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THE INFLUENCE OF ARGUMENT DELAY ON OSCILLATORY PROPERTIES OF A SECOND-ORDER DIFFERENTIAL EQUATION

JÁN OHRISKA

Consider a differential equation

$$u''(t)+p(t)u^{\alpha}(\tau(t))=0 \tag{1}$$

on $[t_0, \infty]$, where

(i) $0\leq p(t)\in C_{[t_0,\infty)}$; $p(t)$ is not identically zero in any neighborhood $o(\infty)$;

(ii) $\tau(t)$ is a nondecreasing continuous function on $[t_0, \infty)$, $\tau(t)\leq t$, and $\lim_{t\to\infty}\tau(t)=\infty$;

(iii) $\alpha=\frac{r}{s}$, where r and s are odd natural numbers.

Without mentioning them again, we shall assume the validity of conditions (i), (ii) and (iii) throughout the paper.

The basic initial-value problem for (1) is defined as follows: *Define a continuous function $\Phi(t)$ on an initial set E_{t_0}. Suppose that u'_0 is an arbitrary real number. Find a solution $u(t)$ of (1) on $[t_0, T)(T\leq\infty)$ which satisfies the initial conditions*

$$u(t_0)=\Phi(t_0), \qquad u'(t_0+0)=u'_0$$

$$u(\tau(t))=\Phi(\tau(t)) \quad \text{for} \quad \tau(t)<t_0.$$

Suppose that there exist solutions of (1) on $[t_0, \infty)$. For this there is a sufficient condition, e.g. that the step method (cf. [2]) be applicable for extending the solutions. In the sequel we shall use the term "*solution*" only to denote a solution which exists on $[t_0, \infty)$. Moreover, we shall exclude from our considerations solutions of the equation of type (1) with the property that $u(t)\equiv 0$ for $t\geq T_1$, where $t_0\leq T_1<\infty$.

Definition 1. *A solution $u(t)$ of (1) is oscillatory for $t\geq t_0$ if there exists an infinite sequence of points $\{t_i\}_{i=1}^{\infty}$ such that $u(t_i)=0$ and $t_i\to\infty$ for $i\to\infty$. A* *solution $u(t)$ of (1) is non-oscillatory if there exists a number T_2 such that $t_0 \leq T_2 < \infty$ and $u(t) \neq 0$ for $t \geq T_2$.*

Definition 2. *Equation (1) is oscillatory if all its solutions are oscillatory. It is non-oscillatory if at least one of its solutions is non-oscillatory.*

Definition 3. *Let (1) be oscillatory for $\tau(t) \equiv t$. We shall say that an argument delay $\tau(t) \not\equiv t$ influences the oscillatory properties of solutions of (1) if for this $\tau(t) \not\equiv t$ equation (1) is non-oscillatory. If, on the other hand, for some argument delay $\tau(t)$, (1) is oscillatory, we shall say that $\tau(t)$ does not influence the oscillatory properties of solutions of (1).*

In [3], H. E. Gollwitzer showed that if $0 < \alpha < 1$ and $0 < t - \tau(t) \leq M$ (where M is a constant), then equation (1) is oscillatory if and only if

$$\int^{\infty} t^{\alpha} p(t)\,\mathrm{d}t = \infty . \tag{2}$$

The following theorem shows that the condition (2) is necessary even if $t - \tau(t) \to \infty$ for $t \to \infty$.

Theorem 1. *Let $0 < \alpha < 1$. A necessary condition for (1) to be oscillatory is that*

$$\int^{\infty} t^{\alpha} p(t)\,\mathrm{d}t = \infty .$$

Proof. The proof is indirect — a modification of the proof given in [4] by Lička and Švec for $\tau(t) \equiv t$. Let $\int^{\infty} t^{\alpha} p(t)\,\mathrm{d}t < \infty$. Then there exists $t_1 > t_0 \geq 0$ such that

$$\int_{t_1}^{\infty} t^{\alpha} p(t)\,\mathrm{d}t < \frac{1}{2}$$

Without loss of generality we can assume that $\tau(t_1) \geq 0$.

Let us investigate a solution $u(t)$ of (1) which satisfies the initial conditions

$$\begin{aligned} u(t) &= 0 \quad \text{for} \quad \tau(t_1) \leq t \leq t_1 , \\ u'(t_1 + 0) &= 1 . \end{aligned} \tag{3}$$

We state that this solution has no zeros on (t_1, ∞), and proceed to prove this assertion.

Let t_2 be the first zero of $u(t)$ greater than t_1. Then $u(t) \geq 0$ for $t \in [\tau(t_1), t_2]$. According to Rolle's theorem, there exists $\xi \in (t_1, t_2)$ such that $u'(\xi) = 0$. However, we can prove that $u'(t) \neq 0$ for $t \in (t_1, t_2)$. Suppose that $t \in (t_1, t_2)$. Then for $x \in (t_1, t]$ we have $\tau(x) \in [\tau(t_1), \tau(t)] \subset [\tau(t_1), t]$ and therefore $u(\tau(x)) \geq 0$. Looking at (1), we see that $u''(x) \leq 0$ for $x \in (t_1, t]$ so that $u'(x)$ is non-increasing on this interval. From (3) we see that $u'(x) \leq 1$ for $x \in (t_1, t]$ and $u'(x) = 0$ for $x \in (\tau(t_1), t_1)$.

Calculate

$$\int_{\tau(t_1)}^{\tau(x)} u'(s)\mathrm{d}s\,.$$

If for $x>t_1$ is $\tau(x)\leqslant t_1$, then

$$\int_{\tau(t_1)}^{\tau(x)} u'(s)\mathrm{d}s=0 \quad \text{and} \quad u(\tau(x))=0\leqslant x\,.$$

If $t_1<\tau(x)\leqslant t$, then

$$\int_{\tau(t_1)}^{\tau(x)} u'(s)\mathrm{d}s=\int_{\tau(t_1)}^{t_1} u'(s)\mathrm{d}s+\int_{t_1}^{\tau(x)} u'(s)\mathrm{d}s=\int_{t_1}^{\tau(x)} u'(s)\mathrm{d}s\leqslant\int_{t_1}^{\tau(x)}\mathrm{d}s\,,$$

whence $u(\tau(x))-u(\tau(t_1))\leqslant\tau(x)-t_1$, or $u(\tau(x))\leqslant\tau(x)\leqslant x$. Thus $u(\tau(x))\leqslant x$ for $x\in(t_1,t]$. Then also

$$\int_{t_1}^{t} p(x)u^\alpha(\tau(x))\mathrm{d}x\leqslant\int_{t_1}^{t} x^\alpha p(x)\mathrm{d}x\,.$$

Integrating (1) from t_1 to t ($t_1\leqslant t<t_2$), we get

$$u'(t)=1-\int_{t_1}^{t} p(x)u^\alpha(\tau(x))\mathrm{d}x\geqslant 1-\int_{t_1}^{t} x^\alpha p(x)\mathrm{d}x\geqslant 1-\frac{1}{2}=\frac{1}{2}\,,$$

which proves our assertion.

Thus $u'(t)$ has no zeros to the right of t_1 which means that any solution which satisfies (3) is non-oscillatory. This completes the proof.

Odarič and Ševelo [5] proved that for $\alpha>0$ the condition

$$\int^{\infty} p(t)\mathrm{d}t=\infty$$

is sufficient for (1) to be oscillatory. Thus in this case argument delay has no influence on the oscillatory properties of solutions of (1). We shall therefore assume in the sequel that $\int^{\infty} p(t)\mathrm{d}t<\infty$.

Theorem 2. *Let* $0<\alpha<1$. *Let* $H(t)$ *be a function such that* $H(t)\in C^1_{[t_0,\infty)}$, $H'(t)\geqslant 0$ *and* $\lim_{t\to\infty} H(t)=\infty$. *Let*

$\tau(t)\geqslant H(t)$ *on some neighborhood* $o_1(\infty)$. (4)

If

$$\int^{\infty} H^\alpha(t)p(t)\mathrm{d}t=\infty\,, \tag{5}$$

then (1) *is oscillatory.*

Proof. The proof will be indirect using the method by which Atkinson [1] proved this Theorem 1. Suppose that the hypotheses of the theorem hold and that (1) has a non-oscillatory solution $u(t)$. Because of (ii) there exists $t_1 \geq t_0$ such that neither $u(t)$ nor $u(\tau(t))$ is zero for $t \geq t_1$. Furthermore, without loss of generality we can assume that $u(t) > 0$, $u(H(t)) > 0$ and $u(\tau(t)) > 0$ for $t \geq t_1$ and $t_1 \in o_1(\infty)$. From (1) we can now see that $u''(t) \leq 0$ for $t \geq t_1$, so that $u'(t)$ is non-increasing for $t \geq t_1$. Since the solution $u(t)$ is assumed to be positive and (i) holds, it is evident that $u'(t)$ is a positive function converging to a nonnegative value as $t \to \infty$ (if this were not true, it would mean that $\lim_{t \to \infty} u'(t) \leq C < 0$ and therefore $u(t) \to -\infty$ for $t \to \infty$, which is a contradiction).

Integrating (1) from t_1 to t ($t \geq t_1$) yields the result

$$u'(t) - u'(t_1) + \int_{t_1}^{t} p(x) u^{\alpha}(\tau(x)) \, dx = 0 .$$

Since $0 \leq \lim_{t \to \infty} u'(t) < \infty$, the last equation yields

$$\int_{t_1}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx < \infty . \tag{6}$$

This enables us to integrate (1) from t to ∞ ($t \geq t_1$) and we have

$$\lim_{z \to \infty} u'(z) - u'(t) + \int_{t}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx = 0$$

and therefore

$$u'(t) \geq \int_{t}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx . \tag{7}$$

Since $u'(t)$ is non-increasing for $t \geq t_1$, we can use (ii) and (4) to obtain

$$u'(H(t)) \geq u'(\tau(t)) \geq u'(t) \quad \text{for} \quad t \geq t_1 \tag{8}$$

(where $u'(r(t))$ denotes the value of the derivative of $u(t)$ at the point $r(t)$). Now, using (7) and (8), it is possible to write

$$u'(H(t)) \geq \int_{t}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx , \qquad t \geq t_1 .$$

Multiplying the last inequality by $H'(t)$ and integrating from t_2 to t ($t \geq t_2 \geq t_1$), we obtain

$$u(H(t)) - u(H(t_2)) \geq \int_{t_2}^{t} H'(s) \int_{s}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx \, ds .$$

Since $u(H(t_2))>0$, we see that

$$u(H(t)) \geqslant \int_{t_2}^{t} H'(s) \int_{s}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx \, ds . \tag{9}$$

Changing the order of integration in the formula (9), we get

$$\begin{aligned} u(H(t)) \geqslant & \int_{t_2}^{t} \int_{t_2}^{x} H'(s) p(x) u^{\alpha}(\tau(x)) \, ds \, dx + \\ & + \int_{t}^{\infty} \int_{t_2}^{t} H'(s) p(x) u^{\alpha}(\tau(x)) \, ds \, dx . \end{aligned} \tag{10}$$

Since the first integral on the right of (10) is positive and

$$u(\tau(x)) \geqslant u(H(x)) \quad \text{for} \quad x \geqslant t_1 \tag{11}$$

because of (4) and the fact that $u'>0$, we have

$$\begin{aligned} u(H(t)) \geqslant & \int_{t_2}^{t} H'(s) \, ds \int_{t}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx \geqslant \\ & \geqslant [H(t) - H(t_2)] \int_{t}^{\infty} p(x) u^{\alpha}(H(x)) \, dx . \end{aligned} \tag{12}$$

Raising both sides of (12) to the power α and multiplying by $p(t)$, we have

$$p(t) u^{\alpha}(H(t)) \left[\int_{t}^{\infty} p(x) u^{\alpha}(H(x)) \, dx \right]^{-\alpha} \geqslant [H(t) - H(t_2)]^{\alpha} p(t) . \tag{13}$$

Integrating (13) from t_3 to t $(t \geqslant t_3 \geqslant t_2)$ yields

$$\begin{aligned} \frac{1}{1-\alpha} \left[\int_{t_3}^{\infty} p(x) u^{\alpha}(H(x)) \, dx \right]^{1-\alpha} - \frac{1}{1-\alpha} \left[\int_{t}^{\infty} p(x) u^{\alpha}(H(x)) \, dx \right]^{1-\alpha} \geqslant \\ \geqslant \int_{t_3}^{t} [H(s) - H(t_2)]^{\alpha} p(s) \, ds . \end{aligned} \tag{14}$$

From (6) and (11) we see that the first term on the left of (14) is positive and finite and the second term converges to zero as $t \to \infty$; thus the left side of (14) is positive and finite for $t \to \infty$. Since the right part is nonnegative, it is also finite, i.e.

$$\int_{t_3}^{\infty} [H(s) - H(t_2)]^{\alpha} p(s) \, ds < \infty .$$

It is easy to show that $\int^{\infty} [H(s) - H(t_2)]^{\alpha} \; p(s) \, ds < \infty$ if and only if $\int^{\infty} H^{\alpha}(s) p(s) \, ds < \infty$, which yields a contradiction with (5) and completes the proof of the theorem.

In [5] it is proved that if $0<\alpha<1$ and $\tau'(t)\geq 0$, then the condition

$$\int^{\infty} \tau'(t)p(t)\,\mathrm{d}t=\infty$$

is sufficient for (1) to be oscillatory.

The following corollary of Theorem 2 shows when it is possible to replace this condition by (2), or what supplementary condition ensures that the condition given by Gollwitzer in [3] remains sufficient when $t-\tau(t)\to\infty$ for $t\to\infty$.

Corollary 2.1. *Let* $0<\alpha<1$ *and* $\tau(t)\in C_{[t_0,\infty)}$. *Let*

$$\tau(t)\geq kt \tag{15}$$

on some neighborhood $o_1(\infty)$ $(0<k\leq 1)$ *and*

$$\int^{\infty} t^{\alpha}p(t)\,\mathrm{d}t=\infty.$$

Then equation (1) *is oscillatory.*

Theorem 1 and Corollary 2.1 furnish the basis for the following

Assertion 1. *Let* $0<\alpha<1$ *and* $\tau(t)\geq kt$ *on some neighborhood* $o_1(\infty)$, *where* $0<k\leq 1$. *Then equation* (1) *is oscillatory if and only if*

$$\int^{\infty} t^{\alpha}p(t)\,\mathrm{d}t=\infty. \tag{16}$$

Ličko and Švec proved in [4] that (16) is a necessary and sufficient condition for the equation

$$y''(t)+p(t)y^{\alpha}(t)=0, \qquad \alpha<1$$

to be oscillatory.

Comparison of our Assertion 1 with the result from [4] shows that an argument delay $\tau(t)$ satisfying (15) has no influence on the oscillatory properties of solutions of (1).

Let us therefore consider the conditions which enable the argument delay to influence the oscillatory properties of solutions of (1).

The following two corollaries of Theorem 2 give us information about such conditions.

Corollary 2.2. *Let* $0<\alpha<1$ *and* $\tau(t)\in C_{[t_0,\infty)}$. *Let* $\tau(t)\geq kt^{1-\beta}$ *on some neighborhood* $o_1(\infty)$ $(0<\beta<1,\ k>0)$ *and*

$$\int^{\infty} t^{(1-\beta)\alpha}p(t)\,\mathrm{d}t=\infty.$$

Then equation (1) *is oscillatory.*

Corollary 2.3. *Let* $0<\alpha<1$ *and* $\tau(t)\in C_{[t_0,\infty)}$. *Let* $\tau(t)\geq k\cdot\ln t$ *on some neighborhood* $o_1(\infty)$ $(k>0)$ *and*

$$\int^{\infty} \ln^{\alpha}t\, p(t)\,\mathrm{d}t=\infty\,.$$

Then equation (1) *is oscillatory.*

The following example shows that the condition (16) alone does not ensure the oscillatoriness of (1) if no further assumptions are made concerning $\tau(t)$.

Example 1. Consider the equation

$$u''(t)+\frac{3}{16}\,\frac{1}{t^{59/40}}\,u^{3/5}(\tau(t))=0\,. \tag{17}$$

This equation is oscillatory for $\tau(t)\equiv t$, since $\int^{\infty} t^{3/5}p(t)\,\mathrm{d}t=\infty$. For $\tau(t)=t^{1/2}$ (17) has a non-oscillatory solution $u(t)=t^{3/4}$. Let us remark that for $\tau(t)=t^{1/2}$ the hypotheses of Corollary 2.2 are not satisfied.

The result of the preceding considerations is the following

Assertion 2. *Let* $0<\alpha<1$ *and* $\int^{\infty} p(t)\,\mathrm{d}t<\infty$. *Let* $\tau(t)\in C^1_{[t_0,\infty)}$ *and* $\tau'(t)\geq 0$. *Then a necessary condition for the oscillatory properties of solution of* (1) *to be influenced by the argument delay* $\tau(t)$ *is that* $\liminf_{t\to\infty}\tau'(t)=0$.

Proof. Since $\tau'(t)\geq 0$ by hypothesis, $\liminf_{t\to\infty}\tau'(t)\geq 0$. Note that $\liminf_{t\to\infty}\tau'(t)$ cannot be greater than 1 because (as can be shown quite easily) if this were the case, we should have $\tau(t)>t$ for sufficiently large t, which contradicts (ii). Suppose that $1\geq\liminf_{t\to\infty}\tau'(t)=c>0$. This means that there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k\to\infty$ for $k\to\infty$ and the sequence $\{\tau'(t_k)\}_{k=1}^{\infty}$ has c as a limit. Moreover, for each $\varepsilon>0$ there exists $T(\varepsilon)$ such that $\tau'(t)>c-\varepsilon$ for $t>T(\varepsilon)$. Putting $\varepsilon=\frac{1}{2}c$, we obtain

$$\tau'(t)>c-\frac{1}{2}c=\frac{1}{2}c>0 \quad \text{for} \quad t>T\left(\frac{1}{2}c\right). \tag{18}$$

From Assertion 1 we know that if (18) holds, then (1) is oscillatory if and only if (16) holds; this, however, is necessary and sufficient for (1) with $\tau(t)\equiv t$ to be oscillatory, which would mean that $\tau(t)$ has no influence on the oscillatory properties of solutions of (1). This contradiction completes the proof of our assertion.

Let us now leave unchanged everything that has been said so far in the paper including the definitions, and investigate the case $\alpha > 1$ or $\alpha \geqslant 1$. As before, let $\int^{\infty} p(t)\,\mathrm{d}t < \infty$.

In [3], H. E. Gollwitzer proved that for $\alpha > 1$ and $0 < t - \tau(t) \leqslant M$ (where M is a constant) (1) is oscillatory if and only if $\int^{\infty} tp(t)\,\mathrm{d}t = \infty$. We shall show that this condition is necessary also if $t - \tau(t) \to \infty$ for $t \to \infty$.

Theorem 3. *Let* $\alpha \geqslant 1$. *A necessary condition for* (1) *to be oscillatory is that*

$$\int^{\infty} tp(t)\,\mathrm{d}t = \infty. \tag{19}$$

Proof. We shall give an indirect proof of the theorem. Let

$$\int^{\infty} tp(t)\,\mathrm{d}t < \infty. \tag{20}$$

Using this assumption, we can show that there exists a solution of (1) such that

$$\lim_{t \to \infty} u(t) = 1, \qquad \lim_{t \to \infty} u'(t) = 0, \tag{21}$$

which is therefore evidently nonoscillatory.

It can be verified directly that if the integral equation

$$u(t) = 1 - \int_t^{\infty} (x - t)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x \tag{22}$$

has a solution $u(t)$ which is continuous and bounded for $t \to \infty$, then it is also a solution of (1) satisfies the condition (21). We shall prove the existence of such a solution of (22) using Banach's fixed-point theorem.

Let $S > 1$. Because of (20) there exists $t_1 > 0$ such that

$$\int_{t_1}^{\infty} (x - t_1)p(x)\,\mathrm{d}x < \frac{S-1}{S^{\alpha}}. \tag{23}$$

Let $t_0 = \tau(t_1)$. Let $\mathscr{A}$ denote the set of all functions $u(t)$ bounded and continuous for $t \in [t_0, \infty)$; this is a Banach space with the norm $\|u\| = \sup_{t \in [t_0, \infty]} |u(t)|$. Let $\mathscr{A}_1$ denote the subset of $\mathscr{A}$ defined as follows:

$$\mathscr{A}_1 = \{u(t) \in \mathscr{A} \mid \|u\| \leqslant S\}.$$

Then $\mathscr{A}_1$ is a complete metric space with the metric $\varrho(u_1, u_2) = \|u_1 - u_2\|$.

On $\mathscr{A}_1$ we define the operator $\mathbf{V}$ using the right part of (22), i.e. for every $u \in \mathscr{A}_1$ we put

$$(\mathbf{V}u)(t)=1-\int_{t_1}^{\infty}(x-t_1)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x \quad \text{for} \quad t\in[t_0, t_1]$$

and

$$(\mathbf{V}u)(t)=1-\int_{t}^{\infty}(x-t)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x \quad \text{for} \quad t\geqslant t_1 .$$

We shall show that for every $u\in\mathscr{A}_1$ also $\mathbf{V}u\in\mathscr{A}_1$. In fact, let $u\in\mathscr{A}_1$. Since $p(t)$, $u(t)$ and $\tau(t)$ are continuous, $(\mathbf{V}u)(t)$ is continuous on $[t_0, \infty)$. For $t\in[t_0, t_1]$, (23) yields

$$|(\mathbf{V}u)(t)|\leqslant 1+\int_{t_1}^{\infty}(x-t_1)p(x)\|u(\tau(x))\|^{\alpha}\,\mathrm{d}x\leqslant$$

$$\leqslant 1+S^{\alpha}\int_{t_1}^{\infty}(x-t_1)p(x)\,\mathrm{d}x<S .$$

Analogously, for $t\geqslant t_1$ we get

$$|(\mathbf{V}u)(t)|\leqslant 1+\int_{t}^{\infty}(x-t)p(x)\|u(\tau(x))\|^{\alpha}\,\mathrm{d}x\leqslant$$

$$\leqslant 1+S^{\alpha}\int_{t}^{\infty}(x-t)p(x)\,\mathrm{d}x\leqslant 1+S^{\alpha}\int_{t_1}^{\infty}(x-t_1)p(x)\,\mathrm{d}x<S .$$

It is now easy to see that $\|\mathbf{V}u\|\leqslant S$. Therefore $u\in\mathscr{A}_1\Rightarrow\mathbf{V}u\in\mathscr{A}_1$.

Finally, we prove that $\mathbf{V}$ is contractive on $\mathscr{A}_1$. Let $F(u)=u^{\alpha}$ for $|u|\leqslant S$. Then

$$\left|\frac{\partial F(u)}{\partial u}\right|\leqslant\alpha|u|^{\alpha-1}\leqslant\alpha S^{\alpha-1},$$

and therefore

$$|F(u_1)-F(u_2)|\leqslant\alpha S^{\alpha-1}|u_1-u_2|$$

for any two elements u_1 and u_2 such that $|u_1|\leqslant S$, $|u_2|\leqslant S$, or

$$|F(u_1(t))-F(u_2(t))|\leqslant\alpha S^{\alpha-1}|u_1(t)-u_2(t)|$$

for $u_1(t)$, $u_2(t)\in\mathscr{A}_1$. For $t\in[t_0, t_1]$ we have

$$\begin{aligned}|(\mathbf{V}u_1)(t)-(\mathbf{V}u_2)(t)|&\leqslant\int_{t_1}^{\infty}(x-t_1)p(x)|u_1^{\alpha}(\tau(x))-u_2^{\alpha}(\tau(x)|\,\mathrm{d}x\leqslant\\ &\leqslant\alpha S^{\alpha-1}\int_{t_1}^{\infty}(x-t_1)p(x)|u_1(\tau(x))-u_2(\tau(x))|\,\mathrm{d}x\leqslant\\ &\leqslant\alpha S^{\alpha-1}\int_{t_1}^{\infty}(x-t_1)p(x)\,\mathrm{d}x\,\|u_1-u_2\|\end{aligned}\tag{24}$$

and analogously for $t\geqslant t_1$

$$|(\mathbf{V}u_1)(t)-(\mathbf{V}u_2)(t)|\leqslant\int_t^\infty (x-t)p(x)|u_1^\alpha(\tau(x))-u_2^\alpha(\tau(x))|\,\mathrm{d}x\leqslant$$
$$\leqslant\alpha S^{\alpha-1}\int_t^\infty (x-t)p(x)\,\mathrm{d}x\,\|u_1-u_2\|\leqslant\alpha S^{\alpha-1}\int_{t_1}^\infty (x-t_1)p(x)\,\mathrm{d}x\,\|u_1-u_2\|. \tag{25}$$

Thus from (24) and (25) it follows that

$$\|\mathbf{V}u_1-\mathbf{V}u_2\|\leqslant\alpha S^{\alpha-1}\int_{t_1}^\infty (x-t_1)p(x)\,\mathrm{d}x\,\|u_1-u_2\|.$$

Clearly it is sufficient to choose $t_1>0$ such that, besides (23), the inequality

$$\int_{t_1}^\infty (x-t_1)p(x)\,\mathrm{d}x<\frac{1}{\alpha S^{\alpha-1}}$$

also holds; the operator $\mathbf{V}$ is then contractive on $\mathscr{A}_1$.

From Banach's fixed-point theorem we see that there exists a unique solution of (1) satisfying the conditions (21). This completes the proof.

A sufficient condition for (1) to be oscillatory is given in the following theorem.

Theorem 4. *Let $\alpha>1$, $H(t)\in C^1_{[t_0,\infty)}$, $H'(t)\geqslant 0$ and $\lim_{t\to\infty} H(t)=\infty$. Furthermore, let $\tau(t)\geqslant H(t)$ on some neighborhood $o_1(\infty)$. If*

$$\int^\infty H(t)p(t)\,\mathrm{d}t=\infty, \tag{26}$$

then (1) is oscillatory.

Proof. The proof will again be indirect. We shall start just as in the proof of Theorem 2, up to inequality (10). The second integral on the right of (10) is positive and (11) holds. This enables us to write

$$u(H(t))\geqslant\int_{t_2}^t\int_{t_2}^x H'(s)p(x)u^\alpha(H(x))\,\mathrm{d}s\,\mathrm{d}x$$

or

$$u(H(t))\geqslant\int_{t_2}^t [H(x)-H(t_2)]p(x)u^\alpha(H(x))\,\mathrm{d}x. \tag{27}$$

Raising both sides of (27) to the power α and multiplying by $[H(t)-H(t_2)]p(t)$ yields

$$[H(t)-H(t_2)]p(t)u^\alpha(H(t))\left[\int_{t_2}^t (H(x)-H(t_2))p(x)u^\alpha(H(x))\,\mathrm{d}x\right]^{-\alpha}\geqslant$$
$$\geqslant[H(t)-H(t_2)]p(t).$$

Integrating this inequality from t_3 to $t\,(t>t_3>t_2)$ we get

$$\frac{1}{\alpha-1}\left\{\frac{1}{[\int_{t_2}^{t_3}(H(x)-H(t_2))p(x)u^\alpha(H(x))\,\mathrm{d}x]^{\alpha-1}}-\right.$$
$$\left.-\frac{1}{[\int_{t_2}^{t}(H(x)-H(t_2))p(x)u^\alpha(H(x))\,\mathrm{d}x]^{\alpha-1}}\right\}\geqslant\int_{t_3}^{t}[H(s)-H(t_2)]p(s)\,\mathrm{d}s\,. \tag{28}$$

The first term on the left of (28) is a finite and positive number, and for $t\to\infty$ the whole left part is finite and positive. The right part of (28) is also positive and therefore finite for $t\to\infty$, i.e.

$$\int_{t_3}^{\infty}[H(s)-H(t_2)]p(s)\,\mathrm{d}s<\infty\,.$$

As $\int_{t_3}^{\infty}[H(s)-H(t_2)]p(s)\,\mathrm{d}s<\infty$ if and only if $\int_{t_3}^{\infty}H(s)p(s)\,\mathrm{d}s<\infty$, we have obtained a contradiction to (26), thus completing the proof.

In [5] it is proved that if $\alpha>1$, $\tau'(t)\geqslant 0$ and $\int^{\infty}\tau(t)p(t)\,\mathrm{d}t=\infty$, then equation (1) is oscillatory. We shall now formulate a corollary of Theorem 4 which is concerned with the possibility to replace the condition $\int^{\infty}\tau(t)p(t)\,\mathrm{d}t=\infty$ by the condition $\int^{\infty}tp(t)\,\mathrm{d}t=\infty$ or, in other words, the supplementary condition necessary for making Gollwitzer's [3] condition sufficient also if $t-\tau(t)\to\infty$ for $t\to\infty$.

Corollary 4.1. *Let $\alpha>1$ and $\tau(t)\in C_{[t_0,\infty)}$. Let*

$$\tau(t)\geqslant kt \quad \text{on some neighborhood} \quad o_1(\infty)(0<k\leqslant 1) \tag{29}$$

and

$$\int^{\infty}tp(t)\,\mathrm{d}t=\infty\,.$$

Then (1) is oscillatory.

Theorem 3 and Corollary 4.1 are the basis for

Assertion 3. *Let $\alpha>1$ and $\tau(t)\geqslant kt$ on some neighborhood $o_1(\infty)$ where $0<k\leqslant 1$. Then (1) is oscillatory if and only if*

$$\int^{\infty}tp(t)\,\mathrm{d}t=\infty\,. \tag{30}$$

In [1], Atkinson proved that (30) is a necessary and sufficient condition for the oscillatoriness of the equation

$$y''(t)+p(t)y^\alpha(t)=0, \qquad \alpha>1\,. \tag{31}$$

Comparison of Assertion 3 with the result given in [1] shows that if (29) is satisfied, then (1) and (31) are equivalent in the sense that (1) is oscillatory if and only if (31) is.

Thus if (29) is satisfied, then the argument delay has no influence on the oscillatory properties of solutions of (1).

Let us now investigate the conditions which permit the argument delay to influence the oscillatory properties of solutions of (1).

Corollary 4.2. *Let $\alpha > 1$ and $\tau(t) \in C_{[t_0,\infty)}$. Let $\tau(t) \geq kt^{1-\beta}$ on some neighborhood $o_1(\infty)$ $(0<\beta<1,\ k>0)$. Then*

$$\int^{\infty} t^{1-\beta} p(t)\,\mathrm{d}t = \infty, \tag{32}$$

is a sufficient condition for (1) to be oscillatory.

Corollary 4.3. *Let $\alpha > 1$ and $\tau(t) \in C_{[t_0,\infty)}$. Let $\tau(t) \geq k \cdot \ln t$ on some neighborhood $o_1(\infty)$ $(k>0)$. Then*

$$\int^{\infty} \ln t\, p(t)\,\mathrm{d}t = \infty,$$

is a sufficient condition for (1) to be oscillatory.

The following example is intended to show that the condition (30) does not ensure the oscillatoriness of (1) unless further assumptions are made concerning $\tau(t)$.

Example 2. Consider the equation

$$u''(t) + \frac{1}{4}\,\frac{1}{t^{15/8}}\,u^3(\tau(t)) = 0. \tag{33}$$

For $\tau(t) \equiv t$ it is oscillatory because $\int^{\infty} tp(t)\,\mathrm{d}t = \infty$. On the other hand, for $\tau(t) = t^{1/4}$ there exists a non-oscillatory solution of (33), namely $u(t) = \sqrt{t}$. Note that condition (32) does not hold for $\tau(t) = t^{1/4}$.

From our investigation of (1) for $\alpha > 1$ the following assertion follows.

Assertion 4. *Let $\alpha > 1$ and $\int^{\infty} p(t)\,\mathrm{d}t < \infty$. Let $\tau(t) \in C^1_{[t_0,\infty)}$ and $\tau'(t) \geq 0$. Then*

$$\liminf_{t\to\infty} \tau'(t) = 0$$

is a necessary condition for $\tau(t)$ to influence the oscillatory properties of (1).

The proof of this Assertion is essentially the same as that of Assertion 2, the only diference being that Assertion 3 is now used instead of Assertion 1.

Finally, let us investigate the question of conditions under which $\tau(t)$ does influence the oscillatory properties of solutions of (1).

Theorem 5. *Let $\alpha > 0$ and $\tau(t) \in C^1_{[t_0, \infty)}$. Let $g(t)$ be locally integrable on $[t_0, \infty)$ and let*

$$0 \leq \tau'(t) \leq g(t) \quad \textit{on some neighborhood } o_1(\infty). \tag{34}$$

Let

$$\int^{\infty} G^{\alpha}(t)p(t)\,\mathrm{d}t < \infty, \quad \textit{where} \quad G(t) = \int^{t} g(s)\,\mathrm{d}s. \tag{35}$$

Then (1) *is non-oscillatory.*

Proof. The proof will be direct. Let us start as in the proof of Theorem 1. (35) ensures the existence of a point $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} G^{\alpha}(t)p(t)\,\mathrm{d}t < \frac{1}{2}.$$

Whithout loss of generality we can assume that $t_1 \in o_1(\infty)$.

Consider a solution of (1) which satisfies the initial conditions

$$\begin{aligned} u(t) &= 0 \quad \text{for} \quad \tau(t_1) \leq t \leq t_1 \\ u'(t_1 + 0) &= 1. \end{aligned} \tag{36}$$

We shall prove that this solution has no zeros on (t_1, ∞). Let t_2 be the first zero of $u(t)$ greater than t_1. Then $u(t) \geq 0$ for $t \in [\tau(t_1), t_2]$. According to Rolle's theorem, there exists $\xi \in (t_1, t_2)$ such that $u'(\xi) = 0$. We shall now prove that $u'(t) \neq 0$ for $t \in (t_1, t_2)$. Let $t \in (t_1, t_2)$. Then for $x \in (t_1, t]$ we have $\tau(x) \in [\tau(t_1), \tau(t)] \subset [\tau(t_1), t]$ so that $u(\tau(x)) \geq 0$. By (1) this implies that $u''(x) \leq 0$ for $x \in (t_1, t]$, i.e. $u'(x)$ does not increase on this interval. From (36) we see that $u'(x) \leq 1$ for $x \in (t_1, t]$ and $u'(x) = 0$ for $x \in (\tau(t_1), t_1)$. It follows that

$$u'(\tau(x)) \leq 1 \quad \text{for} \quad \tau(x) \in (t_1, t]$$

and

$$u'(\tau(x)) = 0 \quad \text{for} \quad \tau(x) \in (\tau(t_1), t_1)$$

(where $u'(\tau(x))$ is the value of the derivative of $u(t)$ for $t = \tau(x)$).

By (34) this means that

$$u'(\tau(x))\tau'(x) \leq g(x) \quad \text{for} \quad \tau(x) \in (t_1, t]$$

and

$$u'(\tau(x))\tau'(x) = 0 \quad \text{for} \quad \tau(x) \in (\tau(t_1), t_1).$$

Let us calculate $\int_{t_1}^{t} u'(\tau(x))\tau'(x)\,\mathrm{d}x$. If $\tau(t) \leq t_1$, then

$$\int_{t_1}^{t} u'(\tau(x))\tau'(x)\,\mathrm{d}x = 0$$

and

$$u(\tau(t)) = 0 \leqslant G(t).$$

If $\tau(t) \in (t_1, t]$, then there exists $\eta \in [t_1, t]$ such that $\tau(\eta) = t_1$ and we can write

$$\int_{t_1}^{t} u'(\tau(x))\tau'(x)\,\mathrm{d}x = \int_{t_1}^{\eta} u'(\tau(x))\tau'(x)\,\mathrm{d}x + \int_{\eta}^{t} u'(\tau(x))\tau'(x)\,\mathrm{d}x =$$

$$= \int_{\eta}^{t} u'(\tau(x))\tau'(x)\,\mathrm{d}x \leqslant \int_{\eta}^{t} g(x)\,\mathrm{d}x = G(t).$$

Hence $u(\tau(t)) - u(\tau(t_1)) \leqslant G(t)$, or $u(\tau(t)) \leqslant G(t)$, for $t \in (t_1, t_2)$. In that case also

$$\int_{t_1}^{t} p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x \leqslant \int_{t_1}^{t} G^{\alpha}(x)p(x)\,\mathrm{d}x\,, \qquad t_1 < t < t_2.$$

Integrating (1) from t_1 to t (where $t_1 < t < t_2$), we obtain (because of (36))

$$u'(t) = 1 - \int_{t_1}^{t} p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x \geqslant 1 - \int_{t_1}^{t} G^{\alpha}(x)p(x)\,\mathrm{d}x \geqslant$$

$$\geqslant 1 - \int_{t_1}^{\infty} G^{\alpha}(x)p(x)\,\mathrm{d}x \geqslant 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus $u'(t)$ has no zeros to the right of t_1, i.e. the solution $u(t)$ of (1) which satisfies the initial conditions (36) is non-oscillatory. This completes the proof.

Corollary 5.1. *Let* $\alpha > 0$, $\tau(t) \in C^1_{[t_0, \infty)}$, $0 \leqslant \tau'(t) \leqslant t^{\ \beta}$ *on some neighborhood* $o_1(\infty)$ $(0 < \beta < 1)$ *and*

$$\int^{\infty} t^{(1-\beta)\alpha} p(t)\,\mathrm{d}t < \infty.$$

Then (1) *is nonoscillatory.*

Corollary 5.2. *Let* $\alpha > 0$, $\tau(t) \in C^1_{[t_0, \infty)}$, $0 \leqslant \tau'(t) \leqslant t^{\ 1}$ *on some neighborhood* $o_1(\infty)$ *and*

$$\int^{\infty} \ln^{\alpha} t\; p(t)\,\mathrm{d}t < \infty.$$

Then (1) *is non-oscillatory.*

Theorem 5 and its corollaries show that the argument delay will influence the oscillatory properties of solutions of (1) if $\tau'(t)$ approaches zero sufficiently quickly, where the "sufficient speed" depends on the function $p(t)$, which was to be expected anyway.

Example 3. Consider the equation

$$u''(t)+\frac{3}{16\sqrt{2}}\,\frac{1}{t^{21/16}}\,u^{\frac{1}{3}}\quad(\tau(t))=0. \tag{37}$$

For $\tau(t)\equiv t$ the equation is oscillatory, because $\int^{\infty} t^{\frac{1}{3}}\,p(t)\,\mathrm{d}t=\infty$. By Corollary 2.2, (37) is oscillatory for $\tau(t)=t^{\frac{15}{16}}$, since

$$\int^{\infty} t^{(1-\frac{1}{16})\frac{1}{3}}p(t)\mathrm{d}t=\infty\,.$$

For $\tau(t)=4t^{\frac{1}{4}}$, (37) has a non-oscillatory solution according to Corollary 5.1, because

$$\int^{\infty} t^{(1-\frac{3}{4})\frac{1}{3}}p(t)\,\mathrm{d}t<\infty\,.$$

A non-oscillatory solution of (37) for $\tau(t)=4t^{\frac{1}{4}}$ is $u(t)=t^{\frac{3}{4}}$.

Example 4. Consider equation

$$u''(t)+\frac{6}{25}\,\frac{1}{t^{\frac{7}{5}}\ln t}\,u^{\frac{5}{3}}\quad(\tau(t))=0. \tag{38}$$

For $\tau(t)\equiv t$ this equation is oscillatory, as $\int^{\infty} tp(t)\,\mathrm{d}t=\infty$. By Corollary 4.2, (38) is also oscillatory for $\tau(t)=\sqrt{t}$, because

$$\int^{\infty} t^{1-\frac{1}{2}}p(t)\,\mathrm{d}t=\infty\,.$$

However, for $\tau(t)=\ln t$ the equation has, according to Corollary 5.2, a non-oscillatory solution, because

$$\int^{\infty} \ln^{\frac{5}{3}}t\;p(t)\,\mathrm{d}t<\infty\,.$$

$u(t)=t^{\frac{3}{5}}$ is a non-oscillatory solution of (38) for $\tau(t)=\ln t$.

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ВЛИЯНИЕ ЗАПАЗДЫВАНИЯ АРГУМЕНТА НА КОЛЕБЛЕМОСТЬ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА

Ян Огриска

Резюме

В работе рассматривается дифференциальное уравнение

$$u''(t)+p(t)u^{\alpha}(\tau(t))=0. \tag{1}$$

Предполагается, что функция $\tau(t)\in C_{[t_0,\infty)}$ неубывающая и

$$\tau(t)\leq t, \qquad \lim_{t\to\infty}\tau(t)=\infty.$$

Теорема 1 (теорема 2) дает необходимое (достаточное) условие колеблемости уравнения (1) если $0<\alpha<1$. Теорема 3 (теорема 4) приводит необходимое (достаточное) условие колеблемости уравнения (1) если $\alpha>1$. Теорема 5, в предложении что $\tau(t)\in C^1_{[t_0,\infty)}$ и $\alpha>0$, дает достаточное условие неколеблемости уравнения (1). При помощи этих теорем автор занимается вопросом каким образом изменяются достаточные условия колеблемости уравнения (1) в зависимости от характера изменения функции $\tau(t)$ и приводит необходимые условия для того, чтобы запаздывание $\tau(t)$ влияло на колеблемость решений уравнения (1).