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COMPLETIONS OF LATTICE ORDERED GROUPS

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In the presented paper there is examined the existence of the largest completion of an archimedean lattice ordered group. The investigation was inspired by a question proposed by M. Kolibiar at the Algebraic Winter School (Krpáčová 1980). The methods used below are analogous to those applied in the author's papers [6] and [7] for examining the existence of free complete lattice ordered groups or free complete vector lattices.

1. Preliminaries

For the terminology concerning lattice ordered groups cf. Conrad [2] and Fuchs [3]. Let us recall some notations we shall need in the sequel.

Let H be a lattice ordered group. H is said to be complete if each nonempty upper bounded subset of H possesses the least upper bound in H. If this is the case, then also the corresponding dual condition is valid. An l-subgroup H_1 of H is called closed in H if, whenever X is a subset of H_1 and x_0 is an element of H such that $x_0 = \sup X$ or $x_0 = \inf X$ holds in H, then x_0 belongs to H_1. Let $Y \subseteq H$. The set y is said to generate (c-generate) the lattice ordered group H if for each (closed) l-subgroup H_1 of H with $Y \subseteq H_1$ we have $H_1 = H$.

A lattice ordered group A is called archimedean if for each $0 < a \in A$ and each $b \in A$ there exists a positive integer n such that $na \not\leq b$. The following results are well known: (i) If H is a complete lattice ordered group, then each l-subgroup of H is archimedean. (ii) For each archimedean lattice ordered group A there exists a complete lattice ordered group H such that A is an l-subgroup of H and A c-generates H. (iii) Each archimedean lattice ordered group is abelian.

Let G be an archimedean lattice ordered group. We denote by $C(G)$ the class of all complete lattice ordered groups H such that

(a) G is an l-subgroup of H, and
(b) the set G c-generates H.

The lattice ordered groups belonging to $C(G)$ will be said to be completions of G.

Let G_1, $G_2 \in C(G)$. If there exists an isomorphism φ of G_1 onto G_2 such that $\varphi(g) = g$ for each $g \in G$, then we shall not distinguish G_1 from G_2 and we write $G_1 = G_2$.

2. Quasiorders $\leqq_1$ and $\leqq_2$ on $C(G)$

Unless otherwise stated, G will always denote an archimedean lattice ordered group. Let T_1, $T_2 \in C(G)$. The lattice ordered group T_1 is called the α-largest completion of G if for each $S \in C(G)$ there exists an isomorphism φ of S into T_1 such that $\varphi(g) = g$ for each $g \in G$. The lattice ordered group T_2 is said to be the β-largest completion of G if for each $S \in C(G)$ there exists a homomorphism Ψ of T_2 onto S such that $\Psi(g) = g$ for each $g \in G$.

The above notions are related to the following binary relations $\leqq_1$ and $\leqq_2$ on the class $C(G)$. Let S, $T \in C(G)$. We put $S \leqq_1 T$ if there exists an isomorphism φ of S into T such that $\varphi(g) = g$ for each $g \in G$. Further we put $S \leqq_2 T$ if there exists a homomorphism Ψ of T onto S such that $\Psi(g) = g$ for each $g \in G$. The relations $\leqq_1$ and $\leqq_2$ are obviously quasiorders on the class $C(G)$.

Analogous quasiorders concerning the situation when a partially ordered set is embedded into a lattice have been investigated by M. Kolibiar [9].

Let us illustrate the quasiorder $\leqq_1$ by the following example.

Example 1. Let I be an infinite set and for each $i \in I$ let G_i and H_i be the additive group of all rationals or all reals, respectively; both G_i and H_i are linearly ordered in the natural way. Put

$$G = \Sigma_{i \in I} G_i, \quad H = \Sigma_{i \in I} H_i, \quad H' = \Pi_{i \in I} H_i.$$

Then H and H' belong to $C(G)$, $H \leqq_1 H'$ and H fails to be isomorphic with H'.

The natural question arises: what are the properties of the quasiordered class $(C(G); \leqq_1)$ or $(C(G); \leqq_2)$? In particular, does G always have the α-largest completion or the β-largest completion? The first question seems to be rather difficult. It will be shown below that the answer to the second question is negative.

We need the following result (it follows from the construction applied in the proof of Thm. 4.7 in [6]):

(A) Let M be a set with card $M = \aleph_0$. Let α be a cardinal. There exists a complete lattice ordered group G_α such that G_α is c-generated by the set M and card $G_\alpha \geqq \alpha$.

The complete lattice ordered groups G_α were constructed in [6] by means of complete Boolean algebras B_α having properties analogous to those of G_α (i.e., B_α is c-generated by a denombrable set and card $B_\alpha \geqq \alpha$); the Boolean algebras B_α have been described by Hales [4]. Let us denote by G'_α the l-subgroup of G_α generated by the set M.

Since the class of all lattice ordered groups is a variety, there exists the free lattice ordered group $FLG(\aleph_0)$ with $\aleph_0$ free generators and clearly card $FLG(\aleph_0) = \aleph_0$. If H is a lattice ordered group having a denombrable subset M_1 such that M_1 generates H, then there is an l-ideal K in $FLG(\aleph_0)$ such that H is isomorphic with $FLG(\aleph_0)/K$. From this it follows that the number of non-isomorphic types of lattice ordered groups with $\aleph_0$ generators is less than or equal to $2^{\aleph_0}$.

The above consideration shows that there is a set $\{H_i\}_{i \in I}$ of lattice ordered groups such that 1) for each $i \in I$ there is $M_i \subseteq H_i$ with card $M_i = \aleph_0$ such that M_i generates H_i; 2) for each pair i, j of distinct elements i, j of I, H_i fails to be isomorphic with H_j; 3) if H is a lattice ordered group with $\aleph_0$ generators, then there is $i \in I$ such that H is isomorphic to H_i.

Let I_1 be the class of all $i \in I$ that have the following property: there is a cardinal $\alpha(i)$ such that G'_β fails to be isomorphic to H_i for each cardinal β with $\beta > \alpha(i)$. Suppose that $I_1 = I$. Then there is a cardinal α_0 with $\alpha_0 > \alpha(i)$ for each $i \in I$; thus for each $i \in I$, G'_{α_0} fails to be isomorphic with H_i, which contradicts 3). Thus there is $i_0 \in I \setminus I_1$; denote $G^0 = H_{i_0}$. Hence we have

2.1. Lemma. *For each cardinal α there is a cardinal β with $\beta > \alpha$ such that G^0 is isomorphic to G'_β.*

2.2. Theorem. *There exists an archimedean lattice ordered group G such that* (i) *G has neither the α-largest completion nor the β-largest completion, and* (ii) *$C(G)$ is a proper class.*

Proof. Put $G = G^0$. Suppose that T_1 is the α-largest completion of G. Denote card $T_1 = \alpha$. According to 2.1 there exists $\beta > \alpha$ such that G^0 is isomorphic with G'_β; thus without loss of generality we can assume that $G^0 = G'_\beta$. Hence G_β is a completion of G^0. Since card $G_\beta \geqq \beta$, there does not exist any isomorphism of G_β into T_1, which is a contradiction.

Next suppose that T_2 is the β-largest completion of G, card $T_2 = \alpha$. Let β be as in the previous consideration. From card $G_\beta \geqq \beta > \alpha$ it follows that there cannot exist any homomorphism of T_2 onto G_β, which is a contradiction.

The assertion (ii) is an immediate consequence of 2.1.

Let $C_1(G)$ be the partially ordered class that we obtain from the quasiordered class $(C(G), \leqq_1)$ by identifying each pair of elements $G_1, G_2 \in C(G)$ which fulfil the relations $G_1 \leqq_1 G_2$, $G_2 \leqq_1 G_1$. Further let $C_2(G)$ be defined analogously. The question whether there must exist maximal elements in $C_1(G)$ or in $C_2(G)$ remains open.

A nonempty subclass A of $C(G)$ is said to be an antichain in $(C(G); \leqq_1)$ if for any pair of distinct elements $G_1, G_2 \in A$ we have neither $G_1 \leqq_1 G_2$ nor $G_2 \leqq_1 G_1$.

2.3. Proposition. *Let α be an infinite cardinal. There exists an archimedean lattice ordered group G' such that there is an antichain A in $(C(G'), \leqq_1)$ with* card $A = \alpha$.

Proof. Let J be a set, card $J=\alpha$. Let G, H and H' be as in the Example 1. For each $j\in J$ let $G'_j=G$; further we put $G'=\Pi_{j\in J}G'_j$. Let k be a fixed element of J. We set $G^0_k=\Pi_{j\in J}G_{kj}$, where $G_{kj}=H$ for $j=k$ and $G_{kj}=H'$ otherwise. Then $G^0_k\in C(G')$ for each $k\in J$ and $A=\{G^0_k\}_{k\in J}$ is an antichain in $(C(G'), \leqq_1)$ with card $A=\alpha$.

3. Completions of linearly ordered groups

In this paragraph it will be shown that for each archimedean linearly ordered group G, $C(G)$ is a one-element set.

An l-subgroup G_1 of a lattice ordered group G_2 will be said to be an rl-subgroup of G_2 if whenever $X\subseteq G_1$, $x_0\in G_1$ and x_0 is the join of the set X in G_1, then x_0 is also the join of the set X in G_2. (This is equivalent to the corresponding dual condition.)

Let us recall the following definition:

3.1. Definition. *The complete lattice ordered group K is called a Dedekind completion of the lattice ordered group G if the following conditions hold:*

(i) *G is an l-subgroup of K.*

(ii) *For each $k\in K$ there are subsets X, Y of G such that* $\sup X=k=\inf Y$ *holds in K.*

It is easy to verify that if G possesses a Dedekind completion, then this is determined uniquely up to isomorphisms. It will be shown below that if K is the Dedekind completion of G, then G is an rl-subgroup of K (cf. 5.2.1). The following theorem has been proved by Clifford (cf. Fuchs [3]):

(B) Let G be an archimedean lattice ordered group. Then G possesses a Dedekind completion.

The Dedekind completion of an archimedean lattice ordered group G will be denoted by $d(G)$.

Let us denote by R the additive group of all reals with the natural linear order. Then we have (cf. [3], Chap. IV, Thm. 1 (Hölder)):

(C) Each archimedean linearly ordered group is isomorphic to an l-subgroup of R.

3.2. Lemma. *Let G be an l-subgroup of a lattice ordered group K. Assume that G is linearly ordered. Let G' be the convex l-subgroup of K generated by G. Then G' is a closed l-subgroup of K.*

Proof. The case $G=\{0\}$ being trivial we can assume that $G\neq\{0\}$. Choose $0<g\in G$. It is easy to verify that G' is the set of all $k\in K$ such that $|k|\leqq|g(k)|$ for some $g(k)\in G$. Let $\emptyset\neq X\subseteq G$, $k\in K$ and suppose that $\sup X=k$ holds in K. If X is upper-bounded in G, then clearly $k\in G'$. *If* X fails to be upper-bounded in G, then we have in K the relation

$$k = \sup X = \sup G = \sup (G - g) = \sup G - g = k - g < k,$$

which is a contradiction.

Let G, K and G' be as in 3.2. For each $k \in G'$ denote $I(k) = \{g \in G, g \leqq k\}$. Let G_1 be the set of all elements $k \in G'$ such that $k = \sup I(k)$ holds in K. Clearly $k \in K$ belongs to G_1 if and only if there exists a nonempty subset X of G such that (i) X is an upper bounded subset of G and (ii) $\sup X = k$ holds in K.

3.2. Lemma. *G_1 is the closed l-subgroup of K generated by G, and G_1 is linearly ordered.*

Proof. From the definition of G_1 it follows that G_1 is linearly ordered. Let $k_1, k_2 \in G_1$. There are nonempty subsets X_1, X_2 of G that are upper-bounded in G such that the relations $k_1 = \sup X_1$ and $k_2 = \sup X_2$ hold in K. Then $X_1 + X_2$ is an upper-bounded subset of G and $k_1 + k_2 = \sup (X_1 + X_2)$ is valid in K; thus G_1 is closed with respect to the group operation. Hence G_1 is an l-subgroup of K, $G \subseteq G_1$. Let $\{k_i\}_{i \in I}$ be a nonempty subset of G_1, $k \in K$ and suppose that $k = \bigvee_{i \in I} k_i$ holds in K. Then in view of 3.2 we have $k \in G'$, hence there is $g \in G$ with $k \leqq g$. For each $i \in I$ there exists a nonempty subset X_i of G such that $\sup X_i = k_i$ is valid in K. Therefore $X = \bigcup_{i \in I} X_i$ is a subset of G that is upper bounded by the element $g \in G$, and $\sup X = k$ holds in K. Hence $k \in G_1$, completing the proof.

By a dual argument we can verify that for each $k \in G_1$ there exists a nonempty subset Y of G such that $k = \inf Y$ is valid in K.

3.4. Proposition. *Let G be an archimedean linearly ordered group. Then $C(G) = \{d(G)\}$.*

Proof. Since G is archimedean, $d(G)$ exists by Theorem (B), and $d(G)$ belongs to $C(G)$ according to the definition 3.1. Let $K \in C(G)$ and let G_1 be as in 3.3. Since K is complete, it follows from 3.3 that G_1 is c-generated by G. From this and from $K \in C(G)$ we obtain $K = G_1$. In view of the definition of G_1 and with respect to 3.3 we infer that the condition (ii) from 3.1 is valid. Therefore $K = d(G)$.

A generalization of this result will be given below (cf. 4.8).

4. Completions of direct products

Let G be a lattice ordered group and let $X \subseteq G$. Put

$$X^\delta = X^{\delta(G)} = \{g \in G: |x| \wedge |g| = 0 \quad \text{for each} \quad x \in X\}.$$

The following results are well known:

(C) (Cf. Šik [10]) X^δ is a closed convex l-subgroup of G.

(D) (Riesz; cf. [3]) If G is complete, then X^δ is a direct factor of G.

Now let G be an archimedean lattice ordered group, $H \in C(G)$, $G = A \times B$. Let A_1 be the set of all elements $h \in H$ such that there exists a subset $X \subseteq A$ having the property that $\sup X = h$ holds in H. Next let A' be the convex l-subgroup of H generated by the set A_1. Analogously we define B_1 and B'. It is a routine to verify that A' can be characterized as the set of all $h \in H$ that fulfil the following condition:

(c) There are sets $X \subseteq A^+$ and $Y \subseteq A^-$ such that (i) $\sup X$ and $\inf Y$ do exist in H, and (ii) $\inf Y \leqq h \leqq \sup X$.

Let $Z \subset A'$, $h \in H$ and suppose that $\sup_H Z = h$. Put $X = \{z \vee 0 : z \in Z\}$, $Y = \{z \wedge 0 : z \in Z\}$. Then $\sup_H X = h \vee 0$ and $\sup_H Y = h \wedge 0$, $X \subseteq A'$, $Y \subseteq A'$. From the convexity of A' in H it follows that $h \wedge 0 \in A'$. For each $x \in X$ there exists a set $P(x) \subseteq A$ with $\sup_H P(x) = x$. Put $X_1 = \bigcup_{x \in X} P(x)$. Hence $\sup_H X = \sup_H X_1 = h \vee 0$, yielding $h \vee 0 \in A'$. By using again the convexity of A' we obtain $h \in A'$. Thus A' is a closed l-subgroup of H. Clearly $A \subseteq A'$. Similarly, B' is a closed l-subgroup of H and $B \subseteq B'$. In view of $G = A \times B$ we have $|a| \wedge |b| = 0$ for each $a \in A$ and each $b \in B$, hence according to (c) we infer that $|a'| \wedge |b'| = 0$ is valid for each $a' \in A'$ and each $b' \in B'$. Thus $a' + b' = b' + a'$ for each $a' \in A'$ and each $b' \in B'$; moreover, $A' \cap B' = \{0\}$. Thus $A' + B' = A' \times B'$ and $A' \times B'$ is a closed convex l-subgroup of H. Since $G \subseteq A' \times B'$, we must have $H = A' \times B'$.

If $0 < a' \in A'$, $b \in B$, then from (c) it follows that $a' \wedge |b| = 0$, hence $A' \subseteq B^{\delta(H)}$. Let $0 \leqq y \in B^{\delta(H)}$. We denote by $y(A')$ and $y(B')$ the component of y in the direct factor A' or B', respectively. Then $y = y(A') + y(B') = y(A') \vee y(B')$. From $y \in B^{\delta(H)}$ and from (c) (applied for B') we obtain $y(B') = 0$, hence $y = y(A') \in A'$. Thus $A' = B^{\delta(H)}$. Similarly, $B' = A^{\delta(H)}$. In view of $H = A' \times B'$ this yields $A' = (B')^{\delta(H)} = A^{\delta(H)\delta(H)}$.

Let A_0 be the closed l-subgroup of H generated by A and let B_0 be defined analogously. Put $C_0 = A_0 + B_0$. Then $C_0 = A_0 \times B_0$ and C_0 is a closed l-subgroup of H, $G \subseteq C_0$. Hence $C_0 = H$. If we have either $A_0 \neq A'$ or $B_0 \neq B'$, then in view of $H = A' \times B'$ we would have $C_0 \subset H$, which is a contradiction. Hence $A_0 = A'$ and $B_0 = B'$.

By summarizing, we obtain

4.1. Proposition. *Let G be an archimedean lattice ordered group, $G = A \times B$, $H \in C(G)$. Then $H = A^{\delta(H)\delta(H)} \times B^{\delta(H)\delta(H)}$. Moreover, $A^{\delta(H)\delta(H)} \in C(A)$ and $B^{\delta(H)\delta(H)} \in C(B)$.*

The following assertion is easy to verify:

4.2. Lemma. *Let G, A, B, H be as in 4.1 and let $g \in G$. Then $g(A) = g(A^{\delta(H)\delta(H)})$.*

By standard induction steps we get from 4.1:

4.3. Theorem. *Let G be an archimedean lattice ordered group, $G = A_1 \times A_2 \times \ldots \times A_n$, $H \in C(G)$. Then $H = A_1^{\delta(H)\delta(H)} \times A_2^{\delta(H)\delta(H)} \times \ldots \times A_n^{\delta(H)\delta(H)}$. Moreover, $A_i^{\delta(H)\delta(H)} \in C(A_i)$ holds for $i = 1, 2, \ldots, n$.*

The following example shows that this theorem cannot be generalized for direct decompositions having infinitely many direct factors.

Example 2. Let G, H and H' be as in Example 1. Let J be an infinite set and for each $j \in J$ let $G'_j = H'$, $G^0_j = G$. Put $K_1 = \Pi_{j \in J} G'_j$ and let K be the set of all $k \in K_1$ having the property that the set $\{j \in J: k(G'_j) \notin H\}$ is finite. Further let $G_0 = \Pi_{j \in J} G^J_0$. Then $K \in C(G_0)$, but K cannot be expressed as a direct product $\Pi_{j \in J}(G^j_0)^{\delta(K)\delta(K)}$.

Nevertheless, from 4.3 we obtain the following

4.4. Corollary. *Let G be an archimedean lattice ordered group, $G = \Pi_{j \in J} A_j$, $H \in C(G)$. Then*

(i) *each $A_j^{\delta(H)\delta(H)}$ is a direct factor of H;*

(ii) *if $i, j \in J$, $i \neq j$, then $A_i^{\delta(H)\delta(H)} \cap A_j^{\delta(H)\delta(H)} = \{0\}$;*

(iii) *for each $j \in J$, $A_j^{\delta(H)\delta(H)} \in C(A_j)$.*

For an archimedean lattice ordered group G we denote by $C_b(G)$ the class of all $H \in C(G)$ which have the property that for each $h \in H$ there exists $g \in G$ with $h \leqq g$ (in other words, the convex l-subgroup of H generated by G coincides with H). Clearly $d(G) \in C_b(G)$.

4.5. Lemma. *Let G, A_j ($j \in J$) and H be as in 4.4. If $H \in C_b(G)$, then $A_j^{\delta(H)\delta(H)} \in C_b(A_j)$ for each $j \in J$.*

Proof. Assume that $H \in C_b(G)$. Denote $B_j = A_j^{\delta(H)\delta(H)}$. Let $0 \leqq b_j \in B_j$. Then $b_j \in H$, hence there is $g \in G$ with $b_j \leqq g$. In view of 4.2, $b_j = b_j(B_j) \leqq g(B_j) = g(A_j) \in A_j$, thus $B_j \in C_b(A_j)$.

4.6. Theorem. *Let G be an archimedean lattice ordered group, $G = \Pi_{j \in J} A_j$, $H \in C(G)$. Assume that $A_j^{\delta(H)\delta(H)}$ belongs to $C_b(A_j)$ for each $j \in J$. Then $H = \Pi_{j \in J} A_j^{\delta(H)\delta(H)}$.*

Proof. In view of 4.4 it suffices to verify that the following conditions are valid (we use the denotation $B_j = A_j^{\delta(H)\delta(H)}$ as above):

a) if $0 \leqq b_j \in B_j$ for each $j \in J$, then $\sup_H \{b_j\}_{j \in J}$ does exist;

b) if $0 \leqq h \in H$, then $h = \bigvee_{j \in J} h(B_j)$.

Let $0 \leqq b_j \in B_j$ ($j \in J$). Because of $B_j \in C_b(A_j)$ there exist elements $a_j \in A_j$ with $0 \leqq b_j \leqq a_j$ for each $j \in J$. Further there exists $g \in G$ such that $g(A_j) = a_j$ holds for each $j \in J$. In view of $g(A_j) = g(B_j)$ (cf. 4.2) we have $b_j \leqq g$ for all $j \in J$. Hence there exists $\sup_H \{b_j\}_{j \in J}$; thus a) is valid.

Let $0 \leqq h \in H$. Put $h(B_j) = b_j$ for each $j \in J$ and let g be as in a). Then in H we have $g = \bigvee_{j \in J} a_j$, hence

$$h = h \wedge g = \bigvee_{j \in J} (h \wedge a_j).$$

Since $h \wedge a_j \in B_j$ and $h \wedge a_j \leqq h$, we get $h \wedge a_j \leqq h(B_j)$. Therefore $h = \bigvee_{j \in J} h(B_j)$.

4.7. Corollary. *Let G, A_j ($j \in J$) and Z be as in 4.4. Then the following conditions are equivalent:* (i) $H \in C_b(G)$; (ii) *for each $j \in J$, $A_j^{\delta(H)\delta(H)}$ belongs to $C_b(A_j)$.*

Proof. (i) implies (ii) by 4.5. Let (ii) be valid. Then from the condition b) in 4.6 we obtain that (i) holds.

4.8. Corollary. *Let G be an archimedean lattice ordered group, $G = \Pi_{j \in J} A_j$. Assume that all A_j are linearly ordered. Let $H \in C(G)$. Then $H = \Pi_{j \in J} d(A_j)$.*

Proof. This follows from 3.4, 4.4 and 4.6.

Now let us consider the question what the structure of H is if G and H are as in 4.4 and if we do not assume that $H \in C_b(G)$.

4.9. Lemma. *Let G and H be as in 4.4, $B_j = A_j^{\delta(H)\delta(H)}$ and let $0 < h \in H$. Then there exists $j \in J$ with $h(B_j) > 0$.*

Proof. Suppose that $h(B_j) = 0$ for each $j \in J$ (under the denotation as above). Then for each $j \in J$ and each $0 \leqslant b_j \in B_j$ we have $h \wedge b_j = 0$. Let $0 \leqq g \in G$. Since $g = \bigvee_{j \in J} g(A_j) = \bigvee_{j \in J} g(B_j)$, we get $h \wedge g = 0$, hence $G \subseteq \{h\}^{\delta(H)} \subset H$. Since $\{h\}^{\delta(H)}$ is a closed l-subgroup of H, we have a contradiction.

4.10. Lemma. *Let G and H be as in 4.4, $B_j = A_j^{\delta(H)\delta(H)}$ and let $0 \leqq h \in H$. Then $h = \bigvee_{j \in J} h(B_j)$.*

Proof. Since H is complete, there exists $h_1 = \bigvee_{j \in J} h(B_j)$ in H and $h_1 \leqq h$. If $j \in J$ and $(h - h_1)(B_j) > 0$, then $h(B_j) < h(B_j) + (h - h_1)(B_j) \leqq h_1 + (h - h_1) =$ h and $h(B_j) + (h - h_1)(B_j) \in B_j$, which is impossible. Therefore $(h - h_1)(B_j) = 0$ for each $j \in J$. Thus in view of 4.9, $h = h_1$.

The notion of the completely subdirect product of lattice ordered groups has been introduced by Šik [11]; cf. also [8], §3.

From 4.4, 4.9 and 4.10 we obtain:

4.11. Theorem. *Let G be an archimedean lattice ordered group, $G = \Pi_{j \in J} A_j$, $H \in C(G)$. Then H is a completely subdirect product of lattice ordered groups $A_j^{\delta(H)\delta(H)}$ ($j \in J$).*

4.12. Corollary. *Let G be an archimedean lattice ordered group, $G = \Pi_{j \in J} A_j$. If the class $C(A_j)$ is a set for each $j \in J$, then $C(G)$ is a set as well.*

4.13. Proposition. *Let G be an archimedean lattice ordered group, $G = \Pi_{j \in J} A_j$. For each $j \in J$, let $B_j \in C(A_j)$, $B = \Pi_{j \in J} B_j$. Then $B \in C(G)$.*

Proof. Since B is a direct product of complete lattice ordered groups, B is complete as well. G is an l-subgroup of B. For each $j \in J$, B_j is the closed l-subgroup of B generated by A_j. Let C be the closed l-subgroup of B generated by G. Then $B_j \subseteq C$ for each $j \in J$. Let $0 \leqq b \in B$. We have $b = \bigvee_{j \in J} b(B_j)$ and $b(B_j) \in C$ for each $j \in J$, whence $b \in C$; thus $B^+ \subseteq C$. From this it follows that $B = C$. Therefore $B \in C(G)$.

4.14. Corollary. *Let G be an archimedean lattice ordered group, $G=\Pi_{j\in J}A_j$, $H\in C(G)$. Then $\Pi_{j\in J}A_j^{\delta(H)\delta(H)}$ belongs to $C(G)$.*

This follows from 4.1 and 4.13.

5. The class $C_0(G)$

Let G be an archimedean lattice ordered group. We denote by $C_0(G)$ the class of all completions H of G such that each element $h\in H$ with $h\geqq 0$ is a join of a subset of G. This class $C_0(G)$ is nonempty, since the Dedekind completion $d(G)$ belongs to $C_0(G)$. The class $C_0(G)$ need not coincide with $C(G)$ (this is a consequence of 2.2).

A subset $\{x_i\}_{i\in I}$ of G is said to be disjoint if $x_i\geqq 0$ for each $i\in I$ and $x_i\wedge x_j=0$ for each pair of distinct elements $i, j\in I$. The lattice ordered group G is said to be laterally complete if each disjoint subset of G possesses the join in G.

We shall apply the following result (cf. [5]):

(D) Let K be a complete lattice ordered group. There exists a complete lattice ordered group K_1 such that

(i) K_1 is laterally complete;

(ii) K is a convex l-subgroup of K_1;

(iii) for each $0<k_1\in K_1$ there exists a disjoint subset X of K such that $\sup X=k_1$ holds in K_1.

It is easy to verify that K_1 is defined uniquely up to isomorphism. K_1 is said to be the lateral completion of K and we write $K_1=l(K)$. Clearly $K_1\in C(K)$. (Lateral completions of lattice ordered groups that are not assumed to be complete have been investigated by several authors, e.g. [1].)

In this paragraph it will be shown that for each $H\in C_0(G)$ the relation

$$d(G)\leqq_1 H\leqq_1 l(d(G))$$

is valid.

The following lemma shows that the definition of $C_0(G)$ is in a certain sense self-dual.

5.1. Lemma. *Let $H\in C_0(G)$, $0>h\in H$. Then there is a subset $S\subset G$ such that $h=\inf S$ holds in H.*

Proof. There is $0\leqq g\in G$. We have $0<g-h$, hence there exists $S_1\subset G$ with $\sup_H S_1=g-h$. Thus $\inf_H(-S_1)=h-g$ and this yields $\inf_H(-S_1+g)=h$. Clearly $-S_1+g\subset G$.

5.2. Lemma. *Let $H\in C_0(G)$. Then G is an rl-subgroup of H.*

Proof. It is easy to verify that G is regular with respect to joins if and only if it is regular with respect to meets. Suppose that G fails to be regular in H. Then there is $S_1\subset G$ such that

$$h_1 = \inf_H S_1 > g = \inf_G S_1 .$$

Put $S_1 - g = S$, $h = h_1 - g$. Then $0 = \inf_G S < h = \inf_H S$. There exists $T \subset G$ with $\sup_H T = h$. Since $h > 0$, there is $t \in T$ with $t \not\leqq 0$. Thus $t_1 = t \vee 0 > 0$, $t_1 \leqq h$, $t_1 \in G$. Further we have $t_1 \leqq s$ for each $s \in S$, hence $\inf_G S \neq 0$, which is a contradiction.

5.2.1. Corollary. *G is an rl-subgroup of $d(G)$.*

5.3. Lemma. *Let $H \in C_0(G)$. Let G_1 be the convex l-subgroup of H generated by G. Then $G_1 = d(G)$.*

Proof. G_1 is a complete lattice ordered group and G is an l-subgroup of G_1. Let $x \in G_1$. There exists $g \in G$ with $-(0 \wedge x) \leqq g$. Since $-(0 \wedge x) = (x \vee 0) - x$, we have $x + g \geqq x \vee 0 \geqq 0$. Thus there is $S \subseteq G$ with $\sup_H S = x + g$. Therefore $\sup_H S_1 = x$, where $S_1 = S - g \subseteq G$. Analogously we can verify that there exists $S_2 \subseteq G$ with $\inf_H S_2 = x$. Clearly $\sup_H S_1 = \sup_{G_1} S_1$ and $\inf_H S_2 = \inf_{G_1} S_2$. Therefore $G_1 = d(G)$.

5.4. Corollary. *Let $H \in C_0(G)$. Then $d(G) \leqq_1 H$.*

5.5. Proposition. *Let G be an archimedean lattice ordered group. Let $H \in C_0(G)$, $0 \leqq h \in H$. Then there exists a disjoint system S of elements of $d(G)$ such that $\sup_H S = h$.*

Before proving 5.5 we need some auxiliary results. In 5.6—5.9 we assume that H is a complete lattice ordered group. For $X \subset H$ we denote

$$[X] = X^{\delta(H)\delta(H)} .$$

For $x \in H$ we write $[x]$ instead of $[\{x\}]$. Let $0 \leqq x \in H$. For each $0 \leqq y \in [x]$ we have $y = \bigvee (nx \wedge y)$ $(n = 1, 2, \ldots)$ (cf. e.g., [12]; in [12] vector lattices are investigated, but the proof remains valid for complete lattice ordered groups as well). From this it follows that

$$z[x] = \bigvee_n (nx \wedge z[x]) = \bigvee_n (nx[x] \wedge z[x]) = \bigvee_n ((nx \wedge z)[x]) = \bigvee_n (nx \wedge z)$$

is valid for each $0 \leqq z \in H$.

5.6. Lemma. *Let I be a nonempty set, $0 \leqq x_i \in H$ for each $i \in I$, $\sup_H x_i = x$, $0 \leqq y \in H$. Then $y[x] = \bigvee_{i \in I} y[x_i]$.*

Proof. We have

$$y[x] = \bigvee_n (y \wedge nx) = \bigvee_n (y \wedge n \bigvee_{i \in I} x_i) = \bigvee_n \bigvee_{i \in I} (y \wedge nx_i) =$$
$$= \bigvee_{i \in I} \bigvee_n (y \wedge nx_i) = \bigvee_{i \in I} y[x_i] .$$

5.7. Lemma. *Let $0 < a \in H$ and let $0(a)$ be the set of all elements $a_i \in H$ having the property that there exists $a'_i \in H$ with $a_i \wedge a'_i = 0$, $a_i \vee a'_i = a$. Then* (i) *$0(a)$ is a Boolean algebra, and* (ii) *$0(a)$ is a closed sublattice of H.*

The first assertion follows immediately from the definition of $0(a)$. The second assertion is a consequence of the infinite distributivity of H.

The following further properties of the elements of $0(a)$ are easy to verify: Let a_i, $a_j \in 0(a)$, $c \in H$. The we have

$$a_i[a_j] = a_i \wedge a_j, \quad a_i[c] \in 0(a), \quad (c[a_i])[a_j] = c[a_i \wedge a_j].$$

Now let $0 < a \in H$, $0 < b \in H$. Put

$$A^0 = \{a_i \in 0(a) : a_i = 0 \text{ or } b[a_i] > 0\}, \quad a^0 = \sup_H A^0.$$

For each positive integer n we denote

$$A_n^0 = \{a_i \in A^0 : na_i \geqq b[a_i]\}, \quad a_n^0 = \sup_H A_n^0.$$

Then from 5.7 we obtain that a_n^0 belongs to $0(a)$ for $n = 1, 2, \ldots$. We also have $a_1^0 \leqq a_2^0 \leqq \ldots \leqq a$. For each positive integer n we put $a_n^1 = a^0 - a_n^0$. This yields $a_n^1 \wedge a_n^0 = 0$ for $n = 1, 2, \ldots$

5.8. Lemma. *Let* $0 < a_i \in 0(a)$, $a_i \leqq a_n^1$. *Then* $na_i < b[a_i]$.

Proof. If $b[a_i] - na_i = 0$, then $a_i \leqq a_n^0$, which is impossible. Thus $b[a_i] - na_i \neq 0$. Suppose that $b[a_i] - na_i \ngtr 0$. Hence $(b[a_i] - na_i)^- = z > 0$. Clearly $z \in [a_i]$. Hence $a_i[z] = a_j \in 0(a)$ and $[z] = [a_j]$. Thus $0 < a_j \leqq a_i \leqq a_n^1$ and $(b[a_i] - na_i)^+ \wedge z = 0$. From this it follows that

$$0 \leqq (b[a_j] - na_j)^+ = (b[a_j] - na[a_j])^+ = (b - na)^+[a_j] =$$
$$= ((b - na)^+[a_i])[a_j] = (b[a_i] - na_i)^+[a_j] = (b[a_i] - na_i)^+[z] = 0,$$

whence $(b[a_j] - na_j)^+ = 0$, implying $b[a_j] \leqq na_j$. Hence $a_j \leqq a_n^0$, $a_n^0 \wedge a_n^1 > 0$, which is a contradiction.

5.9. Lemma. $\bigwedge_n a_n^1 = 0$.

Proof. By way of contradiction, assume that there is $0 < x \in H$ such that $x \leqq a_n^1$ holds for $i = 1, 2, \ldots$ Then according to 5.8 we have $nx \leqq b[a_n^1] \leqq b$ for each positive integer n. This is impossible, because H is archimedean.

Put $a_1 = a_1^n$, and define by induction $a_n = a_n^0 - a_{n-1}$ for each $n > 1$. Then we have

(2) $$a_n^0 = a_1 \vee a_2 \vee \ldots \vee a_n$$

for each positive integer n, and

(3) $$a_n \wedge a_m = 0$$

for each pair of distinct positive integers n, m. Moreover, for each $0 < a_i \leqq a_n$ we have $na_i \geqq b[a_i]$ and $(n-1)a_i < b[a_i]$. Put $a_0' = \bigvee_n a_n^0$. Then $a_0' \in 0(a)$ and clearly $a_0' \leqq a_0$. If we had $a_0' < a_0$, then there would be $a_i \in 0(a)$ with $0 < a_i \leqq a_0 - a_0'$ and hence $a_i \leqq a_n^1$ for $n=1, 2, \ldots$, contradicting 5.9. Hence $\bigvee_n a_n^0 = a_0$. From this and from (2) we obtain

(4) $$\bigvee_n a_n = a_0 .$$

From the definition of a_0 we get $b[a] = a_0$, hence according to (4) and in view of 5 6

(5) $$b[a] = b[a_0] = \bigvee_n b[a_n] .$$

In view of (3) the system $\{b[a_n]\}_{n=1,2,\ldots}$ is disjoint; moreover,

(6) $$b[a_n] \leqq na_n \qquad (n=1, 2, \ldots).$$

Proof of Proposition 5.5:

Let $0 < h \in H$. There exists a subset $X \subset G^+$ such that $\sup X = h$ is valid in H. From this we infer by using the Axiom of Choice that there exists a disjoint subset $\{y_j\}_{j \in J}$ of strictly positive elements of $[X]$ such that (i) $y_j \in G$ for each $j \in J$, and (ii) if $z \in [X]^+$, $z \wedge y_j = 0$ for each $j \in J$, then $z = 0$. Thus $[X] = [\{y_j\}_{j \in J}]$. Because of $h \in [X]$ we have also $h \in [\{y_j\}_{j \in J}] = \bigvee_{j \in J}[y_j]$, hence $h = \bigvee_{j \in J} h[y_j]$.

Let $j \in J$ be fixed. Put $y_j = a$, $h[y_j] = b$ and let us write a_{nj} instead of a_n. Then according to (5) and (6)

$$h[y_j] = \bigvee_n (h[y_j])[a_{nj}], \; (h[y_j])[a_{nj}] \leqq na_{nj} \qquad (n=1, 2, \ldots),$$

hence by 5.2 and 5.3 $(h[y_j])[a_{nj}] \in d(G)$. Further we have

$$h = \bigvee_{j \in J} \bigvee_n (h[y_j])[a_{nj}]$$

and the system $\{(h[y_j])[a_{nj}]\}$ $(j \in J, n = 1, 2, \ldots)$ is disjoint. The proof is complete.

In 5.10—5.16 we assume that K and H are complete lattice ordered groups such that (i) K is a convex l-subgroup of H, and (ii) for each $0 < h \in H$ there exists a disjoint subset X in K with $\sup_H X = h$.

Let us denote $H' = l(K)$. For distinguishing the lattice operations in H and in H' we shall denote the lattice operations in H' by $\wedge'$ and $\vee'$, while $\wedge$, $\vee$ are lattice operations in H. (If $x, y \in K$, then $x \wedge y = x \wedge' y$ and $x \vee y = x \vee' y$.)

Suppose that both $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are disjoint subsets of K.

5.10. Lemma. *Assume that* $h' \in H'$, $h' = \bigvee'_{i \in I} a_i$ *and that* $\bigvee_{i \in I} a_i = \bigvee_{j \in J} b_j$. *Then* $h' = \bigvee'_{j \in J} b_j$.

Proof. From $\bigvee_{i \in I} a_i = \bigvee_{j \in J} b_j$ it follows that

(7) $$a_i = \bigvee_{j \in J}(a_i \wedge b_j) \text{ for each } i \in I,$$

(8) $$b_j = \bigvee_{i \in I}(b_j \wedge a_i) \text{ for each } j \in J.$$

Since K is a convex l-subgroup of H, we obtain

(7′) $$a_i = \bigvee\nolimits'_{j\in J}(a_i \wedge b_j) \text{ for each } i \in I,$$

(8′) $$b_j = \bigvee\nolimits'_{i\in I}(b_j \wedge a_i) \text{ for each } j \in J.$$

According to (8′), $b_j \leqq h'$ is valid for each $j \in J$, hence (in view of the lateral completeness of H') there is $h_1 \in H'$ with $h_1 \leqq h'$ such that

(9) $$h_1 = \bigvee\nolimits'_{j\in J} b_j .$$

From (9) and (7′) we get $h' \leqq h_1$, thus $h_1 = h'$.

If $\{a_i\}_{i\in I}$ are as in 5.10 and if $h = \bigvee_{i\in I} a_i$, then we put $\varphi(h) = \bigvee'_{i\in I} a_i$. From 5.10 it follows that φ is a correctly defined mapping of the set H^+ into $(H')^+$.

5.11. Lemma. *Let $h, h_1 \in H^+$, $h = \bigvee_{i\in I} a_i$, $h_1 = \bigvee_{j\in J} b_j$. Then $h_1 \leqq h \Leftrightarrow \varphi(h_1) \leqq \varphi(h)$.*

Proof. We have

$$h_1 \leqq h \Leftrightarrow (8') \Leftrightarrow (8) \Leftrightarrow \varphi(h_1) \leqq \varphi(h).$$

5.12. Corollary. *φ is a monomorphism, and $\varphi(H^+)$ is an upper directed subset of H'.*

5.13. Lemma. *$\varphi(H^+)$ is a convex sublattice of H'.*

Proof. It is obvious that 0 is the least element of $\varphi(H^+)$. Hence in view of 5.12 it suffices to verify that if $p \in \varphi(H^+)$ and $p_1 \in H'$, $0 \leqq p_1 \leqq p$, then $p \in \varphi(H^+)$. Assume that $h \in H^+$, $\varphi(h) = p$, $h = \bigvee_{i\in I} a_i$. Let $0 \leqq p_1 \leqq p$. Then there is a disjoint subset $\{b_j\}_{j\in J}$ in K with $p_1 = \bigvee'_{j\in J} b_j$. In view of $p = \bigvee'_{i\in I} a_i$ the relation (8′) holds, thus $\{b_j\}_{j\in J}$ is upper bounded in H; hence there exists $h_1 \in H$ such that (9) is valid. Therefore $p_1 = \varphi(h_1) \in \varphi(H^+)$.

Clearly $\varphi(k) = k$ for each $k \in K^+$.

5.14. Lemma. *Let $X \subseteq K^+$, $\sup_H X = h$. Then $\sup_{H'} \varphi(X) = \varphi(h)$.*

Proof. According to the assumption there exists a disjoint subset X_1 of K such that $\sup_H X_1 = h$. Then $\varphi(h) = \sup_{H'} X_1$. Since $x \leqq h$ and $\varphi(x) = x$ for each $x \in X$, in view of 5.11 we have $x \leqq \varphi(h)$ for each $x \in X$. Thus there exists $\sup_{H'} X = q$ and $q \leqq \varphi(h)$. From 5.13 it follows that there exists $h_1 \in H^+$ with $\varphi(h_1) = q$. By using 5.11 again we get $h_1 \leqq h$; moreover, from the fact that $\varphi(x) = x \leqq \varphi(h_1)$ we infer that $x \leqq h_1$ holds for each $x \in X$, yielding $h \leqq h_1$. Thus $h = h_1$, completing the proof.

5.15. Lemma. *Let $h, h_1 \in H^+$. Then $\varphi(h + h_1) = \varphi(h) + \varphi(h_1)$.*

Proof. There are sets $X, X_1 \subseteq K^+$ with $\sup_H X = h$, $\sup_H X_1 = h_1$. Then we have

$$\sup_H (X + X_1) = h + h_1 .$$

In view of 5.14 we obtain

$$\varphi(h + h_1) = \sup_{H'} (X + X_1) = \sup_{H'} X + \sup_{H'} X_1 = \varphi(h) + \varphi(h_1).$$

We have proved that $\varphi(H^+)$ is a convex sublattice and a subsemigroup of $(H')^+$ isomorphic with H^+. From this there follows by routine calculations the

5.16. Corollary. *$\varphi(H^+) - \varphi(H^+)$ is a convex l-subgroup of H' isomorphic with H.*

5.17. Theorem. *Let G be an archimedean lattice ordered group and let $H \in C_0(G)$. Then there exists an isomorphism φ of H into $l(d(G))$ such that* (i) *$\varphi(x) = x$ for each $x \in d(G)$, and* (ii) *$\varphi(H)$ is a convex l-subgroup of $l(d(G))$.*

Proof. This is a consequence of 5.3, 5.5 and 5.16.

5.18. Corollary. *Let G be an archimedean lattice ordered group. Then $l(d(G)) \in C_0(G)$ and $H \leqq_1 l(d(G))$ is valid for each $H \in C_0(G)$.*

If H is a convex l-subgroup of $l(d(G))$ with $G \subseteq H$, then obviously $H \in C_0(G)$. Hence from 5.17 we obtain (in view of identifying certain elements of $C(G)$, cf. the end of § 1):

5.19. Corollary. *Let G be an archimedean lattice ordered group. Then $C_0(G)$ is the set of all convex l-subgroups H of $l(d(G))$ having the property that $G \subseteq H$.*

Our concluding remark concerns the question in what way we can search to generalize the above consideration for lattice ordered groups that need not be archimedean. For a lattice ordered group H we denote by H_D the extension of H described in [3], Chap. V, § 10. (The construction of H_D is due to C. J. Everett.) If H is archimedean, then the following conditions are equivalent: (a) $H_D = H$; (b) H is complete. Let G be a lattice ordered group (here we do not assume that G is archimedean). Let $C_1(G)$ be the class of all lattice ordered groups H such that (i) $H_D = H$; (ii) G is an l-subgroup of H; (iii) H is c-generated by G. The quasiorders $\leqq_1$ and $\leqq_2$ in the class $C_1(G)$ can be defined analogously as we did for $C(G)$. The following problem remains open: Which results concerning $C(G)$ can be extended for $C_1(G)$?

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ПОПОЛНЕНИЯ СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

В этой статье исслеуется класс $C(G)$ всех пополнений архимедовой структурно упорядоченной группы G. Доказано, что в $C(G)$ может отсутствовать наибольший элемент и что $C(G)$ может быть собственным классом. Если G — полное прямое произведение линъйно упорядоченных групп, то card $C(G) = 1$. Рассмотрены соотношения между прямыми разложениями G и прямыми разложениями структурно упорядоченных групп, принадлежащих $C(G)$.