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A NOTE ON CATEGORIES OF PARTIAL ALGEBRAS

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Throughout the paper we shall consider partial algebras of a certain type τ. Four kinds of homomorphisms of partial algebras are defined in [2]:

Firstly, a homomorphism φ: $\mathfrak{A} \to \mathfrak{B}$ between partial algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a map φ: $A \to B$ such that, if $f^{\mathfrak{A}}(a_1, \ldots, a_n)$ is defined in $\mathfrak{A}$, then

(i) $f^{\mathfrak{B}}(a_1\varphi, \ldots, a_n\varphi)$ is defined in $\mathfrak{B}$;

(ii) $f^{\mathfrak{A}}(a_1, \ldots, a_n)\varphi = f^{\mathfrak{B}}(a_1\varphi, \ldots, a_n\varphi)$.

Here $f^{\mathfrak{A}}$ and $f^{\mathfrak{B}}$ denote the corresponding partial operations in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. If f is a nullary operation then we interpret this as follows: If $f^{\mathfrak{A}}(\emptyset)$ is defined, so is $f^{\mathfrak{B}}(\emptyset)$, and $(f^{\mathfrak{A}}(\emptyset))\varphi = f^{\mathfrak{B}}(\emptyset)$.

Secondly, a full homomorphism is a homomorphism φ: $\mathfrak{A} \to \mathfrak{B}$ such that $f^{\mathfrak{B}}(a_1\varphi, \ldots, a_n\varphi) = a\varphi$ implies that there exist $b, b_1, \ldots, b_n \in A$ with $f^{\mathfrak{A}}(b_1, \ldots, b_n) = b$ and $b_1\varphi = a_1\varphi, \ldots, b_n\varphi = a_n\varphi, b\varphi = a\varphi$.

Thirdly, a p-morphism (partial morphism) φ: $\mathfrak{A} \to \mathfrak{B}$ is a partial function φ: $A \to B$ (not necessarily defined on the whole A) such that if $f^{\mathfrak{B}}(a_1\varphi, \ldots, a_n\varphi)$ is defined, then

(i) $a = f^{\mathfrak{A}}(a_1, \ldots, a_n)$ is defined;

(ii) $a \in D(\varphi)$;

(iii) $f^{\mathfrak{B}}(a_1\varphi, \ldots, a_n\varphi) = a\varphi$.

Here $D(\varphi)$ is the domain of φ. For nullary partial operations we interpret the above to mean that if $f^{\mathfrak{B}}$ is defined, so is $f^{\mathfrak{A}}$ and $(f^{\mathfrak{A}}(\emptyset))\varphi = f^{\mathfrak{B}}(\emptyset)$.

Finally, a strong homomorphism is a map which is both a homomorphism and a p-morphism.

Therefore the following four kinds of categories of partial algebras can be considered: All partial algebras of type τ together with all homomorphisms, full homomorphisms, strong homomorphisms or p-morphisms form the category $\mathcal{P}$, the full category $\mathcal{F}$, the strong category $\mathcal{S}$ and the p-category, respectively, of all partial algebras of type τ.

The category $\mathcal{P}$ obviously has some nice properties: it is complete and cocomplete, the monomorphisms coincide with the injective morphisms. Moreover, on any set there is a free algebra. The behaviour of the others in this respect may be of some interest. Here is a report on some observations:

In the categories $\mathscr{F}$ and $\mathscr{S}$ every morphism is a monomorphism if and only if it is injective. In the p-category every monomorphism is injective but not every injective p-morphism is a monomorphism. In the $\mathscr{S}$ and p-categories every morphism is an epimorphism if and only if it is surjective. In the category $\mathscr{F}$ every morphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is an epimorphism if and only if $\mathfrak{B} = \mathrm{Im}^*\varphi$, where $\mathrm{Im}^*\varphi$ is the smallest subalgebra of $\mathfrak{B}$ containing $\mathrm{Im}\,\varphi$.

The categories $\mathscr{F}$, $\mathscr{S}$ and the p-category are closed with respect to products and contain free algebras on arbitrary sets if and only if the type τ is the empty sequence. The category $\mathscr{F}$ is closed with respect to coproducts if and only if the type τ is the empty sequence or $\tau = \langle 0 \rangle$. The category $\mathscr{S}$ is closed with respect to coproducts if and only if there are just unary operations or the type τ is the empty sequence and the p-category is closed with respect to coproducts if and only if there are no nullary operations.

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ЗАМЕТКА О КАТЕГОРИЯХ ЧАСТИЧНЫХ АЛГЕБР

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Резюме

В статье приведены некоторые основные свойства четырех сортов категорий частичных алгебр соответствующих четырем сортам гомоморпфизмов частичных алгебр. Эти свойства касаются существований свободных алгебр, прямых и свободных произведений и основных свойств мономорфизмов и эпиморфизмов.