Mathematica Slovaca

Alfonz Haviar

On G-lattices

Mathematica Slovaca, Vol. 29 (1979), No. 1, 17--24

Persistent URL: http://dml.cz/dmlcz/130977

Math. Slovaca 29, 1979, No. 1, 17—24

ON G-LATTICES

HAVIAR ALFONZ

E. Fried in [2] and H. Skala in [5] have introduced a class $\mathbf{K}_1$ of weakly associative lattices (called trellises in [5]). In [6] a class $\mathbf{K}_2$ of weakly commutative lattices (N-skew lattices) is given. In this paper a class of G-lattices is defined, which is a generalization of both classes $\mathbf{K}_1$ and $\mathbf{K}_2$.

1. Basic definitions and properties

Definition 1. *A trellis (or* WA*-lattice) is an algebra* $(L; \wedge, \vee)$, *where* $\wedge$ *and* $\vee$ *are two binary commutative operations on L, called meet and join, respectively, satisfying the following identities:*

(1) $\quad x\wedge(y\vee x)=x \quad$ *and dually*

(2) $\quad x\wedge((x\vee y)\wedge(x\vee z))=x \quad$ *and dually.*

We obtain the dual identity by changing the operation symbols and reversing the sequence of variables (e. g. the dual ldentity of $x\wedge(y\wedge z) = x\wedge(z\wedge y)$ is $(z\vee y)\vee x = (y\vee z)\vee x$).

Proposition 1. *In a trellis there holds*

(3) $\quad (x\wedge y)\wedge x = x\wedge(y\wedge x) \quad$ *and dually.*

Proof. If follows from the commutativity of the operations $\wedge$ and $\vee$.

Definition 2. *An N-skew lattice is an algebra* $(L; \wedge, \vee)$, *where* $\wedge$ *and* $\vee$ *are two binary associative operations satisfying the following identities:*

(4) $\quad x\wedge(y\wedge z)=x\wedge(z\wedge y) \quad$ *and dually*

(5) $\quad x\wedge x=x \quad$ *and dually*

(6) $\quad x\wedge(x\vee y)=x \quad$ *and dually.*

Proposition 2. (1) *and* **(2)** *hold in teach N-skew lattice.*

Proof. The verification of (1) is in [6]. The associativity of both operations and the identity (6) imply the identity (2).

Definition 3. *Let* $(L\,;\ \wedge,\ \vee)$ *be an algebra, where* $\wedge$ *and* $\vee$ *are two binary operations on L called meet and join, respectively. Let us call the algebra* $(L\,;\ \wedge,\ \vee)$ *a G-lattice (a generalized lattice) if the operations* $\wedge$ *and* $\vee$ *satisfy the identities* (1), (2), (3), *and* (4).

Proposition 3. *If an algebra is a trellis or an N-skew lattice, then it is a G-lattice.*
Proof. It suffices to use Proposition 1 and Proposition 2.

Proposition 4. *For every* x, y *of a G-lattice the identities* (5), (6) *and the following identities are satisfied:*

(7) $(x \wedge y) \wedge x = x \wedge y$ *and dually*
(8) $(x \wedge y) \wedge y = x \wedge y$ *and dually*
(9) $x \wedge (x \wedge y) = x \wedge y$ *and dually*
(10) $x \wedge y = x$ *if and only if* $x \vee y = y$.

Proof. Proof of (5). Using (1) and (2) we get

$$x \wedge x = x \wedge (((x \wedge x) \vee (x \wedge x)) \vee x) = x\,.$$

$x \vee x = x$ can be proved dually. Proof of (6) follows from (5) and (2). It is obvious that (1) and (6) imply (7), (8), and (10). Proof of (9) follows from (4), (3) and (7).

2. G-ordered set

Definition 4. *A binary relation R is called weakly transitive if*

$$R\,.(R \cap R^{-1}) \subseteq R \quad \textit{and} \quad (R \cap R^{-1})\,.R \subseteq R\,.$$

Let L be a non-void set. The identity on L will be denoted by 1_L.

Definition 5. *We call a G-ordered set a relational system* $(L\,;\ R,\ R_1,\ R_2)$, *where* R, R_1, R_2 *are binary relations on L satisfying the following conditions:*

(a) *R is reflexive and weakly transitive*
(b) *R_1 and R_2 are reflexive and antisymmetric*
(c) $R_1 \cap R$ *and* $R_2 \subseteq R$
(d) $R_1 R_1^{-1} \cap R \cap R^{-1} \subseteq 1_L$ and $R_2^{-1} R_2 \cap R \cap R^{-1} \subseteq 1_L$.

Let $L\,;\ R,\ R_1,\ R_2)$ be a G-ordered set, n a natural number and $(a_1,\ \dots,\ a_n) \in L^n$. An element $v \in L$ is called a G-lower bound of $(a_1,\ \dots,\ a_n)$ if

$$v\ R\ a_j \quad \text{for all} \quad j \in \{1,\ \dots,\ n\}\,.$$

An element $u \in L$ is called a G-upper bound of $(a_1,\ \dots,\ a_n)$ if

$$a_j\ R\ u \quad \text{for all} \quad j \in \{1,\ \dots,\ n\}\,.$$

Definition 6. *An element $i \in L$ is called a G-infimum of $(a_1, \ldots, a_n)$ if the following conditions are satisfied:*

(i) $i\ R_1\ a_1$
(ii) *i is a G-lower bound of $(a_1, \ldots, a_n)$*
(iii) *$v\ R\ i$ holds for every G-lower bound v of $(a_1, \ldots, a_n)$.*

An element $s \in L$ is called a G-supremum of $(a_1, \ldots, a_n)$ if the following conditions are satisfied:

(j) $a_n\ R_2\ s$
(jj) *s is a G-upper bound of $(a_1, \ldots, a_n)$*
(jjj) *$s\ R\ u$ holds for every G-upper bound u of $(a_1, \ldots, a_n)$.*

Proposition 5. *Let $(L; R, R_1, R_2)$ be a G-ordered set. If a G-infimum (or a G-supremum) of $(a_1, \ldots, a_n)$ exists, then it is unique.*

Proof. Assume that i_1 and i_2 are G-infima of $(a_1, \ldots, a_n)$. Then $i_1\ R_1\ a_1$, $i_2\ R_1\ a_1$, $i_1\ R\ i_2$, and $i_2\ R\ i_1$ are valid. It follows that $(i_1, i_2) \in R_1 R_1^{-1} \cap R \cap R^{-1}$, hence $i_1 = i_2$ by (d). The verification for G-supremum is analogous.

The G-infimum of $(a_1, \ldots, a_n)$ (if it exists) will be denoted by $\inf(a_1, \ldots, a_n)$. We write $\sup(a_1, \ldots, a_n)$ for the G-supremum of $(a_1, \ldots, a_n)$.

Proposition 6. *Let $(L; R, R_1, R_2)$ be a G-ordered set. For all $a, b \in L$ the following conditions hold:*

(e) $\inf(a, a) = a$ *and* $\sup(a, a) = a$
(f) $a\ R\ b \Leftrightarrow \inf(a, b) = a$ *and* $a\ R\ b \Leftrightarrow \sup(a, b) = b$
(g) $a\ R_1\ b \Leftrightarrow \inf(b, a) = a$ *and* $a\ R_2\ b \Leftrightarrow \sup(b, a) = b$.

Proposition 6 follows directly from the Definitions.

3. GR-lattice

Definition 7. *Let $(L; R, R_1, R_2)$ be a G-ordered set. If each pair of elements of L has a G-infimum and a G-supremum, then $(L; R, R_1, R_2)$ will be called a GR-lattice.*

Proposition 7. *Let $\mathcal{R} = (L; R, R_1, R_2)$ be a GR-lattice. Let us define the operations $\wedge$ and $\vee$ on L in the following way*

(0) $a \wedge b = \inf(a, b)$, $a \vee b = \sup(a, b)$.

Then $(L; \wedge, \vee)$ is a G-lattice.

Proof. The identities (1) and (2) follow directly from the definitions. Now we will verify the identity (4). Let $i_1 = \inf(a, \inf(b, c))$, $i_2 = \inf(a, \inf(c, b))$, $v_1 =$

$\inf(b, c)$, $v_2 = \inf(c, b)$. Then $i_1\, R\, v_1$, $v_1\, R\, v_2$, $v_2\, R\, v_1$ are true and imply $(i_1, v_2) \in R(R \cap R^{-1})$ and using (a) we get $i_1\, R\, v_2$. $i_1\, R\, a$ is evident, hence $i_1\, R\, i_2$. A similar argument shows $i_2\, R\, i_1$. From $i_1\, R_1\, a$, $i_2\, R_1\, a$, $i_1\, R\, i_2$, $i_2\, R\, i_1$ by (d) we claim $i_1 = i_2$. The second identity of (4) can be proved similarly. For the verification of (3) it is sufficient to prove $(x \wedge y) \wedge x = x \wedge y$ and $x \wedge (y \wedge x) = x \wedge y$. The first identity holds by (f). Using (4) we obtain $x \wedge (y \wedge x) = x \wedge (x \wedge y)$ and by (g) $\inf(x, \inf(x, y)) = \inf(x, y)$, which completes the proof.

Denote by $\mathscr{R}^+ = (L\,;\, \wedge, \vee)$ the G-lattice corresponding to a GR-lattice $\mathscr{R} = (L\,;\, R, R_1, R_2)$, which operations $\wedge$, $\vee$ are defined by (0).

Proposition 8. *Let $\mathscr{L} = (L\,;\, \wedge, \vee)$ be a G-lattice. Let us define binary relations R, R_1, R_2 on L in the following way:*

(r) $a\, R\, b \Leftrightarrow a \wedge b = a$

(p) $a\, R_1\, b \Leftrightarrow b \wedge a = a$

(q) $a\, R_2\, b \Leftrightarrow b \vee a = b$.

Then $(L\,;\, R, R_1, R_2)$ is a GR-lattice.

Proof. First we prove the conditions (a)—(d) of Definition 5. (a). From (5) it immediately follows that R is a reflexive relation. If $(a, b) \in R(R \cap R^{-1})$, then there exists such $c \in L$ that $a\, R\, c$, $c\, R\, b$, $b\, R\, c$ hold. Thus by (r) and (4) we get $a \wedge b = a \wedge (b \wedge c) = a \wedge (c \wedge b) = a \wedge c = a$, which imply $a\, R\, b$. The inclusion $(R \cap R^{-1})R \subseteq R$ can be proved dually. (b). By (5) it follows that R_1 and R_2 are reflexive relations. We assume $a\, R_1\, b$ and $b\, R_1\, a$. Then by (p) and (7) $b = a \wedge b = (b \wedge a) \wedge b = b \wedge a = a$, i. e. R_1 is an antisymmetrical relation. A similar argument shows that R_2 is also antisymmetrical. (c). $a\, R_1\, b$ implies that $a \wedge b = (b \wedge a) \wedge b = b \wedge a = a$, hence $a\, R\, b$. $R_2 \subseteq R$ can be proved dually.

(d). Let $a\, R\, b$, $b\, R\, a$, $a\, R_1\, c$, $b\, R_1\, c$ be valid; then by (4) $a = c \wedge a = c \wedge (a \wedge b) = c \wedge (b \wedge a) = c \wedge b = b$. The second condition can be proved similarly.

It remains to prove that all pairs of elements of L have both a G-infimum and a G-supremum. We will show that

(s) $\inf(a, b) = a \wedge b, \qquad \sup(a, b) = a \vee b.$

From (9) and (8) we obtain $a \wedge b\; R_1\; a$ and $a \wedge b\; R\; b$. From $x\, R\, a$ and $x\, R\, b$ by (2) and (10) we get $x \wedge (a \wedge b) = x \wedge ((x \vee a) \wedge (x \vee b)) = x$, i. e. $x\, R\, a \wedge b$. The other equality can be proved in the same way.

A GR-lattice correspoding to a G-lattice $\mathscr{L} = (L\,;\, \wedge, \vee)$, and the relations of which are given as in the conditions (r), (p), (q), will be denoted by $\mathscr{L}^*$.

Theorem 1. *Let $\mathscr{L} = (L\,;\, \wedge, \vee)$ be a G-lattice and $\mathscr{R} = (L\,;\, R, R_1, R_2)$ be a GR-lattice. Then*

$$(\mathscr{L}^*)^+ = \mathscr{L} \quad \text{and} \quad (\mathscr{R}^+)^* = \mathscr{R}.$$

Proof. Let us denote $(\mathscr{L}^*)^+ = (l\,;\, \cap, \cup)$, $(\mathscr{R}^+)^* = (L\,;\, S, S_1, S_2)$. From (0) and

(s) it follows that $a \cap b = \inf(a, b) = a \wedge b$. Similarly $a \cup b = a \vee b$. By (r) and (0) and (f)

$$a \; S \; b \Leftrightarrow a \wedge b = a \Leftrightarrow \inf(a, b) = a \Leftrightarrow a \; R \; b.$$

We can prove analogously that $S_1 = R_1$ and $S_2 = R_2$.

Thus, we are justified to speak of a G-lattice without specifying whether one is defined by relations or by operations.

Proposition 9. *A G-lattice* $(L; \wedge, \vee)$ *is*

a) *a trellis if and only if in* $\mathscr{L}^* = (L; R, R_1, R_2)$ *is* $R = R_1 = R_2$,
b) *an N-skew lattice if and only if the relations* R, R_1, R_2 *are transitive,*
c) *a lattice if and only if it is both a trellis and an N-skew lattice.*

Proof. a) Let a G-lattice $(L; \wedge, \vee)$ be a trellis and let there hold $a \; R_1 \; b$. By (p) $b \wedge a = a$ and so $a \wedge b = a$, which implies $a \; R \; b$. The equality $R_2 = R$ can be proved similarly. Conversely, let $R = R_1 = R_2$. It is enough to verify that $a \wedge b = b \wedge a$, $a \vee b = b \vee a$. Because of

$$\inf(a, b) R_1 a \wedge \inf(a, b) R b \Leftrightarrow \inf(a, b) R a \wedge \inf(a, b) R_1 b$$

we have $\inf(a, b) = \inf(b, a)$. In a similar manner we obtain $a \vee b = b \vee a$.
b) This part follows from [6].
c) This is an immediate consequence of the definitions.

4. Some properties of G-lattices

Proposition 10. *The elements* x, y *of any G-lattice satisfy the following identities*

(11) $\quad x \wedge (y \wedge x) = x \wedge y \quad$ *and dually*
(12) $\quad (x \wedge y) \wedge (y \wedge x) = x \wedge y \quad$ *and dually.*

Proof. The identity (11) follows immediately from (3) and (7). The identity (12) holds by (4) and (5).

Proposition 11. *Let* $(L; \wedge, \vee)$ *be a G-lattice. If* $a \; R \; b$ *and* $b \; R \; a$ *hold for* a, $b \in L$, *then the following identities are valid:*
(13) $\quad x \wedge a = x \wedge b$ *and* $a \vee x = b \vee x$ *for each* $x \in L$,

(14) $(a \wedge x) \wedge (b \wedge x) = a \wedge x$ *and* $(x \vee b) \vee (x \vee a) = x \vee a$ *for each* $x \in L$.

Proof. By (r) $a \wedge b = a$ and $b \wedge a = b$, therefore by (4) $x \wedge a = x \wedge (a \wedge b) = x \wedge (b \wedge a) = x \wedge b$. Similarly $a \vee x = b \vee x$. Further, $(a \wedge x) \wedge b = (a \wedge x) \wedge a = a \wedge x$ by (13) and (7), hence $(a \wedge x) \vee b = b$ by (10). It implies $(a \wedge x) \wedge (b \wedge x)$

$= (a\wedge x)\wedge(((a\wedge x)\vee b)\wedge((a\wedge x)\vee x)) = a\wedge x$ by (6) and (2). The second identity in (14) can be proved dually.

Theorem 2. Let $\mathscr{L}=(L;\wedge,\vee)$ *be a G-lattice. A relation* $\equiv$ *on L defined in the following way*

$$(15)\qquad a\equiv b \text{ if and only if } a\wedge b=a \text{ and } b\wedge a=b$$

is a congruence relation of $\mathscr{L}$.

Proof. It can be easily shown that the relation $\equiv$ is reflexive and symmetric. If $a\equiv b$ and $b\equiv c$, then by Proposition 11 $a\wedge c = a\wedge b = a$ and $c\wedge a = c\wedge b = c$ and so $a\equiv c$. If $a\equiv b$, then by Proposition 11 $x\wedge a = x\wedge b$ (hence $x\wedge a \equiv x\wedge b$ too) and $a\wedge x \equiv b\wedge x$ for all $x\in L$. Similarly $a\vee x = b\vee x$ and $x\vee a \equiv x\vee b$. This completes the proof.

Remark. From (12) it follows that $x\wedge y\equiv y\wedge x$, hence the quotient algebra $\mathscr{L}/\equiv$ is a trellis. Every congruence class (with $\wedge$ and $\vee$) is a nest.

5. Examples

1. Let Z be the set of all integers. We define binary relations R, R_1, R_2 on Z as follows

$1\ R\ x$ and $-1\ R\ x$ and $x\ R\ 0$ for all $x\in Z$,

$x\ R\ x$ if and only if $\dfrac{|y|}{|x|}$ is either 1 or a prime number, for $x\neq \pm 1$ and $y\neq 0$,

$x\ R_1\ y$ if and only if $x\ R\ y$ and (either $xy>0$ or $x\geqq 0$, $y=0$),

$x\ R_2\ y$ if and only if $x\ R\ y$ and $xy\geqq 0$.

Then $(Z;R,R_1,R_2)$ is a G-lattice. It is neither an N-skew lattice nor a trellis. For instance

$$(4\wedge 6)\wedge 12=2\wedge 12=1,\qquad 4\wedge(6\wedge 12)=4\wedge 6=2,$$
$$(-4)\vee 6=12,\qquad 6\vee(-4)=-12.$$

2. Let C be the set of all complex numbers. Define the relations R, R_1 on C in the following way

$$(a,b)\ R\ (c,d) \text{ if and only if } |a|\leqq|c| \text{ and } \begin{cases} b\leqq d \text{ if } b\neq 0 \text{ and } d\neq 0\\ d\leqq 0 \text{ if } b=0\\ b\geqq 0 \text{ if } d=0\end{cases}$$

$(a,b)\ R_1\ (c,d)$ if and only if $(a,b)\ R\ (c,d)$ and $ac\geqq 0$. Let $R_1=R_2$. Then $(C;R,R_1,R_2)$ is a G-lattice which is neither an N-skew lattice nor a trellis.

3. Let $\mathscr{L}=(L;\leqq)$ be a trellis ($x\leqq y$ means $x\wedge y=x$) and A be a finite set $A=\{a_1,\ldots,a_n\}$. Let $(f_1,\ldots,f_n)$ and $(g_1,\ldots,g_n)$ be n-tuple mappings of L onto A such that

$$f_i(x) \in A \setminus \{f_1(x), \ldots, f_{i-1}(x)\},$$

$$g_i(x) \in A \setminus \{g_1(x), \ldots, g_{i-1}(x)\}$$

for all $x \in L$, $i \in \{2, \ldots, n\}$. Define on $L \times A$ the relations R, R_1, R_2 as follows

$(x, a_i)\ R\ (y, a_j)$ if and only if $x \leqq y$,
$(x, a_i)\ R_1\ (y, a_j)$ if and only if $x \leqq y$ and there exists an f_k such that $f_k(x) = a_i$ and $f_k(y) = a_j$,
$(x, a_i)\ R_2\ (y, a_j)$ if and only if $x \leqq y$ and there exists an g_h such that $g_h(x) = a_i$ and $g_h(y) = a_j$,

$i, j, k, h \in \{1, \ldots, n\}$. Then $\mathscr{L}_a = (L \times A;\ R, R_1, R_2)$ is a G-lattice and

$$(x, a_i) \wedge (y, a_j) = (x \wedge y,\ f_r(x \wedge y)),$$

$$(x, a_i) \vee (y, a_j) = (x \vee y,\ g_s(x \vee y))$$

if $f_r(x) = a_i$ and $g_s(y) = a_j$.

Remark. If $\mathscr{L}_a$ is a G-lattice given as in Example 3 and $\equiv$ is the congruence relation given by (15) of $\mathscr{L}_a$, then it is possible to show that $\mathscr{L}_a/\equiv$ is a trellis isomorphic with $\mathscr{L}$.

REFERENCES

[1] FRIED, E.—GRÄTZER, G.: Some examples of weakly associative lattices, Coll. Math., 27, 1973, 215—221.
[2] FRIED, E.: Tournaments and non-associative lattices, Ann. Univ. Sci. Budapest Sect. Math., 13, 1970, 151—164.
[3] GERHARDTS, M. D.: Schrägverbände und Quasiordnungen, Math. Ann., 181, 1969, 65—73.
[4] GERHARDTS, M. D.: Zerlegungshomomorphismen in Schrägverbänden, Arch. Math., 21, 1970, 116—122.
[5] SKALA, H.: Trellis theory, Algebra Univ., 1, 1971, 218—233.
[6] HAVIAR, A.: N-Schrägverbände und Quasiordnungen, Mat. Čas., 23, 1973, 240—248.

Received August 30, 1976

Katedra matematiky
Pedagogickej fakulty
Tajovského 40
975 49 Banská Bystrica

О G—СТРУКТУРАХ

А. Гавьяр

Резюме

В работе определяется понятие G-структуры. Это алгебра типа $\langle 2, 2\rangle$, основные операции которой связаны на основном множестве тождествами (1), (2), (3), (4). Каждую G-структуру можно рассматривать и как G-упорядоченное множество.