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MAXIMAL ADDITIVE AND MAXIMAL MULTIPLICATIVE FAMILY FOR THE FAMILY OF $\mathcal{B}$-DARBOUX BAIRE ONE FUNCTIONS

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1. In his book [1], p. 14, A. M. Bruckner defines the maximal additive and the maximal multiplicative family for a given family F of real functions in this way: *A subfamily F_0 of the family F is called the maximal additive (multiplicative) family for F iff F_0 is the set of all functions f of F such that $f+g\in F$ ($fg\in F$) for all $g\in F$.*

In [2], p. 109, A. M. Bruckner and J. G. Ceder proved that *the maximal additive family for the family of all real Darboux Baire one functions of a real variable is the family of all real continuous functions of a real variable.*

In the cited book [1], p. 15, A. M. Bruckner presents the problem of finding the maximal multiplicative family for the same family. Recently, R. Fleissner solved this problem in [3]. *The maximal multiplicative family for the family of all real Darboux Baire one functions of a real variable is the family of all real Darboux Baire one functions f of a real variable which have the following property:*

If f is discontinuous from the right (from the left) at a, then $f(a)=0$ and there exists a decreasing (an increasing) sequence $\{a_n\}_{n=1}^{\infty}$ converging to a such that $f(a_n)=0$ for all n.

Let X be a topological space and let $\mathcal{B}$ be a base for the topology in X. In [4] there is given the following definition: *A real function f defined on X is called $\mathcal{B}$-Darboux iff for each $A\in\mathcal{B}$, every $x, y\in\bar{A}$ ($\bar{A}$ denotes the closure of A) and each $c\in(\min(f(x), f(y)), \max(f(x), f(y)))$ there exists a point $z\in A$ such that $f(z)=c$.*

It is natural to ask whether similar characterizations as the above one hold also for the maximal additive family and for the maximal multiplicative family for some families of all real $\mathcal{B}$-Darboux Baire one functions. In this paper it will be demonstrated that similar characterizations hold for such families of functions if X is a finite-dimensional Banach space with a strictly convex norm and if $\mathcal{B}$ is the base of all spherical neighbourhoods. The characterization of the maximal multiplicative family and the maximal additive family for the family of all $\mathcal{B}$-Darboux Baire one functions if X is an euclidean space and $\mathcal{B}$ is the base of all open intervals in X is given in [6].

2. The proofs of the cited propositions on the maximal additive family and on the maximal multiplicative family in the case of real functions of a real variable are based on the following three facts:

a) Let $a \in (-\infty, \infty)$. If f is a discontinuous function from the right (from the left) at a, then there exists a closed interval $I = \langle a, b \rangle$ ($I = \langle c, a \rangle$) and α, β such that $\alpha < \beta$ and for each decreasing (increasing) sequence $\{a_n\}_{n=1}^{\infty}$ contained in I and converging to a, there holds: $\alpha = \sup_n \inf f((a, a_n)) < \inf_n \sup f((a, a_n)) = \beta$

($\alpha = \sup_n \inf f((a_n, a)) < \inf_n \sup f((a_n, a)) = \beta$).

b) Each real Darboux Baire one function defined on a closed interval I possesses an extension in the family of all real Darboux Baire one functions of a real variable.

c) For the family of all real Baire one functions of a real variable the Young criterion states the condition under which a real Baire one function has or has not the Darboux property.

We recall that the generalization of the Young criterion for real Baire one functions in the case of $\mathcal{B}$-Darboux functions was proved in [5]. This generalization of Young's criterion is as follows:

Theorem 1. **(Satz 9,** p. 425 in [5]) *Let X be a complete metric space and let $\mathcal{B}$ be a base in X having the following two properties:*

(1*) *For each open neighbourhood U of a point $x \in X$ and for each $B \in \mathcal{B}$ satisfying $x \in \bar{B}$ there exists a $C \in \mathcal{B}$ such that $C \subset U \cap B$ and $x \in \bar{C} - C$.*

(2) *For each $B \in \mathcal{B}$ and for each decomposition of B into two non empty disjoint sets A_1 and A_2 such that $\bar{U} \cap B \subset A_1$, resp. $\bar{U} \cap B \subset A_2$ for each $U \in \mathcal{B}$, which is contained in A_1, resp. A_2, the sets $A_1' \cap A_2$ and $A_1 \cap A_2'$ are non empty (A_1' denotes the derivative set of A_1).*

Then a real Baire one function f defined on X is $\mathcal{B}$-Darboux iff for each $B \in \mathcal{B}$ and for each $x \in X$ satisfying $x \in \bar{B} - B$, there exists a simple sequence $\{x_n\}_{n=1}^{\infty}$ converging to x such that $x_n \in B$ for $n = 1, 2, 3, \ldots$ and $\lim_{n \to \infty} f(x_n) = f(x)$.

3. Now we give some propositions concerning strictly convex Banach spaces. We recall that a Banach space X is strictly convex iff for every $x, y \in X$ the equality $\|x + y\| = \|x\| + \|y\|$ implies that there exists a non negative number λ such that $x = \lambda y$.

Lemma 1. *Let X be a strictly convex Banach space, let $U_r = \{x \in X: \|x\| < r\}$ and $V = b + U_r$ and $W = a + U_p$, where r and p are positive. Let $x \in X$ and $x \in \bar{V} - V$ and $x \in \bar{W} - W$. Then $W \subset V$ holds iff $p \leqq r$ and $a = \lambda b + (1 - \lambda)x$ for appropriate $\lambda \in (0, 1)$.*

Proof. Let $W \subset V$. Then $2p = \operatorname{diam} W \leqq \operatorname{diam} V = 2r$ (diam W is the diameter of W) and thus $p \leqq r$. There holds $r-p = \|b-x\| - \|a-x\| \leqq \|b-a\|$. If $b-a=0$, we have $p=r$ and $a=b$. Let $\|b-a\|>0$. Then we have $b-\frac{r}{\|b-a\|}(b-a) \notin V$ and therefore also $b-\frac{r}{\|b-a\|}(b-a) \notin W$. This gives: $r-\|b-a\| = \left(\frac{r}{\|b-a\|} - 1\right)\|b-a\| = \left\|\left(\frac{r}{\|b-a\|}-1\right)(b-a)\right\|$ $= \left\|a-b+\frac{r}{\|b-a\|}(b-a)\right\| \geqq p$. Thus we have that $\|b-a\| \leqq r-p$ and therefore there holds that $\|(b-a) + (a-x)\| = \|b-x\| = r = \|b-a\| + p = \|b-a\| + \|a-x\|$. Thus there exists a non negative number α such that $b-a = \alpha(a-x)$, which implies $a=\frac{1}{1+\alpha}b + \frac{\alpha}{1+\alpha}x$.

Let $p \leqq r$ and $a = \lambda b + (1-\lambda)x$ for $\lambda \in (0, 1\rangle$. Let $u \in W$. Then $\|u-a\| < p = \|a-x\| = \lambda r$. Therefore holds that $\|b-u\| \leqq \|b-a\| + \|a-u\| = (1-\lambda) r + \|a-u\| < r$. Thus $u \in V$. Therefore $W \subset V$.

Lemma 2. *Let X be a strictly convex Banach space, let $x \in X$, $a_n \in X$, $b_n \in X$, $b \in X$, $r_n>0$, $p_n>0$ and $r>0$ for all n. Let $V = b + U_r$, $V_n = b_n + U_{r_n}$, $W_n = a_n + U_{p_n}$, $x \in \bar{V} - V$, $x \in \bar{V}_n - V_n$, $x \in \bar{W}_n - W_n$, $V_{n+1} \subset V_n \subset V$, $W_{n+1} \subset W_n \subset$ V for all n and $\lim_{n\to\infty} \operatorname{diam} W_n = \lim_{n\to\infty} \operatorname{diam} V_n = 0$. Then for each $n=1, 2, 3, \ldots$ there exists k_n and l_n such that $W_{k_n} \subset V_n$ and $V_{l_n} \subset W_n$.*

Proof. From Lemma 1 we have: $b_n = \lambda_n b + (1-\lambda_n)x$ and $a_n = \mu_n b + (1-\mu_n)x$ for some $\lambda_n, \mu_n \in (0, 1\rangle$. There holds: $2p_n = \operatorname{diam} W_n = 2\mu_n r$, $2r_n = \operatorname{diam} V_n = 2\lambda_n r$ and therefore $\lim_{n\to\infty} p_n = \lim_{n\to\infty} r\mu_n = \lim_{n\to\infty} r_n = \lim_{n\to\infty} r\lambda_n = 0$. Thus for each $n=1, 2, 3, \ldots$, there exists k_n and l_n such that $\mu_{k_n} < \lambda_n$ and $\lambda_{l_n} < \mu_n$. Then $p_{k_n} < r_n$, $r_{l_n} < p_n$, $\frac{\mu_{k_n}}{\lambda_n}, \frac{\lambda_{l_n}}{\mu_n} \in (0, 1)$ and $a_{k_n} = \frac{\mu_{k_n}}{\lambda_n} b_n + \left(1-\frac{\mu_{k_n}}{\lambda_n}\right)x$, $b_{l_n} = \frac{\lambda_{l_n}}{\mu_n} a_n + \left(1-\frac{\lambda_{l_n}}{\mu_n}\right)x$. From Lemma 1 we get that $W_{k_n} \subset V_n$ and $V_{l_n} \subset W_n$.

4. Let X be a metric space and let $\mathcal{B}$ be a base in X. Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in \bar{B} - B$. *We shall say that a sequence $\{C_n\}_{n=1}^{\infty}$ of elements of $\mathcal{B}$ converges from B to x iff $x \in \bar{C}_n - C_n$, $C_{n+1} \subset C_n \subset B$ for $n=1, 2, 3, \ldots$ and $\lim_{n\to\infty} \operatorname{diam} C_n = 0$. We shall say that a real function f defined on X is $\mathcal{B}$-discontinuous from B at x iff there exists a sequence $\{C_n\}_{n=1}^{\infty}$ converging from B to x such that* $\sup_n \inf f(C_n) < \inf_n \sup f(C_n)$.

We shall say that a metric space X and its base $\mathcal{B}$ have the property (a) *iff for* each $x \in X$, for each $B \in \mathcal{B}$ satisfying $x \in \bar{B} - B$ and for each real function f $\mathcal{B}$-discontinuous from B at x there exists $D \in \mathcal{B}$ and α, β such that $D \subset B$, $x \in \bar{D} - D$ and for each sequence $\{C_n\}_{n=1}^{\infty}$ converging from D to x we have $\alpha = \sup_n \inf f(C_n) < \inf_n \sup f(C_n) = \beta$.

Let X be a topological space and $\mathcal{B}$ be a base in X. *We shall say that a real function defined on $\bar{B}$, where $B \in \mathcal{B}$, is $\mathcal{B}$-Darboux on $\bar{B}$ iff for each $U \in \mathcal{B}$ contained in B, for each x, $y \in \bar{U}$ and for each $c \in (\min(f(x), f(y)), \max(f(x), f(y)))$, there exists a point $z \in U$ such that $f(z) = c$.*

We shall say that a metric space X and its base $\mathcal{B}$ have the property (b) *iff for each real $\mathcal{B}$-Darboux Baire one function φ defined on $\bar{B}$, where $B \in \mathcal{B}$, there exists an extension in the family of all real $\mathcal{B}$-Darboux Baire one functions defined on X.*

Lemma 3. *Let X be a separable Banach space and let S be a sphere $\{x \in X: \|x - a\| = r\}$, where $a \in X$ and $r > 0$. Let $\varepsilon > 0$. Then there exists a subset $A \subset S$ such that $\|a - b\| > \varepsilon$ for every $a, b \in A$, $a \neq b$ and for each $x \in S$, there exists an $a \in A$ such that $\|x - a\| < 2\varepsilon$.*

Proof. As X is separable, there exists a countable dense set H in S. By mathematical induction it is easy to see that there exists a subset A of H such that

(i) for every b, $c \in A$, $b \neq c$, we have $\|b - c\| > \varepsilon$,

(ii) for each $x \in H$, there exists a $y \in A$ such that $\|x - y\| \leqq \varepsilon$.

Now let $x \in S$. Then there exists a $y \in H$ such that $\|x - y\| < \varepsilon$. By (ii) there exists a $b \in A$ such that $\|y - b\| \leqq \varepsilon$. But then we have $\|x - b\| < 2\varepsilon$.

Lemma 4. *Let X be a separable Banach space of dimension at least two, let $\varepsilon > 0$ and let n be a positive integer. Let S be a sphere $\{x \in X: \|x - a\| = r\}$, where $a \in X$, $r > 0$. Then there exists a continuous function f_n defined on S such that $|f_n(x)| \leqq n$ for each $x \in S$ and such that for each $u \in S$ there is $-n = \min f_n(D) < \max f_n(D) = n$, where $D = \{z \in S: \|z - u\| \leqq \varepsilon\}$.*

Proof. Let A be a set of Lemma 3 for $\frac{\varepsilon}{3}$. It is easy to see that there exists a subset B of S disjoint with A such that there is one and only one $c \in A$ for each $b \in B$ such that $\|b - c\| < \frac{\varepsilon}{8}$ and such that there exists one and only one $b \in B$ for each $c \in A$ such that $\|b - c\| < \frac{\varepsilon}{8}$.

Let $F = A \cup B$. Then F is a closed subset of S. Indeed, let $\eta = \frac{\varepsilon}{24}$ and $u \in \bar{F}$. Let D be a subset $\{z \in S: \|z - u\| \leqq \eta\}$ of S. Then it is evident that the intersection $D \cap F$ has at most two points. Therefore $u \in F$.

By the construction of A and B, it is evident that A and B are closed subsets of S. Let φ_n be a function defined on $A \cup B$ as follows: $\varphi_n(b) = -n$ for $b \in B$ and

$\varphi_n(c) = n$ for each $c \in A$. By the Tietze extension theorem, there exists a continuous function f_n defined on S such that $|f_n(z)| \leqq n$ for each $z \in S$ and $f_n(z) = \varphi_n(z)$ for each $z \in A \cup B$.

It is easy to prove that f_n is a desired function in the lemma.

Proposition 1. *Let* $(X, \|.\|)$ *be a strictly convex Banach space of finite dimension. Let* $\mathscr{B}$ *be the family of all sets of form* $a + U_r$*, where* $a \in X$*,* $r > 0$ *and* $U_r = \{x \in X: \|x\| < r\}$.

Then for X and $\mathscr{B}$ (1*), (2), (a) *and* (b) *are satisfied.*

Proof. The property (1*) is evident.

(2) Let $B = a + U_r$, where $a \in X$ and $r > 0$. Let $B = A_1 \cup A_2$, where A_1 and A_2 are non empty disjoint subsets of B satisfying $\bar{U} \cap B \subset A_1$, resp. $\bar{U} \cap B \in A_2$, for each $U \in \mathscr{B}$ contained in A_1, resp. A_2. It is easy to prove that $A_1 \subset A_1'$ and $A_2 \subset A_2'$. Let $A_1' \cap A_2 = \emptyset$. Then $A_1 \subset B \cap A_1' \subset A_1$ and the set A_1 is closed relatively to B. Then A_2 is a non empty open set relatively to B. From the connectivity of B it follows that $A_1 \cap A_2' \neq \emptyset$. Let $u \in A_1 \cap A_2'$. Then there exists a positive number ϱ such that $u + U_{2\varrho} \subset B$. Then there exists a point v such that $v \in A_2 \cap (u + U_\varrho)$. The point v is an interior point of A_2. Therefore the set $W = \cup\{v + U_\tau: \tau > 0, v + U_\tau \subset A_2\}$ is a set of the form $v + U_\varepsilon$ for some positive number ε. There holds $\varepsilon < \|u - v\|$, because $u \in A_1$. Since $\varepsilon < \|u - v\|$, the set $K = A_1 \cap (\overline{(v + U_{\|u-v\|})} - W)$ is a non empty compact set and $K \cap (\bar{W} - W) = \emptyset$ (there holds $\bar{W} - W \subset A_2$). Therefore there must exist a positive number η such that $\|x - y\| \geqq \eta$ for each $x \in K$ and each $y \in \bar{W} - W$. But then $v + U_{\varepsilon+\eta} \subset A_2$. This gives $W = v + U_\varepsilon \subsetneqq v + U_{\varepsilon+\eta} \subset W$, which is impossible.

(a) Let $x \in X$, $B \in \mathscr{B}$ and $x \in \bar{B} - B$. Then $B = a + U_r$ and $\|x - a\| = r > 0$. Let f be a real function $\mathscr{B}$-discontinuous from B at x. Then there exists a sequence $\{C_n\}_{n=1}^{\infty}$ such that $x \in \bar{C}_n - C_n$, $C_{n+1} \subset C_n \subset B$ for $n = 1, 2, 3, \ldots$, $\lim_{n\to\infty} \operatorname{diam} C_n = 0$ and $\alpha = \sup_n \inf f(C_n) < \inf_n \sup f(C_n) = \beta$. Then there exist an $a_n \in X$ and a $r_n > 0$ such that $\lim_{n\to\infty} r_n = 0$ and $C_n = a_n + U_{r_n}$ for all n.

From Lemma 1 we get: $r_{n+1} \leqq r_n$ and $a_n = \frac{r_n}{r} a + \left(1 - \frac{r_n}{r}\right)x$. We put $D = B$. Then for each sequence $\{D_n\}_{n=1}^{\infty}$ of elements of $\mathscr{B}$ converging from B to x we have: $\alpha = \sup \inf f(D_n) < \inf \sup f(D_n) = \beta$, since there exist, by Lemma 2, positive integers p_n and q_n such that $D_{q_n} \subset C_n$ and $C_{p_n} \subset D_n$.

b) This follows from the following extension theorem:

Theorem 2. (Extension theorem) *Let X be a separable Banach space and let* $\mathscr{B}$ *be the system of all sets* $a + U_r$*, where* $a \in X$*,* $U_r = \{x \in X: \|x\| < r\}$ *and* $r > 0$*. Let* $B \in \mathscr{B}$*. Let* φ *be a real* $\mathscr{B}$*-Darboux Baire one function on* $\bar{B}$*. Then there exists a* $\mathscr{B}$*-Darboux Baire one function defined on X which is an extension of* φ.

Proof. Let $B=a+U_r$, where $a\in X$ and $r>0$. Let $S=a+\{x\in X: \|x\|=r\}$. Then $S=\bar{B}-B$. Let $B_n = \left\{x\in X: \|x-a\|<r\left(1-\frac{1}{n+1}\right)\right\}$ and $S_n = \left\{x\in X: \|x-a\| = r\left(1+\frac{1}{n}\right)\right\}$.

If X is a one-dimensional Banach space, then the theorem is evidently true.

Let X be an at least two-dimensional space. Then let f_n be a function defined on S_n from Lemma 4 for $\varepsilon=\frac{r}{n}$. Since the function φ is on $\bar{B}$ of the Baire class one, there exists a sequence $\{h_n\}_{n=1}^{\infty}$ of continuous functions defined on $\bar{B}$ such that $\lim_{n\to\infty} h_n(x) = \varphi(x)$ and $|h_n(x)|\leqq n$ for each $x\in\bar{B}$. By the Tietze extension theorem, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of continuous functions defined on X such that $|g_n(x)|\leqq n$ for each $x\in X$, $g_n(x)=f_n(x)$ for each $x\in S_n$, $g_n(x)=h_n(x)$ for each $x\in\bar{B}_n\cup S$ and $g_{n+1}(x) = g_n(x)$ for each $x\in X$ satisfying the inequality $\|x-a\|\geqq r\left(1+\frac{1}{n}\right)$ and for $n = 1, 2, 3, \ldots$. It is easy to prove that the limit $\lim_{n\to\infty} g_n(x)$ exists for each $x\in X$. Let $f(x) = \lim_{n\to\infty} g_n(x)$ for each $x\in X$. Then $f(x)=\varphi(x)$ for each $x\in\bar{B}$. For $x\in X$, which satisfies the inequality $\|x-a\| \geqq r\left(1+\frac{1}{n}\right)$, there holds $f(x) = g_n(x)$. Therefore f is of the first class of Baire and it is an extension of φ.

Let $C\in\mathscr{B}$, $x, y\in\bar{C}$ and $\min(f(x), f(y)) < c < \max(f(x), f(y))$. If $\bar{C}\subset\bar{B}$, then $f(x) = \varphi(x)$, $f(y) = \varphi(y)$ and there exists a $z\in C$ such that $\varphi(z)=c$. But then $f(z)=\varphi(z)$ and therefore $f(z)=c$.

If $\bar{C}\subset X-\bar{B}$, then the function f is continuous on $\bar{C}$ and therefore there exists a $z\in C$ such that $f(z)=c$.

If $\bar{C}-\bar{B}\neq\emptyset$ and $\bar{C}\cap\bar{B}\neq\emptyset$, then $C-\bar{B}$ is a non empty open set. Let n be a positive integer such that $-n<c<n$. We can easily prove that there exist a positive integer k and a point u such that $u\in S_k$, $k\geqq n$ and $\overline{u+U_{r/k}}-\bar{B}\subset C$. Then $D=S_k\cap\overline{(u+U_{r/k})} = \left\{v\in S_k: \|v-u\| \leqq \frac{r}{k}\right\}$. From Lemma 4 it follows that $-k = \min f_k(D) = \min f(D) < \max f(D) = \max f_k(D) = k$. Therefore there exists a $z\in D$ such that $f(z) = f_k(z) = c$. It is evident that $z\in C$. We have proved that the function f is $\mathscr{B}$-Darboux on X, and thus the extension theorem is proved.

Proposition 2. *Let X be a strictly convex Banach space of finite dimension. Let $\mathscr{B} = \{a + \{x\in X: \|x\|<r\} : a\in X, r>0\}$. Let f be a real $\mathscr{B}$-Darboux Baire one function on X. Then f is discontinuous at x iff it is $\mathscr{B}$-discontinuous from some* B at x.

Proof. If X is a one-dimensional Banach space, it is evident. If f is $\mathscr{B}$-discontinuous from some B at x, then it is obvious that f is discontinuous at x.

Let X be a strictly convex Banach space of dimension at least two and let f be discontinuous at x. Since f is a $\mathscr{B}$-Darboux Baire one function on X which is discontinuous at x, there holds: $\alpha = \sup_{r>0} \inf f(x + U_r) < \inf_{r>0} \sup f(x + U_r) = \beta$ and $\alpha \leqq f(x) \leqq \beta$. Let $S = \{z \in X\colon \|z - x\| = 1\}$. Since S is compact, there exists a finite subset A of S such that for each $z \in S$ there exists an $a \in A$ such that $\|z - a\| < 1$. For each $a \in A$ we put $S_a = \{u \in X\colon \|u - a\| < 1\}$. Then $x \in \bar{S}_a - S_a$ for each $a \in A$. Let $\{C_{a,n}\}_{n=1}^{\infty}$ be a sequence of elements of $\mathscr{B}$ such that $\{C_{a,n}\}_{n=1}^{\infty}$ converges from S_a to x. Let $\alpha_a = \sup_n \inf f(C_{a,n}) \leqq \inf_n \sup f(C_{a,n}) = \beta_a$ for each $a \in A$. Since f is $\mathscr{B}$-Darboux, we have $\alpha_a \leqq f(x) \leqq \beta_a$ for each $a \in A$.

We shall assume that f is not $\mathscr{B}$-discontinuous from any B of $\mathscr{B}$ at x. Then $\alpha_a = \beta_a = f(x)$ for each $a \in A$. Let η be a positive number that satisfies $(\alpha, \beta) - (f(x) - \eta, f(x) + \eta) \neq \emptyset$. Since A is finite and since $\alpha_a = \beta_a = f(x)$ for each $a \in A$, there exists an n such that $C_{a,n} \subset S_a$ and $f(C_{a,n}) \subset (f(x) - \eta, f(x) + \eta)$ for each $a \in A$. Let $\varrho = \min\{\operatorname{diam} C_{a,n}\colon a \in A\}$. Let $u \in x + U_\varrho$, $u \neq x$. Then $\|x - u\| > 0$ and $v = x + \dfrac{1}{\|x - u\|}(u - x) \in S$. There exists an $a \in A$ such that $v \in S_a$. Let $C_{a,n} = b_a + U_{r_a}$, $b_a = \lambda_a a + (1 - \lambda_a)x$, $r_a = \|b_a - x\| = \lambda_a \geqq \varrho > \|x - u\|$. We put $c = \dfrac{\|x - u\|}{\lambda_a} v + \left(1 - \dfrac{\|x - u\|}{\lambda_a}\right)x$. Since $v \in S_a$, $x \in \bar{S}_a - S_a$, $0 < \dfrac{\|x - u\|}{\lambda_a} < 1$ and since X is a strictly convex Banach space, we have: $c \in S_a$. But then $\|b_a - u\| = \|\lambda_a(a - c)\| < \lambda_a$. Therefore $u \in C_{a,n}$. Thus we have proved that $(x + U_\varrho) - \{x\} \subset \cup\{C_{a,n}\colon a \in A\}$. Then we get: $f(x + U_\varrho) = f((x + U_\varrho) - \{x\}) \cup \{f(x)\} \subset \cup\{f(C_{a,n})\colon a \in A\} \cup \{f(x)\} \subset (f(x) - \eta, f(x) + \eta)$. Therefore there holds: $f(x) - \eta \leqq \inf f(x + U_\varrho) \leqq \alpha < \beta \leqq \sup f(x + U_\varrho) \leqq f(x) + \eta$. Thus $(\alpha, \beta) - (f(x) - \eta, f(x) + \eta) = \emptyset$. But this is impossible. Therefore f must be $\mathscr{B}$-discontinuous from some B at x.

5. **Theorem 3.** (The maximal additive family for the family of all $\mathscr{B}$-Darboux Baire one functions). *Let X be a finite dimensional strictly convex Banach space and let $\mathscr{B}$ be the system of all sets $a + U_r$, where $a \in X$, $U_r = \{x \in X\colon \|x\| < r\}$ and $r > 0$. The maximal additive family for the family of all $\mathscr{B}$-Darboux Baire one functions defined on X is the family of all continuous functions.*

Proof. Let f be a continuous function on X. According to the theorem 13 (Satz 13) in [5], p. 427, $f + g$ is a real $\mathscr{B}$-Darboux Baire one function for each $\mathscr{B}$-Darboux Baire one function g. Therefore f belongs to the maximal additive family for the family of all $\mathscr{B}$-Darboux Baire one functions defined on X.

Now let f be a function from the maximal additive family for the family of all $\mathscr{B}$-Darboux Baire one functions defined on X. Then f is evidently a real

$\mathscr{B}$-Darboux Baire one function, since $f+0=f$ is a real $\mathscr{B}$-Darboux Baire one function.

We shall assume that f is discontinuous at x. According to Proposition 2, it is $\mathscr{B}$-discontinuous from some B, $B \in \mathscr{B}$ at x. According to Proposition 1 (a) is satisfied. Therefore there exist a $D \in \mathscr{B}$ and two numbers α, β such that $\alpha<\beta$, $D \subset B$ and for each sequence $\{C_n\}_{n=1}^{\infty}$ converging from D to x we have: $\alpha = \sup_n \inf f(C_n) < \inf_n \sup f(C_n) = \beta$. There also holds that $\alpha \leqq f(x) \leqq \beta$, since f is a $\mathscr{B}$-Darboux function. Let g be a function defined on $\bar{B}$ as follows: $g(u) = f(u)$ for $u \in \bar{B} - \{x\}$ and $g(x) \in (\alpha, \beta) - \{f(x)\}$.

The function g is a Baire one function on $\bar{B}$ and we shall prove that it is also $\mathscr{B}$-Darboux on $\bar{B}$. Let $C \in \mathscr{B}$, $C \subset B$, u, $v \in \bar{C}$ and let $\min(g(u), g(v)) < c < \max(g(u), g(v))$. If $u \neq x$ and $v \neq x$, then there exists a point $z \in C$ such that $f(z)=c$, since $g(u)=f(u)$ and $g(v)=f(v)$. But there is also $z \neq x$ (z is in C) and therefore $g(z) = f(z) = c$. If $u=x$, then $x \in \bar{C} - C$ and $C \subset B$. From Lemma 1 we get that there exists an integer n such that $C_k \subset C$ for all $k \geqq n$. There exists a k such that $k \geqq n$ and $g(x) \in f(C_k) = g(C_k)$. Since $C_k \subset C$, it is $g(x) \in f(C)$. But then there exists a $z \in C$ such that $g(z)=f(z)=c$. In the case $v=x$ we proceed similarly.

The function $-g$ is also a $\mathscr{B}$-Darboux Baire one function on B. From the extension theorem there exists a function h which extends the function $-g$ and which is a $\mathscr{B}$-Darboux Baire one function on X. Therefore the function $k=f+h$ must be a $\mathscr{B}$-Darboux Baire one function on X. But $k(u) = f(u) + h(u) = g(u) + (-g(u)) = 0$ for each $u \in \bar{B} - \{x\}$ and $k(x) = f(x) + h(x) = f(x) - g(x) \neq 0$. Therefore the function k can not be a $\mathscr{B}$-Darboux function.

Thus we have proved that f cannot be $\mathscr{B}$-discontinuous from any B of $\mathscr{B}$ at any point of X. According to Proposition 2 the function f is continuous.

Theorem 4. (The maximal multiplicative family for the family of all $\mathscr{B}$-Darboux Baire one functions) *Let X be a finite dimensional strictly convex Banach space and let $\mathscr{B}$ be the system of all sets $a+U_r$, where $a \in X$, $U_r = \{x \in X: \|x\|<r\}$, $r>0$. The function f belongs to the maximal multiplicative family for the family of all $\mathscr{B}$-Darboux Baire one functions defined on X iff*

(i) *f is a $\mathscr{B}$-Darboux Baire one function on X*

(ii) *if it is discontinuous from B, $B \in \mathscr{B}$, at x, $x \in X$, then $f(x)=0$ and there exists a simple sequence $\{x_k\}_{k=1}^{\infty}$ of points of B such that $f(x_k)=0$ for $k=1,2,3,\ldots$ and $\lim_{k\to\infty} x_k = x$.*

Proof. Let f be an element of the maximal multiplicative family for the family of all $\mathscr{B}$-Darboux Baire one functions defined on X. Then f is a $\mathscr{B}$-Darboux Baire one function on X, since $f \cdot 1=f$ is a $\mathscr{B}$-Darboux Baire one function.

Let f be $\mathscr{B}$-discontinuous from B, $B \in \mathscr{B}$, at x, $x \in X$. From the property (a) there exist a $D \in \mathscr{B}$ and two numbers α, β such that $D \subset B$ and for each sequence $\{C_n\}_{n=1}^{\infty}$ converging from D to x we have: $\alpha = \sup_n \inf f(C_n) < \inf_n \sup f(C_n) = \beta$.

Let $f(x) \neq 0$. We can assume that $f(x) > 0$ (by multiplying by -1 we can transfer the case $f(x) < 0$ to the case $f(x) > 0$). For the number α either $\alpha > 0$ or $\alpha \leqq 0$ can hold.

We treat the case $\alpha > 0$. There exists a $C \in \mathscr{B}$ such that $f(C) \subset \left(\frac{\alpha}{2}, 2\beta\right)$. The function φ defined on $\bar{C}$ by $\varphi(u) = f(u)$ for $u \in \bar{C} - \{x\}$ and $\varphi(x) \in (\alpha, \beta) - \{f(x)\}$ is a $\mathscr{B}$-Darboux Baire one function on $\bar{C}$. According to the extension theorem there exists a $\mathscr{B}$-Darboux Baire one function g on X which extends φ. Let $h = \max\left(\frac{\alpha}{2}, g\right)$. According to Theorem 13 (Satz 13, [5], p. 427), the function h is a $\mathscr{B}$-Darboux Baire one function on X. For $u \in \bar{C}$ we have: $h(u) = g(u) = \varphi(u)$. The function $\frac{1}{h}$ is also a $\mathscr{B}$-Darboux Baire one function on X. In fact, it is a Baire one function, since h is a Baire one function and $h \geqq \frac{\alpha}{2}$. Let $B \in \mathscr{B}$, $u, v \in \bar{B}$ and $\min\left(\frac{1}{h(u)}, \frac{1}{h(v)}\right) < c < \max\left(\frac{1}{h(u)}, \frac{1}{h(v)}\right)$. Then $\min(h(u), h(v)) < \frac{1}{c} < \max(h(u), h(v))$. But h is a $\mathscr{B}$-Darboux function on X, therefore there exists a $z \in C$ such that $h(z) = \frac{1}{c}$. This gives that $\frac{1}{h(z)} = c$. Therefore $\frac{1}{h}$ is also a $\mathscr{B}$-Darboux function.

The function $\frac{f}{h}$ must be a $\mathscr{B}$-Darboux Baire one function, since f belongs to the maximal multiplicative family for the family of all $\mathscr{B}$-Darboux Baire one functions on X. But the function $\frac{f}{h}$ is not a $\mathscr{B}$-Darboux function on X, since $\left(\frac{f}{h}\right)(x) = \frac{f(x)}{\varphi(x)} \neq 1$ and $\left(\frac{f}{h}\right)(u) = \frac{f(u)}{\varphi(u)} = 1$ for all $u \in \bar{C} - \{x\}$.

Therefore the case $\alpha > 0$ is impossible.

Let $\alpha \leqq 0$. Then we have: $\alpha \leqq 0 < f(x) \leqq \beta$. Let ε be a such positive number that $0 < \varepsilon < \frac{f(x)}{2}$. Let φ be a function defined on $\bar{D}$ by the equality: $\varphi(u) = \max(\varepsilon, f(u))$ for $u \in \bar{D} - \{x\}$ and $\varphi(x) = \frac{f(x)}{2}$. The function φ is a Baire one function on $\bar{D}$. It is also a $\mathscr{B}$-Darboux function on $\bar{D}$. In fact, let $C \in \mathscr{B}$, $C \subset D$, $u, v \in \bar{C}$ and $\min(\varphi(u), \varphi(v)) < c < \max(\varphi(u), \varphi(v))$. Then we have: $\min(f(u), f(v)) \leqq \min(\varphi(u), \varphi(v)) < c < \max(\varphi(u), \varphi(v)) = \max(f(u), f(v))$ and $\varepsilon < c$. There exists a $z \in C$ such that $f(z) = c$. But there is $\varphi(z) = \max(\varepsilon, f(z))$

$= \max(\varepsilon, c) = c$. From the extension theorem we get a $\mathscr{B}$-Darboux Baire one function h on X which extends φ. Let $g = \max(\varepsilon, h)$. Then g is also a $\mathscr{B}$-Darboux Baire one function on X. It is also $\frac{1}{g}$ a $\mathscr{B}$-Darboux Baire one function. Therefore $\frac{f}{g}$ must be a $\mathscr{B}$-Darboux Baire one function on X. But the function $\frac{f}{g}$ is not a $\mathscr{B}$-Darboux function on X, since $\left(\frac{f}{g}\right)(u) = 1$ for each $u \in \bar{D} - \{x\}$ satisfying $\varepsilon \leqq f(u)$, $\left(\frac{f}{g}\right)(u) = \frac{f(u)}{\varepsilon} < 1$ for each $u \in \bar{D} - \{x\}$ satisfying $f(u) < \varepsilon$ and $\left(\frac{f}{g}\right)(x) = 2$. Therefore the case $\alpha \leqq 0$ is also impossible.

Therefore we cannot have $f(x) \neq 0$. Also we have proved that $f(x) = 0$.

If there does not exist a simple sequence $\{x_k\}_{k=1}^{\infty}$ of points of B converging to x such that $f(x_k) = 0$ for $k = 1, 2, 3, \ldots$, then there exists $C \in \mathscr{B}$ such that $C \subset B$, $x \in \bar{C} - C$ and $f(C) \subset (0, \infty)$ or $f(C) \subset (-\infty, 0)$. It is sufficient to treat the case $f(C) \subset (0, \infty)$. There exists an $E \in \mathscr{B}$ such that $x \in \bar{E} - E$, $E \subset C$ and $\operatorname{diam} E < \operatorname{diam} C$. Then $\bar{E} - \{x\} \subset C$. We define a function φ as follows: $\varphi(u) = f(u)$ for $u \in \bar{E} - \{x\}$ and $\varphi(x) \in (\alpha, \beta)$. Then we have $\varphi(x) \neq f(x)$. The function φ is a $\mathscr{B}$-Darboux Baire one function on $\bar{E}$. From $f(C) \subset (0, \infty)$, $\bar{E} - \{x\} \subset C$ and $\varphi(x) \in (\alpha, \beta)$ it follows that $\varphi(u) > 0$ for all $u \in \bar{E}$. According to the extension theorem there exists a $\mathscr{B}$-Darboux Baire one function g on X which extends $\frac{1}{\varphi}$. Therefore the function gf must be a $\mathscr{B}$-Darboux Baire one function on X. But there holds: $(gf)(u) = \frac{f(u)}{\varphi(u)} = 1$ for $u \in \bar{E} - \{x\}$ and $(gf)(x) = \frac{f(x)}{\varphi(x)} = 0$. Therefore the function gf can not be a $\mathscr{B}$-Darboux function. Thus there exists in B a simple sequence $\{x_k\}_{k=1}^{\infty}$ such that $\lim_{k \to \infty} x_k = x$ and $f(x_k) = 0$ for $k = 1, 2, 3, \ldots$.

Let f be a $\mathscr{B}$-Darboux Baire one function which satisfies: if $B \in \mathscr{B}$, $x \in X$ and f is $\mathscr{B}$-discontinuous from B at x, then $f(x) = 0$ and there exists a simple sequence $\{x_k\}_{k=1}^{\infty}$ of points of B such that $\lim_{k \to \infty} x_k = x$ and $f(x_k) = 0$ for $k = 1, 2, 3, \ldots$. Let g be a $\mathscr{B}$-discontinuous one function on X. Then gf is a Baire one function on X. To prove that gf is also $\mathscr{B}$-Darboux, we use the generalization of the Young theorem. Let $B \in \mathscr{B}$, $x \in X$, $x \in \bar{B} - B$. Let f be not $\mathscr{B}$-discontinuous from B at x. Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathscr{B}$ converging from B to x. Then $\sup_n \inf f(C_n) = \inf_n \sup f(C_n)$ holds. From the generalization of the Young theorem it follows that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in C_n$ and $\lim_{n \to \infty} g(x_n)$

$= g(x)$. From $x_n \in C_n$ and $f(x) = \sup_n \inf f(C_n) = \inf_n \sup f(C_n)$ it follows that $\lim_{n\to\infty} f(x_n) = f(x)$. Thus we have: $x_n \in B$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} (gf)(x_n) = (gf)(x)$.

Now let f be $\mathcal{B}$-discontinuous from B at x. Then $f(x)=0$ and there exists a simple sequence $\{x_n\}_{n=1}^{\infty}$ of points of B such that $f(x_n)=0$ for $n=1, 2, 3, \ldots$. Therefore we have: $\lim_{n\to\infty} (gf)(x_n) = 0 = (gf)(x)$. From the generalization of the theorem of Young it follows that the function gf is $\mathcal{B}$-Darboux. Thus we have proved that gf is a $\mathcal{B}$-Darboux Baire one function and therefore f belongs to the maximal multiplicative family for the family of all $\mathcal{B}$-Darboux Baire one functions defined on X.

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МАКСИМАЛЬНЫЙ АДДИТИВНЫЙ И МУЛЬТИПЛИКАТИВНЫЙ КЛАСС ДЛЯ КЛАССА ФУНКЦИЙ $\mathcal{B}$-ДАРБУ 1-ОГО КЛАССА БЭРА

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Резюме

В работе рассматривается максимальный аддитивный и максимальный мультипликативный класс для класса функций $\mathcal{B}$-Дарбу 1-ого класса Бэра, определенных на конечномерном строго выпуклом пространстве Банаха X, причем $\mathcal{B}$ является базисом шаровых окрестностей в X.