa)} \le c_5\|\Box_\kappa N_\kappa\varphi\|_{-\frac12(D_\kappa)} = c_5\|\varphi\|_{-\frac12(D_\kappa)}\,, \tag{3.6}$$

where c_5 is independent of φ and κ. It follows from Theorem 2.6 that there exists a linear extension operator

$$P_\kappa\colon H^{\frac12}(D_\kappa) \to H^{\frac12}(\mathbb{C}^n)$$

such that for each $\varphi \in H^{\frac12}(D_\kappa)$, $P_\kappa\varphi = \varphi$ on D_κ and

$$\|P_\kappa\varphi\|_{\frac12(\mathbb{C}^n)} \le c_5\|\varphi\|_{\frac12(D_\kappa)} \tag{3.7}$$

for some positive constant c_5. The constant c_5 in (3.7) can be chosen independent of κ since an extension exists for any Lipschitz domain (see E. Stein [22] or Grisvard [11; Theorem 1.4.3.1]). By applying P_κ to $N_\kappa\varphi$ componentwise and by using (3.6) and (3.7), there exist a positive constant C independent of κ such that

$$\|P_\kappa N_\kappa\varphi\|_{\frac{1}{2}(D)} \le \|P_\kappa N_\kappa\varphi\|_{\frac{1}{2}(\mathbb{C}^n)} \le c_7\|N_\kappa\varphi\|_{\frac{1}{2}(D_\kappa)} \le C\|\varphi\|_{-\frac{1}{2}(D_\kappa)}\,.$$

Let P be the extension operator of Theorem 2.6 applied to D. Since $D_\kappa \to D$ converges uniformly, then $P_\kappa \to P$ converges uniformly also. Also since $\lim\limits_{\kappa\to\infty} N_\kappa\varphi = N\varphi$, then $\lim\limits_{\kappa\to\infty} P_\kappa N_\kappa\varphi = PN\varphi = N\varphi$. Then (0.1) is proved by taking the limit in the above inequality. Thus N can be extended as a bounded operator from $H^{-\frac{1}{2}}_{(r,q)}(D)$ to $H^{\frac{1}{2}}_{(r,q)}(D)$.

To prove that N is compact, since N is bounded from $H^{-\frac{1}{2}}_{(r,q)}(D)$ into $H^{\frac{1}{2}}_{(r,q)}(D)$, and by Theorem 2.6, the inclusions

$$H^{\frac{1}{2}}(D) \hookrightarrow L^2(D) \hookrightarrow H^{-\frac{1}{2}}(D)$$

and

$$H^{\frac{1}{2}}(D) \hookrightarrow H^{-\frac{1}{2}}(D)$$

are compact; since a composition of a bounded and a compact operator is compact, the compactness of N on $H^{-\frac{1}{2}}(D)$ and $L^2(D)$ follows.

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